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A Note on Invariant Submanifolds of CR-Integrable Almost Kenmotsu Manifolds

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Abstract. The present paper deals with invariant submanifolds of CR-integrable almost Kenmotsu manifolds. Among others it is proved that every invariant submanifold of a CR-integrable $(k,\mu)'$-almost Kenmotsu manifold with $k < -1$ is totally geodesic. Finally, we construct an example of an invariant submanifold of a CR-integrable $(k,\mu)'$-almost Kenmotsu manifold which is totally geodesic.

1. Introduction

In modern analysis the geometry of submanifolds has become a subject of growing interest for its significant applications in applied mathematics and theoretical physics [16]. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system. The study of geometry of invariant submanifolds was initiated by Bejancu and Papaghuic [1]. In general the geometry of an invariant submanifold inherits almost all properties of the ambient manifold. Invariant manifolds has many applications in functional differential equations [11]. On the other hand one of the recent topics in the theory of almost contact metric manifolds is the study of so-called nullity distributions. The notion of k-nullity distribution ($k \in \mathbb{R}$) was introduced by Gray [10] and Tanno [19] in the study of Riemannian manifolds (M, g), which is defined for any point $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{1}$$

for any $X, Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$.

Recently Blair, Koufogiorgos and Papantoniou [4] introduced a generalized notion of the k-nullity distribution named the (k,μ)-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any point $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$N_p(k,\mu) = \{Z \in T_pM^{2n+1} : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY]\}, \tag{2}$$

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where $h = \frac{1}{2}\pounds_\xi \phi$ and £ denotes the Lie differentiation.

In [8], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, another generalized notion of the k-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any point $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$N_p(k,\mu)' = \{Z \in T_pM^{2n+1} : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \tag{3}$$

where $h' = h \circ \phi$.

On the other hand, Kenmotsu [13] introduced a special class of almost contact metric manifolds named Kenmotsu manifolds nowadays. Almost Kenmotsu manifolds satisfying some nullity conditions were investigated by De and Mandal [6], Dileo and Pastore [7, 8], Wang and Liu [20–23] and many others. We refer the reader to ([7],[23]) for more related results on $(k,\mu)'$-nullity distribution on almost Kenmotsu manifolds. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold [12], thus, it is of interest to generalize some results on Kenmotsu manifolds to almost Kenmotsu manifolds.

An invariant submanifold of an almost contact manifold is a submanifold for which the structure tensor field ϕ maps tangent vectors into tangent vectors. There is a well-known result of Kon [14] that an invariant submanifold of a Sasakian manifold is totally geodesic, provided the second fundamental form of the immersion is covariantly constant. Generally an invariant submanifold of a Sasakian manifold needs not to be totally geodesic. In a recent paper De and Majhi [5] studied invariant submanifolds of Kenmotsu manifolds and obtain a necessary condition for a three dimensional invariant submanifold of a Kenmotsu manifold to be totally geodesic. Also in [15], Mangione studied invariant submanifolds of Kenmotsu manifolds and some conditions that these submanifolds are totally geodesic. Besides these invariant submanifolds of Lorentzian para-Sasakian manifolds were studied by Özgür and Murathan [18]. Moreover invariant submanifolds have been studied by Endo [9], Murathan et al. [17] and many others.

As far as we know, submanifolds of almost Kenmotsu manifolds have not yet been studied. In this paper, we initiate the study of submanifolds of almost Kenmotsu manifolds. In fact, in this paper, we classify invariant submanifolds of CR-integrable almost Kenmotsu manifolds:

Theorem 1.1. *Any invariant submanifold of a CR-integrable almost Kenmotsu manifold is a minimal submanifold.*

Theorem 1.2. *Every invariant submanifold of a CR-integrable $(k,\mu)'$-almost Kenmotsu manifold with $k < -1$ is totally geodesic.*

This paper is organized in the following way. In Section 2, we discuss about submanifolds. In the next section we recall some well known basic formulas and properties of almost Kenmotsu manifolds. We study invariant submanifolds of CR-integrable almost Kenmotsu manifolds in Section 4. Next, Section 5 is devoted for the detailed proof of Theorems 1.1 and 1.2 respectively. This paper ends with an example of an invariant submanifold of a CR-integrable almost Kenmotsu manifold with $(k,\mu)'$-nullity distribution which is totally geodesic.

