A NOTE ON HURWITZIAN NUMBERS

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Abstract. In this note Hurwitzian numbers are defined for the nearest integer, and backward continued fraction expansions, and Nakada's α-expansions. It is shown that the set of Hurwitzian numbers for these continued fractions coincides with the classical set of such numbers.

1. Introduction

A real number x with a continued fraction expansion of the form

$$x = [a_0; a_1, \dots, a_n, \overline{a_{n+1}(k), \dots, a_{n+p}(k)}]_{k=0}^{\infty}, \tag{1}$$

is called *Hurwitzian* if a_0 is an integer, a_i's are all positive integers, $a_{n+1}, \dots, a_{n+p}$ (called *a quasi period* of x) are polynomials with rational coefficients which take positive integral values for $k = 0, 1, 2, \dots$, and at least one of them is not constant. A well-known example of such numbers is $e = [2; \overline{1, 2k+2, 1}]_{k=0}^{\infty}$; see [P] for more examples. Hurwitzian numbers are generalizations of numbers with an eventually periodic continued fraction expansion. An old and classical result states, that a number x is a quadratic irrational (that is, an irrational root of a polynomial of degree 2 with integer coefficients) if and only if x has a continued fraction expansion which is eventually periodic, i.e., if x is of the form

$$x = [a_0; a_1, \dots, a_p, \overline{a_{p+1}, \dots, a_{p+\ell}}], \quad p \geq 0, \ \ell \geq 1, \tag{2}$$

where the bar indicates the period, see [HW], [O] or [P] for various classical proofs of this result.

Apart from the regular continued fraction (RCF) expansion of x there are very many other—classical—continued fraction expansions of x, such as the nearest integer continued fraction (NICF) expansion, the 'backward' continued fraction expansion, and Nakada's α-expansions. In this note we will define what Hurwitzian numbers are for such continued fraction expansions and show that their set of Hurwitzian numbers coincides with the classical set of Hurwitzian numbers. As a by-product quadratic irrationals will have an eventually period expansion for each of these expansions.

2. Hurwitzian numbers for the NICF

Every $x \in \mathbb{R} \setminus \mathbb{Q}$ can be expanded in a unique continued fraction expansion

$$x = b_0 + \cfrac{e_1}{b_1 + \cfrac{e_2}{b_2 + \ddots + \cfrac{e_n}{b_n + \ddots}}} =: [\, b_0;\, e_1/b_1,\, e_2/b_2,\, \cdots,\, e_n/b_n,\, \cdots],$$

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satisfying $b_0 \in \mathbb{Z}$, $x - b_0 \in [-\frac{1}{2}, \frac{1}{2})$, $e_n = \pm 1$, $b_n \in \mathbb{N}$ and $e_{n+1} + b_n \geq 2$ for $n \geq 1$. This continued fraction expansion is known as the *nearest integer continued fraction* (NICF) expansion of x.

In [K] it is shown that the NICF expansion can be obtained from the RCF by singularizing the first, the third, etc. 1's in every block of consecutive 1's preceded by either a partial quotient different from 1 or preceded by a_0. This singularization process is based upon the identity

$$A + \cfrac{e}{1 + \cfrac{1}{B + \xi}} = A + e + \frac{-e}{B + 1 + \xi}.$$

Example 1. The NICF expansion of e is given by

$$[3; -1/4, \overline{-1/2,\ 1/(2k+5)}]_{k=0}^{\infty}.$$

In view of this example we have the following definition.

Definition 1. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then x has an NICF-Hurwitzian expansion if

$$x = [\, b_0;\, e_1/b_1,\, \dots,\, e_n/b_n,\, \overline{e_{n+1}/b_{n+1}(k),\, \dots,\, e_{n+p}/b_{n+p}(k)}\,]_{k=0}^{\infty} \tag{3}$$

where $b_0 \in \mathbb{Z}$, $x - b_0 \in [-\frac{1}{2}, \frac{1}{2})$, $e_n = \pm 1$, $b_n \in \mathbb{N}$ and $e_{n+1} + b_n \geq 2$ for $n \geq 1$. Moreover, for $i = 1, \dots, p$ we have that b_{n+i} are polynomials with rational coefficients which take positive integral values for $k = 0, 1, 2, \dots,$ and at least one of them is non-constant.

The following result gives the necessary and sufficient condition for an irrational number to have an NICF-Hurwitzian expansion.

