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A NOTE ON IDEMPOTENT SEPARATING CONGRUENCES ON A REGULAR SEMIGROUP

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1. INTRODUCTION

Bohdan Zelinka [6] has shown that a compatible tolerance on a group is a congruence. We give an example which shows that a compatible tolerance on a regular semigroup is not a congruence. We prove that a compatible tolerance on a regular semigroup, which is contained in $\mathscr{H}$, is a congruence and hence an idempotent separating congruence [3]. In [4] Meakin defined a certain relation and proved that it is a maximum idempotent separating congruence on a regular semigroup. By making use of the notion of sandwich sets introduced by K. S. S. Nambooripad [5] we give an alternative elegant proof for the above. Throughout this paper we follow the terminology and notations of [1] and [3].

2. DEFINITIONS AND PRELIMINARY RESULTS

A reflexive and symmetric relation defined on a semigroup is called a *tolerance relation*. A tolerance ξ on a semigroup S is *left weakly compatible* if $(a, b) \in \xi$ implies $(ra, rb) \in \xi$ for every r in S. A *right weakly compatible* tolerance is defined dually. A tolerance on a semigroup is called *weakly compatible* if it is both left and right weakly compatible. A tolerance ξ in a semigroup S is called *strongly compatible* if $(a, b) \in \xi$ and $(c, d) \in \xi$ imply $(ac, bd) \in \xi$. Strong compatibility of a tolerance implies its weak compatibility whereas weak compatibility of a tolerance does not imply its strong compatibility. This is illustrated by the following example.

Let $S = \{e, a, f, b\}$ be a semigroup with the multiplication table:

	e	a	f	b
e	e	a	f	b
a	a	e	b	f
f	f	b	f	b
b	b	f	b	f

A relation $\varrho = \{(e, e), (a, a), (f, f), (b, b), (e, a), (a, e), (f, b), (b, f), (a, b), (b, a), (e, f), (f, e)\}$ is a tolerance on S and it is obviously weakly compatible, $(a, b) \in \varrho$ and $(a, e) \in \varrho$ but (a^2, be), that is, $(e, b) \notin \varrho$. So ϱ is not strongly compatible. It can be seen that transitivity and weak compatibility of a binary relation on a semigroup implies its strong compatibility. Zelinka's [6] notion of compatibility is compatibility in the strong sense.

Hereafter S stands for a regular semigroup and E_s denotes its set of idempotents. If $a \in S$ then $V(a)$ denotes the set of inverses of a in S. For $e, f \in E_s$ the sandwich set of e, f as introduced by K. S. S. Nambooripad [5] is $S(e,f) = \{g \in E_s: ge = fg = g, egf = ef\}$.

The following result is due to K. S. S. Nambooripad [5] and A. H. Clifford [2].

Lemma 2.1. *Suppose* $e, f, h, k \in E_s$, $a, b \in S$, $a' \in V(a)$ *and* $b' \in V(b)$. *Then* (i) $S(e, f) \neq \square$; (ii) *if* $e\mathscr{L}h$ *and* $f\mathscr{R}k$ *then* $S(e, f) = S(h, k)$; (iii) *if* $g \in S(a'a, bb')$ *then* $agb = ab$ *and* $b'ga' \in V(ab)$.

An interesting consequence of the above lemma is the following.

Corollary 2.2. *If* $g \in S(a'a, bb')$ *then* (i) $b'g \in V(gb)$, (ii) $ga' \in V(ag)$.

Following Meakin [4] we define for $a \in S$,

$$E\,L(a) = \{e \in E_s : L_e \leqq L_a\},$$

$$E\,R(a) = \{e \in E_s : R_e \leqq R_a\}.$$

Clearly $E\,L(a) \neq \square$ and $E\,R(a) \neq \square$ for every a in S.

Lemma 2.3. *If* $g \in S(a'a, bb')$ *and* $e \in E\,L(ab)$ *then* (i) $beb' \in E_s$, (ii) $(gb)\,e(gb)' \in E\,L(a)$.

Proof. $e \in E\,L(ab)$ implies $e \in Sab \subseteq Sb$. So there exists u in S such that $e = ub$. $beb'beb' = bubb'bubb' = bubub b' = beeb' = beb'$. Now $e \in EL(ab) = EL(agb)$ implies $e \in Sagb \subseteq Sgb$. Hence there exists v in S such that $e = vgb$. $(gb)\,e(gb)' \,.\, (gb)\,e(gb)' = gbvgbb'ggbvgbb'g$ (using Cor. 2.2) $= gbvgggbvgbb'g = gbvgbvgbb'g = gbvgbb'g = gbe(gb)'$. Now $L_{gbeb'g} \leqq L_g = L_{ga'a} \leqq L_a$ and so gbeb'g $\in L(a)$. Therefore $gbe(gb)' \in EL(a)$.