2. Basic Concepts

Let (M, g) be an n-dimensional Riemannian submanifold of an $(n+d)$-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. We denote by $\tilde{\nabla}$ the operator of covariant differentiation on $\tilde{M}$. We write

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y), \tag{4}$$

where $\nabla_X Y$ is the tangential component of $\tilde{\nabla}_X Y$ and $\alpha(X,Y)$ is the normal component of $\tilde{\nabla}_X Y$. Then it has been proved that ∇ is the operator of covariant differentiation with respect to the induced metric g on M. We call ∇ the induced connection and α the second fundamental form of M (or, of the corresponding immersion i).

Now, let N be a normal vector field on M and X be a tangent vector field on M, we put

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{5}$$

where $A_N X$ and $\nabla^{\perp}_X N$ are the tangential and normal components of $\tilde{\nabla}_X N$, respectively. A and α are related by the equation

$$g(\alpha(X,Y),N) = g(A_N X,Y),$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$. A is called the associated second fundamental form to α, or, simply the second fundamental form of M. It is known that $\nabla^{\perp}$ is a metric connection in $T^{\perp}M$.

Equations (4) and (5) are called the Gauss and Weingarten formula respectively.

A submanifold M is called totally geodesic if $\alpha(X,Y) = 0$, for all $X, Y \in TM$. It means that the geodesics in M are also geodesic in $\tilde{M}$.

A submanifold M is called totally umbilical if $\alpha(X,Y) = g(X,Y)H$, where H is called the mean curvature vector. If $H = 0$, we say the submanifold is minimal. From the definition it is clear that any totally geodesic submanifold is obviously a minimal submanifold.

3. Almost Kenmotsu Manifolds

Let us consider $\tilde{M}^{2n+1}$ be an almost contact metric manifold with almost contact structure (ϕ, ξ, η, g) given by a $(1,1)$-tensor field ϕ, a characteristic vector field ξ, a 1-form η and a compatible metric g satisfying the conditions [2, 3]

$$\phi^2 = -I + \eta \otimes \xi,\ \phi(\xi) = 0,\ \eta(\xi) = 1,\ \eta \circ \phi = 0,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X and Y of $T\tilde{M}$. The fundamental 2-form Φ is defined by $\Phi(X,Y) = g(X, \phi Y)$ for any vector fields X and Y of $T\tilde{M}^{2n+1}$. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_{ϕ}, defined by $N_{\phi} = [\phi,\phi] + 2d\eta \otimes \xi$, where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ [2].

An almost contact metric manifold $\tilde{M}^{2n+1}$ with almost contact structure (ϕ, ξ, η, g) is said to be an almost Kenmotsu manifold if the 1-form η is closed and $d\Phi = 2\eta \wedge \Phi$. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold.
Let $\tilde{M}^{2n+1}$ be an almost Kenmotsu manifold. Let in an almost Kenmotsu manifold the two tensor fields h and l are defined by $h = \frac{1}{2}\pounds_{\xi}\phi$ and $l = R(\cdot,\xi)\xi$. The tensor fields l and h are symmetric and satisfy the following relations [7]:

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{6}$$

$$\tilde{\nabla}_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_{\xi}\xi = 0), \tag{7}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{8}$$

$$\tilde{R}(X,Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\tilde{\nabla}_Y \phi h)X - (\tilde{\nabla}_X \phi h)Y, \tag{9}$$

for any vector fields X, Y of $T\tilde{M}$ and $\tilde{R}$ is the curvature tensor of $\tilde{M}$. It is well known that a normal almost contact manifold is a CR-manifold. Now considering an almost Kenmotsu manifold we have $[X,Y] - [\phi X, \phi Y] \in \mathcal{D}$ for any $X, Y \in \mathcal{D}$, since $d\eta = 0$, where $\mathcal{D}$ is the contact distribution. Hence the structure is CR-integrable if and only if $[\phi X, \phi Y] - [X,Y] - \phi([\phi X, Y] + [X, \phi Y]) = 0$ on $\mathcal{D}$, which is equivalent to the vanishing of N_{ϕ} on $\mathcal{D}$, that is to the request that the integral manifolds of $\mathcal{D}$ are Kähler. Now we present some properties of CR-integrable almost Kenmotsu manifolds as follows:

Lemma 3.1. *([7]) An almost Kenmotsu manifold $(\tilde{M}^{2n+1}, \phi, \xi, \eta, g)$ is a Kenmotsu manifold if and only if $h = 0$ and the integral submanifolds of the contact distribution $\mathcal{D}$ are Kählerian manifolds.*

Lemma 3.2. *([8]) In an almost Kenmotsu manifold $(\tilde{M}^{2n+1}, \phi, \xi, \eta, g)$ the distribution $\mathcal{D}$ has Kähler leaves if and only if*

$$(\tilde{\nabla}_X\phi)(Y) = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX), \tag{10}$$

for any $X, Y \in T\tilde{M}$.