Theorem 1. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then x is Hurwitzian if and only if x has an NICF-Hurwitzian expansion.*

Proof. Let x be a Hurwitzian number with RCF expansion (1). Let $m_0 \in \mathbb{N}$, $m_0 \geq n$, be such that $a_{m_0} > 1$, and that for all $m \geq m_0$ all non-constant polynomials a_m are greater than 1.

For $i \in \{1, \dots, p-1\}$, we consider 2 cases:

Case (i): $a_{m_0+i} = 1$. By definition of a Hurwitzian number there exist numbers $j_1 \in \{0, 1, \dots, i-1\}$ and $j_2 \in \{i+1, \dots, p\}$ for which $a_{m_0+j_1} > 1$, $a_{m_0+j_2} > 1$, and

$$a_{m_0+j_1+1} = \cdots = a_{m_0+i} = \cdots = a_{m_0+j_2-1} = 1.$$

In case $i - j_1$ is odd the digit $a_{m_0+i} = 1$ will be singularized, and in case $i - j_1$ is even it will not be singularized, but it will either change into $-1/2$ if $j_2 = i+1$, or into $-1/3$ if $j_2 \geq i+2$. Due to the quasi-periodicity we have for each $k \in \mathbb{N}$ that

$$a_{m_0+j_1+kp+1} = \cdots = a_{m_0+i+kp} = \cdots = a_{m_0+j_2+kp-1} = 1,$$

and each of these blocks is singularized in the same way as the block $a_{m_0+j_1+1} = \cdots = a_{m_0+i} = \cdots = a_{m_0+j_2-1}$ was singularized, which means the same thing will happen to $a_{m_0+i+(k-1)p} = 1$ for all $k \in \mathbb{N}$.

Case (ii): $a_{m_0+i} > 1$ (a_{m_0+i} is either a constant or a polynomial). We have 4 possible cases:

(a) $a_{m_0+i-1} = 1 = a_{m_0+i+1}$. In this case, $a_{m_0+i-1} = 1$ belongs to a block of 1's and will be singularized if and only if this block has odd length. On the other hand, $a_{m_0+i+1} = 1$ will always be singularized, so that a_{m_0+i} will either become $-1/(a_{m_0+i} + 2)$ (if the block of 1's 'before' a_{m_0+i} has odd length), or becomes $1/(a_{m_0+i} + 1)$.

(b) $a_{m_0+i-1} \neq 1 = a_{m_0+i+1}$. In this case, a_{m_0+i} becomes $1/(a_{m_0+i}+1)$, due to the singularization of $a_{m_0+i+1} = 1$.
(c) $a_{m_0+i-1} = 1 \neq a_{m_0+i+1}$. In this case, a_{m_0+i} becomes either $-1/a_{m_0+i}+2$, or remains unchanged, depending on whether $a_{m_0+i-1} = 1$ is singularized or not.
(d) $a_{m_0+i-1} \neq 1 \neq a_{m_0+i+1}$. In this case, it is obvious that a_{m_0+i} will remain unchanged.

Due to the periodicity the same thing will happen to $a_{m_0+i+(k-1)p} > 1$ for all $k \in \mathbb{N}$.

To conclude, from (i) and (ii) we see that for each $i \in \{1, \dots, p\}$ and for all $k \in \mathbb{N}$ one has exactly one of the following possibilities:

— $a_{m_0+i+(k-1)p} = 1$ always disappears due to a singularization;
— $a_{m_0+i+(k-1)p} > 1$ always remains unchanged;
— $a_{m_0+i+(k-1)p} > 1$ always becomes $-1/(a_{m_0+i+(k-1)p}+1)$ due to the singularization of a digit 1 before it;
— $a_{m_0+i+(k-1)p} = 1$ always becomes $1/(a_{m_0+i+(k-1)p}+1)$ due to the singularization of a digit 1 after it;
— $a_{m_0+i+(k-1)p} = 1$ always becomes $-1/(a_{m_0+i+(k-1)p}+2)$ due to the singularization of a digit 1 before and after it.

Thus we obtain a quasi-period for the NICF expansion of x.

Conversely, since the singularization process can be reversed in a unique way, we see that a NICF-Hurwitzian number x is also Hurwitzian. □

Applying the procedure given in the proof of Theorem 1 yields that the NICF-expansion of e is given by $e = [3; -1/4, -1/2, \overline{1/(2k+5), -1/2}]_{k=0}^{\infty}$, which is another way of writing e in Example 1.

From the proof of Theorem 1 it is at once clear that x is a quadratic irrational if and only if the NICF-expansion of x is eventually periodic.

