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SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70^{TH} BIRTHDAY

A Note on Kaehler Manifolds

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Abstract. The object of the present paper is to prove that in a Kaehler manifold of dimension $n \geq 4$, div R $= 0$ and div $C = 0$ are equivalent, where 'div' denotes divergence and R and C denote the curvature tensor and Weyl conformal curvature tensor, respectively.

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1. Introduction

Let M be an n-dimensional Kaehler manifold. Then the Kaehler metric g of M satisfies $g(JX, JY) = g(X, Y)$ and $\nabla J = 0$, where J and ∇ denote the complex structure and the covariant differentiation of M, respectively. Let R, S and C denote the curvature tensor, Ricci tensor and Weyl conformal curvature tensor of M, respectively. It is well known that a Kaehler manifold with parallel Ricci tensor is Einstein if M is irreducible. In a Riemannian manifold it can be easily verified from the differential Bianchi identity that div R $= 0$ holds if and only if $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$, where 'div' denotes divergence. It is well known [1] that if the Ricci tensor S satisfies $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$ in a Kaehler manifold, then the Ricci tensor is parallel. In a Riemannian manifold it is also known [1] that the statements
(i) div R $= 0$, (ii) div $C = 0$ and the scalar curvature is constant are equivalent.

In the present paper we prove that in a Kaehler manifold of dimension $n \geq 4$, div R$= 0$ and div C$= 0$ are equivalent.

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2. Preliminaries

In a Riemannian manifold Weyl conformal curvature tensor C is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY \\ &+ S(Y,Z)X - S(X,Z)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and r denotes the scalar curvature. It is well known [3] that in a Riemannian manifold of dimension n> 3,

$$\begin{aligned} (\mathrm{divC})(X,Y)Z &= \frac{n-3}{n-2}[\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} \\ &+ \frac{1}{2(n-1)}\{dr(X)g(Y,Z) - dr(Y)g(X,Z)\}]. \end{aligned} \tag{2}$$

In a Kaehler manifold the following relations hold [5]:

$$g(X,JY) = -g(JX,Y), \tag{3}$$

$$S(X,JY) = -S(JX,Y), \tag{4}$$

$$\nabla_X JY = J\nabla_X Y. \tag{5}$$

3. Main Result

Theorem 1. *Let M be a Kaehler manifold of dimension $n \geq 4$. Then div $R = 0$ and div $C = 0$ are equivalent.*

To prove the theorem we first state and prove the following:

Lemma 1. *In a Kaehler manifold $(\nabla_Z S)(JX,Y) = -(\nabla_Z S)(X,JY)$ holds.*

Proof. In a Kaehler manifold the Ricci tensor S satisfies $S(JX,Y) = -S(X,JY.)$ Now

$$\begin{aligned} (\nabla_Z S)(JX,Y) &= \nabla_Z S(JX,Y) - S(\nabla_Z JX,Y) - S(JX,\nabla_Z Y) \\ &= -\nabla_Z S(X,JY) - S(J\nabla_Z X,Y) - S(X,J\nabla_Z Y), \text{using (5)} \\ &= -\nabla_Z S(X,JY) + S(\nabla_Z X,JY) + S(X,\nabla_Z JY), \text{by (5)} \\ &= -(\nabla_Z S)(X,JY). \end{aligned}$$

This completes the proof.

Lemma 2. *In a Kaehler manifold the Ricci tensor S satisfies the condition $\sum_{i=1}^{n}(\nabla_{e_i} S)(JX,e_i) = \frac{1}{2}dr(JX)$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold.*

Proof. From $S(X,Y) = g(QX,Y)$ we easily get [2]

$$(\nabla_Z S)(X,Y) = g((\nabla_Z Q)X,Y). \tag{6}$$

Replacing X by JX in (6) yields

$$(\nabla_Z S)(JX,Y) = g((\nabla_Z Q)JX,Y). \tag{7}$$

Putting $Y = Z = e_i$ in (7) and taking summation over i, $i = 1,2,\ldots,n$, we get

$$(\nabla_{e_i} S)(JX,e_i) = g((\nabla_{e_i} Q)JX,e_i).$$

We know

$$\begin{aligned}(\mathrm{div}Q)(X) &= \mathrm{tr}(Z \to (\nabla_Z Q)(X)) \\ &= \textstyle\sum_i g((\nabla_{e_i} Q)(X),e_i).\end{aligned}$$

But it is known [4] that (div Q)(X) $= \frac{1}{2}dr(X)$. Hence $(\nabla_{e_i} S)(JX,e_i) = \frac{1}{2}dr(JX)$, which completes the proof.

Proof. [of the main theorem] Suppose div C = 0. Then from (2) we have

$$(\nabla_Z S)(X,Y) - (\nabla_X S)(Z,Y) = \frac{1}{2(n-1)}[dr(Z)g(X,Y) - dr(X)g(Z,Y)]. \tag{8}$$

It is known [5] that in a Kaehler manifold the Ricci tensor S satisfies

$$(\nabla_Z S)(X,Y) = (\nabla_X S)(Z,Y) + (\nabla_{JY} S)(JX,Z). \tag{9}$$

Using (9) in (8) we obtain

$$(\nabla_{JY} S)(JX,Z) = \frac{1}{2(n-1)}[dr(Z)g(X,Y) - dr(X)g(Z,Y)]. \tag{10}$$

Replacing Y by JY in (10) we obtain

$$-(\nabla_Y S)(JX,Z) = \frac{1}{2(n-1)}[dr(Z)g(X,JY) - dr(X)g(Z,JY)]. \tag{11}$$

Using (3) and Lemma 1 we get from (11)

$$(\nabla_Y S)(X,JZ) = \frac{1}{2(n-1)}[dr(Z)g(X,JY) + dr(X)g(JZ,Y)]. \tag{12}$$

Taking $X = Y = e_i$ in (12) we get

$$\frac{1}{2}dr(JZ) = \frac{1}{2(n-1)}dr(JZ)$$

which implies $dr(JZ) = 0$, since $n \geq 4$. Hence $dr(Z) = 0$, that is, r =constant. Using r =constant in (8) we get

$$(\nabla_Z S)(X,Y) = (\nabla_X S)(Z,Y).$$

Therefore div R $= 0$. This completes the proof.

From Theorem 1 and the known result mentioned in the introduction we obtain that if the conformal curvature tensor is divergence free in a Kaehler manifold of dimension ≥ 4, then the Ricci tensor is parallel.

Conversely, if the Ricci tensor is parallel, then from (2) it follows that div $C = 0$. Thus we conclude that in a Kaehler manifold of dimension ≥ 4, the statements (i) div C $= 0$ and (ii) the Ricci tensor is parallel are equivalent.

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