From Lemma 3.1 it follows that an almost Kenmotsu manifold with CR-integrable structure is a Kenmotsu manifold if and only if $h = 0$. That is, a Kenmotsu manifold is an almost Kenmotsu manifold with CR-integrable structure, but the converse is not necessary true.

An almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution is called $(k, \mu)'$-almost Kenmotsu manifold. In a $(k, \mu)'$-almost Kenmotsu manifold we have [8]

$$\tilde{R}(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] - 2[\eta(Y)h'X - \eta(X)h'Y], \tag{11}$$

where $k \in \mathbb{R}$. The $(1, 1)$-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with ϕ and $h'\xi = 0$. Also it is clear that

$$h = 0 \Leftrightarrow h' = 0, \quad h'^2 = (k + 1)\phi^2 (\Leftrightarrow h^2 = (k + 1)\phi^2) \tag{12}$$

and the tensor field h maps tangent vectors into tangent vectors. Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (12) it is clear that $\lambda^2 = -(k+1)$, a constant. Hence $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution. We denote the eigenspaces associated with h' by $[\lambda]'$ and $[-\lambda]'$ corresponding to the non-zero eigen values λ and $-\lambda$ of h' respectively.

4. Invariant Submanifolds of *CR*-Integrable Almost Kenmotsu Manifolds

A submanifold M of a CR-integrable almost Kenmotsu manifold $\tilde{M}$ is invariant if $\phi(TM) \subset TM$. Let us assume that M be an invariant submanifold of a CR-integrable almost Kenmotsu manifold $\tilde{M}$. From (4) we obtain

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y). \tag{13}$$

Putting $Y = \xi$ and using (7) we have

$$X - \eta(X)\xi - \phi hX = \nabla_X\xi + \alpha(X, \xi). \tag{14}$$

Comparing tangential part and normal part we get

$$\nabla_X\xi = X - \eta(X)\xi - \phi hX \tag{15}$$

and

$$\alpha(X, \xi) = 0. \tag{16}$$

Making use of (13) we obtain

$$\tilde{\nabla}_X\phi Y - \phi(\tilde{\nabla}_X Y) = \nabla_X\phi Y - \phi(\nabla_X Y) + \alpha(X, \phi Y) - \phi\alpha(X, Y), \tag{17}$$

which implies that

$$(\tilde{\nabla}_X\phi)Y = (\nabla_X\phi)Y + \alpha(X, \phi Y) - \phi\alpha(X, Y). \tag{18}$$

Using (10) we obtain

$$g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) = (\nabla_X\phi)Y + \alpha(X, \phi Y) - \phi\alpha(X, Y). \tag{19}$$

Comparing tangential part and normal part we have

$$(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \tag{20}$$

and

$$\alpha(X, \phi Y) - \phi\alpha(X, Y) = 0. \tag{21}$$

By the above discussions we can state the following:

Proposition 4.1. *Let M be an invariant submanifold of a CR-integrable almost Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M. Then the following equations hold on M*

$$\alpha(X, \xi) = 0, \tag{22}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi hX, \tag{23}$$

$$(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX), \tag{24}$$

$$\alpha(X, \phi Y) = \phi\alpha(X, Y), \tag{25}$$

for $X, Y \in TM$.

Proposition 4.2. *An invariant submanifold of a CR-integrable almost Kenmotsu manifold is an almost Kenmotsu manifold.*

5. Proof of the Main Theorems

Proof of Theorem 1.1. Let M be an invariant submanifold of a CR-integrable almost Kenmotsu manifold $\tilde{M}$ and $\{e_1, e_2, ..., e_n, \phi e_1, \phi e_2, ..., \phi e_n, \xi\}$ be a ϕ-basis of TM.
Now we get

$$trace\ \alpha = \sum_{i=1}^{n}[\alpha(e_i, e_i) + \alpha(\phi e_i, \phi e_i)] + \alpha(\xi, \xi). \tag{26}$$

By Proposition 4.1 we obtain

$$\alpha(X, \xi) = 0, \ \text{ which implies } \alpha(\xi, \xi) = 0, \tag{27}$$

and

$$\alpha(\phi e_i, \phi e_i) = \phi\alpha(\phi e_i, e_i) = \phi^2\alpha(e_i, e_i) = -\alpha(e_i, e_i). \tag{28}$$

Substituting (27) and (28) in (26) yields

$$trace\ \alpha = \sum_{i=1}^{n}[\alpha(e_i, e_i) + \alpha(\phi e_i, \phi e_i)] = 0. \tag{29}$$

Therefore M is a minimal submanifold. This completes the proof.