3. Hurwitzian numbers for the backward continued fraction

Every $x \in \mathbb{R} \setminus \mathbb{Q}$ can be expanded in a unique continued fraction expansion

$$c_0 - \cfrac{1}{c_1 - \cfrac{1}{c_2 - \ddots - \cfrac{1}{c_n - \ddots}}} =: [\, c_0;\, -1/c_1,\, -1/c_2,\, \cdots,\, -1/c_n,\, \cdots] \,,$$

where $c_0 \in \mathbb{Z}$ such that $x - c_0 \in [-1, 0)$ and c_i's are all integers greater than 1. This continued fraction is known as the *backward continued fraction* expansion of x; see [DK] for details.

Proposition 2 in [DK] gives an algorithm yielding the backward continued fraction expansion from the regular one using singularizations and insertions. The latter is based on the following identity.

$$A + \frac{1}{B+\xi} = A + 1 + \cfrac{-1}{1 + \cfrac{1}{B - 1 + \xi}} \quad .$$

From this algorithm it follows that $x = [a_0; a_1, a_2, \dots]$ has as backward expansion

$$[a_0 + 1;\, (-1/2)^{a_1 - 1},\, -1/(a_2+2),\, (-1/2)^{a_3-1},\, -1/(a_4+2),\, \dots] \tag{4}$$

where $(-1/2)^t$ is an abbreviation of $\underbrace{-1/2, \dots, -1/2}_{t-times}$ for $t \geq 1$. In case $t = 0$, the term $(1/2)^t$ should be omitted.

Example 2. The backward expansion of e is given by

$$[3; \overline{-1/(4k+4),\, -1/3,\, (-1/2)^{4k+3},\, -1/3}\,]_{k=0}^{\infty}\,.$$

This example leads to the following definition.

Definition 2. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then x has a backward-Hurwitzian expansion if

$$\begin{aligned} x \;=\; & [\, c_0; (-1/c_1)^{r_1}, \dots, (-1/c_n)^{r_n}, \\ & \overline{(-1/c_{n+1}(k))^{r_{n+1}(k)}, \dots, (-1/c_{n+p}(k))^{r_{n+p}(k)}}\,]_{k=0}^{\infty} \end{aligned}$$

where $c_0 \in \mathbb{Z}$ such that $x - c_0 \in [-1, 0)$; $(c_i, r_i) = (c, 1)$ or $(2, r)$ for $i = 1, \dots, n$, where c is an integer greater than 2 and r a positive interger. We call p the 'length' of the quasi-period. Moreover,

$$(c_{n+i}(k), r_{n+i}(k)) = (f_i(k), 1) \text{ or } (2, g_i(k))$$

for $i = 1, \dots, p$ where $f_i(k)$ and $g_i(k)$ are polynomials with rational coefficients which take positive integral values for $k = 0, 1, 2, \dots$ and at least one of them is not constant. Here $(-1/c)^r$ is an abbreviation of $\underbrace{-1/c, \dots, -1/c}_{r-times}$.

The following result gives the necessary and sufficient condition for an irrational number to have a backward-Hurwitzian expansion.

Theorem 2. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then x is Hurwitzian if and only if x has a backward-Hurwitzian expansion.*

Proof. Let x be a Hurwitzian number, with RCF-expansion (1). We first notice in (4) that a_n in the RCF-expansion of x becomes $(-1/2)^{a_n - 1}$ in the backward expansion of x if n is odd, and becomes $-1/(a_n + 2)$ if n is even. Let m_0 be defined as in the proof of Theorem 1. Then for all $i > m_0$ we observe the following:

(i) If $a_i = 1$, then it either disappears in case i is odd, or becomes $-1/3$ in case i is even.

(ii) If $a_i > 1$, then it either becomes $(-1/2)^{a_i - 1}$ in case i is odd, or $-1/(a_i + 2)$ in case i is even.

Let p be the length of the quasi-period of the RCF-expansion of x. We see that for all $k \in \mathbb{N}$ the same thing will happen to each $a_{i+(k-1)p}$ if p is even or to each $a_{i+2(k-1)p}$ if p is odd, which yields a quasi-periodicity for the backward expansion of x.

Conversely, since the singularization and insertion processes can be reversed in a unique way, we see that a backward-Hurwitzian number x is also Hurwitzian. $\square$

Clearly x is a quadratic irrational if and only if the backward-expansion of x is eventually periodic. The next section gives a generalization of Section 2.