3. A COUNTER-EXAMPLE AND THE MAIN THEOREM

Example 3.1. A strongly compatible tolerance on a regular semigroup need not be a congruence. The following simple and elegant counterexample is due to Dr. Boris M. Schien.

$L = \{e, f, g\}$ is a commutative idempotant semigroup with the multiplication table given below:

	e	f	g
e	e	e	e
f	e	f	e
g	e	e	g

A relation $\varrho = \{(e, e), (e, g), (g, e), (g, g), (e, f), (f, e), (f, f)\}$ is obviously a strongly compatible tolerance on L. However, it is not a congruence since $(g, e), (e, f) \in \varrho$ but $(g, f) \notin \varrho$.

Theorem 3.2. *If ξ is a strongly compatible tolerance on S such that $\xi \subseteq \mathscr{H}$ then ξ is an idempotent separating congruence on S.*

Proof. Let $(a, b), (b, c) \in \xi$. The condition $\xi \subseteq \mathscr{H}$ implies (a, b) and $(b, c) \in \mathscr{H}$. There exist $a' \in V(a)$, $b^*, b' \in V(b)$ and $c^* \in V(c)$ such that $a'a = b'b$ and $aa' = bb'$ and $b^*b = c^*c$ and $bb^* = cc^*$. $\mathscr{H}$ being transitive we have $(a, c) \in \mathscr{H}$. Hence there exist $a^* \in V(a)$ and $c' \in V(c)$ such that $a'a = b'b = c'c$; $aa' = bb' = cc'$; $a^*a = b^*b = c^*c$; $aa^* = bb^* = cc^*$. Now $(a, b) \in \xi$ and $(b', b') \in \xi$ imply $(ab', bb') \in \xi$. From $(ab', bb') \in \xi$ and $(b, c) \in \xi$ we get $(ab'b, bb'c) \in \xi$. Hence $(aa'a, cc'c) \in \xi$. That is $(a, c) \in \xi$.

Now ξ, being a congruence contained in $\mathscr{H}$, is an idempotent separating congruence [3].

We observe that only reflexivity and strong compatibility of ξ have been used in the above proof.

Meaking [4] defined a relation $\mu = \{(a, b) \in S \times S$: there are inverses a' of a and b' of b such that $aea' = beb'$, $\forall e \in EL(a) \cup EL(b)$ and $a'fa = b'fb$ $\forall f \in ER(a) \cup ER(b)\}$. Clearly μ is a tolerance in S. It was proved that $\mu \subseteq \mathscr{H}$ [4]. Making use of sandwich sets we prove that μ is strongly compatible; whereas Meakin proves its transitivity and weak compatibility separately [4].

Proposition 3.3. *μ is strongly compatible.*

Proof. Let $(a, b), (c, d) \in \mu$. Now $(a, b) \in \mathscr{H} \subseteq \mathscr{L}$ implies $a'a = b'b$ and $(c, d) \in \mathscr{H} \subseteq \mathscr{R}$ implies $cc' = dd'$ where $a' \in V(a)$, $b' \in V(b)$, $c' \in V(c)$ and $d' \in V(d)$. Hence we get $a'a \mathscr{L} b'b$ and $cc' \mathscr{R} dd'$ which by Lemma 2.1 imply (ii) $S(a'a, cc') = S(b'b, dd')$. Let $e \in EL(ac) \cup EL(bd)$ and $g \in S(a'a, cc') = S(b'b, dd')$.

Now $L_e \leqq L_{ac} \leqq L_c$ or $L_e \leqq L_{bd} \leqq L_{bd}$. Therefore $e \in EL(c) \cup EL(d)$, consequently $cec' = ded'$ and $gcec'g = gded'g$. By Lemma 2.3, $gcec'g \in EL(a)$ and $gded'g \in EL(b)$. Hence $gcec'g = gded'g \in EL(a) \cup EL(b)$ which in turn implies $agcec'ga' = bgded'gb'$. That is, $(ac)\, e(ac)' = (bd)\, e(bd)'$. Similarly we get $(ac)' f(ac) = (bd)' f(bd)$ where $f \in ER(ac) \cup ER(bd)$. Hence $(ac, bd) \in \mu$. By Theorem 3.2, μ is an idempotent separating congruence on S. It was proved that μ is a maximum idempotent separating congruence on S [4].

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