Proof of Theorem 1.2. Let M be an invariant submanifold of a CR-integrable $(k, \mu)'$-almost Kenmotsu manifold $\tilde{M}$. Let $X, Y \in TM$. Since the tensor field h' maps tangent vectors into tangent vectors, by (11) it follows that $\tilde{R}(X, Y)\xi$ is a vector field tangent to the submanifold. Then we have from the equation of Gauss ([24], pp. 70)

$$(\nabla_X \alpha)(Y, \xi) = (\nabla_Y \alpha)(X, \xi), \tag{30}$$

which implies

$$\nabla^{\perp}_{X}\alpha(Y,\xi) - \alpha(\nabla_X Y,\xi) - \alpha(Y,\nabla_X\xi) = \nabla^{\perp}_{Y}\alpha(X,\xi) - \alpha(\nabla_Y X,\xi) - \alpha(X,\nabla_Y\xi). \quad (31)$$

Using (22) we get

$$\alpha(Y,\nabla_X\xi) = \alpha(X,\nabla_Y\xi). \quad (32)$$

Taking account of (23) we obtain

$$\alpha(X,Y) - \eta(X)\alpha(Y,\xi) + \alpha(Y,h'X) = \alpha(X,Y) - \eta(Y)\alpha(X,\xi) + \alpha(X,h'Y), \quad (33)$$

which implies that

$$\alpha(h'X,Y) = \alpha(h'Y,X). \quad (34)$$

Case 1: Let us consider that $X \in [\lambda]'$ and $Y \in [-\lambda]'$. Then

$$\alpha(h'X,Y) = \lambda\alpha(X,Y) \text{ and } \alpha(X,h'Y) = -\lambda\alpha(X,Y). \quad (35)$$

Making use of (34) and (35) we have

$$\lambda\alpha(X,Y) = -\lambda\alpha(Y,X), \quad (36)$$

that is,

$$2\lambda\alpha(X,Y) = 0.$$

Since $\lambda \neq 0$, $\alpha(X,Y) = 0$, which implies that M is totally geodesic.
Case 2: Let us consider that $X \in [-\lambda]'$ and $Y \in [\lambda]'$. Similarly, we can conclude that M is totally geodesic.
Case 3: Let us consider that $X, Y \in [\lambda]'$. Since $[\lambda]' = \phi[-\lambda]'$, we can state $Y = \phi Z$ with $Z \in [-\lambda]$. Then $\alpha(X,Y) = \alpha(X,\phi Z) = \phi\alpha(X,Z) = 0$, because of Case 1, which implies that M is totally geodesic. This completes the proof.

6. Example

In this section, we construct an example of an invariant submanifold of an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. We consider 5-dimensional manifold $\tilde{M} = \{(x,y,z,u,v) \in \mathbb{R}^5\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. Let ξ, e_2, e_3, e_4, e_5 are five vector fields in $\mathbb{R}^5$ which satisfies [8]

$$[\xi,e_2] = -2e_2,\ [\xi,e_3] = -2e_3,\ [\xi,e_4] = 0,\ [\xi,e_5] = 0,$$
$$[e_i,e_j] = 0, \text{ where } i,j = 2,3,4,5.$$

Let g be the Riemannian metric defined by

$$g(\xi,\xi) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = g(e_5,e_5) = 1$$
$$\text{and } g(\xi,e_i) = g(e_i,e_j) = 0 \text{ for } i \neq j;\ i,j = 2,3,4,5.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,\xi)$, for any $Z \in T\tilde{M}$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi) = 0,\ \phi(e_2) = e_4,\ \phi(e_3) = e_5,\ \phi(e_4) = -e_2,\ \phi(e_5) = -e_3.$$

Using the linearity of ϕ and g we have

$$\eta(\xi) = 1,\ \phi^2 Z = -Z + \eta(Z)\xi \text{ and } g(\phi Z,\phi U) = g(Z,U) - \eta(Z)\eta(U),$$

for any $Z, U \in T\tilde{M}$. Moreover we obtain,

$$h'\xi = 0,\ h'e_2 = e_2,\ h'e_3 = e_3,\ h'e_4 = -e_4,\ h'e_5 = -e_5 \text{ and } h\xi = 0,\ he_2 = e_4,\ he_3 = e_5,\ he_4 = e_2,\ he_5 = e_3.$$