4. Hurwitzian Numbers for α-expansions

In this section we will define Hurwitzian numbers for the so-called α-expansions, of which the nearest interger continued fraction expansion is an example. These α-expansions were introduced and studied by H. Nakada in 1981 ([N]). We will show that Hurwitzian numbers for these α-expansions also coincide with the classical Hurwitzian numbers.

For $\alpha \in [1/2, 1]$, let $x \in [\alpha - 1, \alpha]$ and define

$$(5)\qquad \begin{aligned} f_1 = f_1(x) &:= \lfloor |1/x| + 1 - \alpha \rfloor, \qquad x \neq 0,\\ f_n = f_n(x) &:= f_1(T_\alpha^{n-1}(x)), \qquad n \geq 2, \qquad T_\alpha^{n-1}(x) \neq 0, \end{aligned}$$

where $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ is defined by

$$T_\alpha = |1/x| - \lfloor |1/x| + 1 - \alpha \rfloor$$

and $\lfloor \xi \rfloor$ denotes the largest integer not exceeding ξ.

Every $x \in \mathbb{R} \setminus \mathbb{Q}$ can be expanded in a continued fraction expansion

$$x = [\, f_0;\, e_1/f_1,\, e_2/f_2,\, \dots\, ,\, e_n/f_n,\, \dots],$$

where $f_0 \in \mathbb{Z}$, $x - f_0 \in [\alpha - 1, \alpha)$, $e_n = \pm 1$, $f_n \in \mathbb{N}$, $n \geq 1$, are given by (5). We call this continued fraction *α-expansion* of x.

Remark. Note that for $\alpha = 1/2$ one has the NICF-expansion, while $\alpha = 1$ is the RCF case.

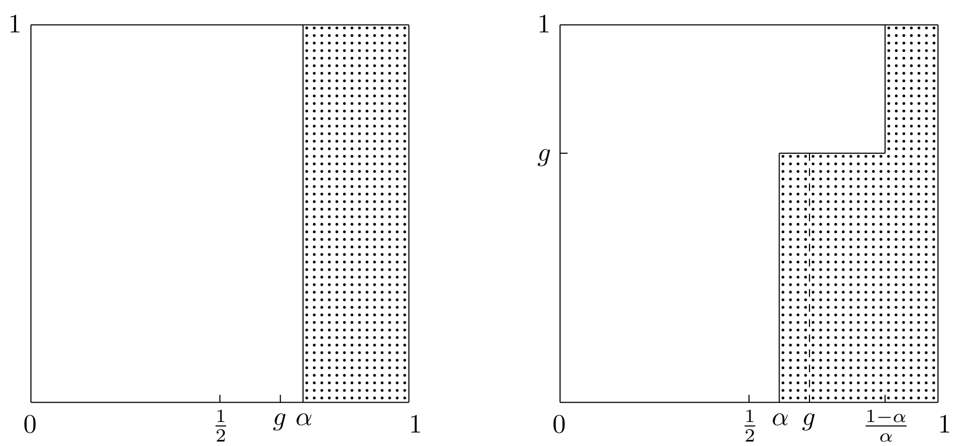
In [K] it is shown that α-expansions can be viewed as S-expansions, with singularization areas

$$S_\alpha = [\alpha, 1] \times [0, 1], \qquad \text{if } g < \alpha \leq 1$$

and

$$S_\alpha = [\alpha, g) \times [0, g) \cup [g, (1-\alpha)/\alpha] \times [0, g] \cup ((1-\alpha)/\alpha, 1] \times [0, 1]$$

in case $1/2 \leq \alpha \leq g$, where $g = (\sqrt{5} - 1)/2$; see Figure 1. In general a singularization



(i) $g < \alpha \leq 1$ (ii) $1/2 \leq \alpha \leq g$

Figure 1. Singularization areas for α-expansions

area S is a subset of the so-called natural extension $[0, 1) \times [0, 1]$ of the RCF-expansion, which satisfies the following three conditions:

(i): $S \subset [\frac{1}{2}, 1) \times [0, 1]$; (ii): $\mathcal{T}(S) \cap S = \emptyset$ and (iii): $\lambda(\partial S) = 0$.

Here λ is Lebesgue measure on $[0, 1) \times [0, 1]$, and $\mathcal{T} : [0, 1) \times [0, 1] \to [0, 1) \times [0, 1]$ is the natural extension map of the RCF-expansion, given by

$$\mathcal{T}(x, y) = \left(\frac{1}{x} - \lfloor \frac{1}{x} \rfloor, \frac{1}{\lfloor \frac{1}{x} \rfloor + y} \right),\ (x, y) \in (0, 1) \times [0, 1];\ \mathcal{T}(0, y) = (0, 0),\ y \in [0, 1].$$

Let $x \in [0,1)$, with RCF-expansion $[a_0; a_1, a_2, \dots]$. Then the S-expansion of x is obtained via the following algorithm:

$$\text{singularize } a_{n+1} = 1 \text{ if and only if } (T_n, V_n) \in S_\alpha,$$

where $T_n = [0; a_{n+1}, a_{n+2}, \dots]$ and $V_n = [0; a_n, \dots, a_1]$, i.e., $(T_n, V_n) = \mathcal{T}^n(x, 0)$, for more details, see [K].

The following lemma is very handy.

Lemma 1. *Let $x, y \in [0,1)$, with RCF-expansions*

$$x = [0; a_1(x), a_2(x), \dots], \quad y = [0; a_1(y), a_2(y), \dots].$$

Let $x \neq y$ and $k \in \mathbb{N} \cup \{0\}$ be such that

$$a_1(x) = a_1(y), \dots, a_{k-1}(x) = a_{k-1}(y), \text{ and } a_k(x) \neq a_k(y).$$

Then one has

$$x > y \quad \text{if and only if} \quad \begin{cases} a_k(x) < a_k(y) & \text{if } k \text{ is odd,} \\ a_k(x) > a_k(y) & \text{if } k \text{ is even.} \end{cases}$$

Proof. For $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{N}$, define cylinders $\Delta_n(a_1, \dots, a_n)$ by

$$\Delta_n(a_1, \dots, a_n) = \{x \in [0,1) \; ; \; a_1(x) = a_1, \dots, a_n(x) = a_n\}.$$

For $x, y \in \Delta_{k-1}(a_1, \dots, a_{k-1})$, $x < y$, one has by definition of the RCF-map $T = T_1$ that $T(x), T(y) \in \Delta_{k-2}(a_2, \dots, a_{k-1})$, and $T(x) > T(y)$. Repeating this argument $k-2$-times, we find that $T^{k-2}(x), T^{k-2}(y) \in \Delta_1(a_{k-1})$, and that $T^{k-2}(x) < T^{k-2}(y)$ if and only if k is even. Since $T(\Delta_1(a_{k-1})) = [0,1)$ and $a_k(x) \neq a_k(y)$, it follows from the definition of T that $T^{k-1}(x) > T^{k-1}(y)$ if and only if k is even. Since $T^{k-1}(x) \in \Delta_1(a_k(x)) = \left(\frac{1}{a_k+1}, \frac{1}{a_k}\right]$, and $T^{k-1}(y) \in \Delta_1(a_k(y))$, it follows that $a_k(x) < a_k(y)$ if and only if k is even. □

We now define Hurwitzian numbers for α-expansions.

Definition 3. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then, for a fixed $\alpha \in [1/2, 1]$, x has an α-Hurwitzian expansion if

$$x = [\, f_0; \, e_1/f_1, \, \dots, \, e_n/f_n, \, \overline{e_{n+1}/f_{n+1}(k), \, \dots, \, e_{n+p}/f_{n+p}(k)} \,]_{k=0}^{\infty} \tag{6}$$

is the α-expansion of x, where $f_0 \in \mathbb{Z}$, $x - f_0 \in [\alpha - 1, \alpha)$, $e_n = \pm 1$, $f_n \in \mathbb{N}$, $n \geq 1$, are given by (5). Moreover, for $i = 1, \dots, p$ we have that f_{n+i} are polynomials with rational coefficients which take positive integral values for $k = 0, 1, 2, \dots$, and at least one of them is non-constant.

We have the following theorem.

Theorem 3. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then x is Hurwitzian if and only if x has a α-Hurwitzian expansion.*

Proof. **Case 1:** $g < \alpha \leq 1$. Let $m_0 \in \mathbb{N}$ be such that for all $m \geq m_0$ all non-constant polynomials a_m are greater than 1. Let $k \in \{m, m+1, \dots, m+p\}$ be such that $a_k = 1$. Then

$$T^{k-1}(x) = [0; 1, \, a_{k+1}, \, \dots].$$

So $a_k = 1$ must be singularized if and only if $T^{k-1}(x) \geq \alpha$.