The Levi-Civita connection $\tilde{\nabla}$ of the metric tensor g is given by Koszul's formula which is given by

$$2g(\tilde{\nabla}_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul's formula we get the following:

$$\begin{aligned}
&\tilde{\nabla}_\xi \xi = 0,\ \tilde{\nabla}_\xi e_2 = 0,\ \tilde{\nabla}_\xi e_3 = 0,\ \tilde{\nabla}_\xi e_4 = 0,\ \tilde{\nabla}_\xi e_5 = \xi,\\
&\tilde{\nabla}_{e_2} \xi = 2e_2,\ \tilde{\nabla}_{e_2} e_2 = -2\xi,\ \tilde{\nabla}_{e_2} e_3 = 0,\ \tilde{\nabla}_{e_2} e_4 = 0,\ \tilde{\nabla}_{e_2} e_5 = 0,\\
&\tilde{\nabla}_{e_3} \xi = 2e_3,\ \tilde{\nabla}_{e_3} e_2 = 0,\ \tilde{\nabla}_{e_3} e_3 = -2\xi,\ \tilde{\nabla}_{e_3} e_4 = 0,\ \tilde{\nabla}_{e_3} e_5 = 0,\\
&\tilde{\nabla}_{e_4} \xi = 0,\ \tilde{\nabla}_{e_4} e_2 = 0,\ \tilde{\nabla}_{e_4} e_3 = 0,\ \tilde{\nabla}_{e_4} e_4 = 0,\ \tilde{\nabla}_{e_4} e_5 = 0,\\
&\tilde{\nabla}_{e_5} \xi = 0,\ \tilde{\nabla}_{e_5} e_2 = 0,\ \tilde{\nabla}_{e_5} e_3 = 0,\ \tilde{\nabla}_{e_5} e_4 = 0,\ \tilde{\nabla}_{e_5} e_5 = 0.
\end{aligned}$$

In view of the above relations we have

$$\tilde{\nabla}_X \xi = -\phi^2 X + h'X \text{ and } (\tilde{\nabla}_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX),$$

for any $X, Y \in T\tilde{M}$.

Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that $\tilde{M}$ is a CR-integrable almost Kenmotsu manifold.

Let f be an isometric immersion from M to $\tilde{M}$ defined by $f(x, y, z) = (x, y, z, 0, 0)$.

Let $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$ where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields in $\mathbb{R}^3$ which satisfies

$$[\xi, e_2] = -2e_2,\ [\xi, e_3] = -2e_3,\ [e_i, e_j] = 0, \text{ where } i, j = 2, 3.$$

Let g be the Riemannian metric defined by

$$g(\xi, \xi) = g(e_2, e_2) = g(e_3, e_3) = 1 \text{ and } g(\xi, e_i) = g(e_i, e_j) = 0 \text{ for } i \neq j;\ i, j = 2, 3.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, \xi)$, for any $Z \in TM$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi) = 0,\ \phi(e_2) = -e_3,\ \phi(e_3) = e_2.$$

Using the linearity of ϕ and g we have

$$\eta(\xi) = 1,\ \phi^2 Z = -Z + \eta(Z)\xi \text{ and } g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any $Z, U \in TM$. Moreover we obtain,

$$h'\xi = 0,\ h'e_2 = e_2,\ h'e_3 = e_3 \text{ and } h\xi = 0,\ he_2 = e_3,\ he_3 = -e_2.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul's formula we get the following:

$$\begin{aligned}
&\nabla_\xi \xi = 0,\ \nabla_\xi e_2 = 0,\ \nabla_\xi e_3 = 0,\\
&\nabla_{e_2} \xi = 2e_2,\ \nabla_{e_2} e_2 = -2\xi,\ \nabla_{e_2} e_3 = 0,\\
&\nabla_{e_3} \xi = 2e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -2\xi.
\end{aligned}$$

In view of the above relations we have

$$\nabla_X \xi = -\phi^2 X + h'X \text{ and } (\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX),$$

for any $X, Y \in TM$.

Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is a CR-integrable almost Kenmotsu manifold. It is obvious that the manifold M under consideration is a submanifold of the manifold $\tilde{M}$.

Again M is invariant.

Let $U = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in TM$ and $V = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 \in TM$ where λ_i and μ_i are scalars, $i = 1, 2, 3$ and $\xi = e_1$.

Then

$$\begin{aligned} \alpha(U, V) &= \sum \lambda_i \mu_j \alpha(e_i, e_j) \\ &= \sum \lambda_i \mu_j (\tilde{\nabla}_{e_i} e_j - \nabla_{e_i} e_j) \\ &= 0. \end{aligned}$$

Hence the submanifold is totally geodesic.

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