Clearly there exists a minimal $i \in \{1, \dots, p\}$ such that a_{k+i} is a non-constant polynomial. Further, let $j \in \mathbb{N} \cup \{\infty\}$ be such that

$$a_{k+j} \neq a_{j+1}(\alpha)$$

where $\alpha = [0; 1, \, a_2(\alpha), \, \dots]$.

In case $j \geq i$, there exists an $\ell_0 \geq 0$ such that, by Lemma 1 for all $\ell \geq \ell_0$

$$T^{k+\ell p-1}(x) > \alpha \Longleftrightarrow j \text{ is odd},$$

implying $a_{k+\ell p} = 1$ must be singularized for all $\ell \geq \ell_0$. Otherwise, they are never singularized.

If $1 \leq j \lneqq i$, then a_{k+j} is a constant different from $a_{j+1}(\alpha)$, so

$$T^{k+\ell p-1}(x) \geq \alpha \Longleftrightarrow j \text{ is odd} \quad \text{and} \quad a_{k+j} > a_{j+1}(\alpha).$$

Case 2: $1/2 \leq \alpha \leq g$. In this case we have to consider (T_n, V_n). It is clear that there exist an $\ell_1 \geq 1$ and a minimal $h \in \{1, \ldots, p\}$ such that for all $\ell \geq \ell_1$, one has $a_{k+\ell p-h} > 1$. If h is odd implying $V_{k-1} < g$, then $a_k = 1$ must be singularized if and only if $T_{k-1} > \alpha$. In this case, let i and j be defined as in Case 1. If $j \geq i$, then there exists an $\ell_2 \geq \ell_1$ such that for all $\ell \geq \ell_2$ one has

$$T_{k+\ell p-1} > \alpha \Longleftrightarrow j \text{ is odd}.$$

If $1 \leq j \lneqq i$, one has

$$T_{k+\ell p-1} \geq \alpha \Longleftrightarrow j \text{ is odd} \quad \text{and} \quad a_{k+j} > a_{j+1}(\alpha).$$

On the other hand, if h is even implying $V_{k-1} > g$, then $a_k = 1$ must be singularized if and only if $T_{k-1} > (1-\alpha)/\alpha$. Again let i be defined as in Case 1, but j be such that

$$a_{k+j} \neq a_{j+1}((1-\alpha)/\alpha).$$

If $j \geq i$, then there exists an $\ell_2 \geq \ell_1$ such that for all $\ell \geq \ell_2$ one has

$$T_{k+\ell p-1} > (1-\alpha)/\alpha \Longleftrightarrow j \text{ is odd}.$$

If $1 \leq j \lneqq i$, one has

$$T_{k+\ell p-1} \geq (1-\alpha)/\alpha \Longleftrightarrow j \text{ is odd} \quad \text{and} \quad a_{k+j} > a_{j+1}((1-\alpha)/\alpha).$$

□

Example 3. Here we give α-expansions of e for some values of α.

(i) For $\alpha = 0.7$,

$$e = [3;\, -1/3,\, 1/2,\, \overline{-1/(2k+5),\, 1/2}]_{k=0}^{\infty}.$$

(ii) For $\alpha = 0.52$,

$$e = [3; -1/4,\, -1/2,\, 1/5,\, -1/2,\, 1/7,\, -1/2,\, 1/9,\, -1/2,\, 1/10,\, 1/2,\, \overline{-1/(2k+13),\, 1/2}]_{k=0}^{\infty}.$$

(iii) For $\alpha = 0.53$,

$$e = [3; -1/4,\, -1/2,\, 1/5,\, -1/2,\, 1/6,\, 1/2,\, \overline{-1/(2k+9),\, 1/2}]_{k=0}^{\infty}.$$

Remarks. 1. From the proof of Theorem 3 it is at once clear that x is a quadratic irrational if and only if the α-expansion of x is eventually periodic.

2. Analogous to Definitions 1 and 3 we can define S-Hurwitzian number for any S-expansion. In case the singularization-area is 'nice' (such as the singularization-areas for Nakada's α-expansion, or for Minkowski's diagonal continued fraction expansion, see [H]), one can show that being S-Hurwitzian is equivalent to being Hurwitzian. However, it is possible to find singularization-areas S and numbers x such that x is Hurwitzian, but not S-Hurwitzian. Consider for example the following singularization-area S:

$$S = \bigcup_{p \text{ prime}} (2p+2, 2p+1] \times \left(\tfrac{1}{2}, 1\right).$$

One easily convinces oneself that e does not have an S-expansion which is S-Hurwitzian.

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