

*Pacific
Journal of
Mathematics*

**A NOTE ON LAGRANGIAN COBORDISMS BETWEEN
LEGENDRIAN SUBMANIFOLDS OF \mathbb{R}^{2n+1}**

ROMAN GOLOVKO

A NOTE ON LAGRANGIAN COBORDISMS BETWEEN LEGENDRIAN SUBMANIFOLDS OF \mathbb{R}^{2n+1}

ROMAN GOLOVKO

We study the relation of an embedded Lagrangian cobordism between two closed, orientable Legendrian submanifolds of \mathbb{R}^{2n+1} . More precisely, we investigate the behavior of the Thurston–Bennequin number and (linearized) Legendrian contact homology under this relation. The result about the Thurston–Bennequin number can be considered as a generalization of the result of Chantraine which holds when $n = 1$. In addition, we provide a few constructions of Lagrangian cobordisms and prove that there are infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n -tori in \mathbb{R}^{2n+1} .

1. Introduction

Basic definitions. A contact manifold (M, ξ) is a $(2n + 1)$ -dimensional manifold M equipped with a smooth maximally nonintegrable hyperplane field $\xi \subset TM$, that is, locally $\xi = \ker \alpha$, where α is a 1-form which satisfies $\alpha \wedge (d\alpha)^n \neq 0$. ξ is a contact structure and α is a contact 1-form which locally defines ξ . The Reeb vector field R_α of a contact form α is uniquely defined by the conditions $\alpha(R_\alpha) = 1$ and $d\alpha(R_\alpha, \cdot) = 0$. The most basic contact manifold is (\mathbb{R}^{2n+1}, ξ) , where \mathbb{R}^{2n+1} has coordinates $(x_1, y_1, \dots, x_n, y_n, z)$, and ξ is given by $\alpha = dz - \sum_{i=1}^n y_i dx_i$. Note that $R_\alpha = \partial_z$. From now on, for ease of notation, we write \mathbb{R}^{2n+1} instead of (\mathbb{R}^{2n+1}, ξ) .

A Legendrian submanifold of \mathbb{R}^{2n+1} is an n -dimensional submanifold Λ which is everywhere tangent to ξ , that is, $T_x \Lambda \subset \xi_x$ for every $x \in \Lambda$. The Lagrangian projection is a map $\Pi : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n}$ defined by

$$\Pi(x_1, y_1, \dots, x_n, y_n, z) = (x_1, y_1, \dots, x_n, y_n).$$

Moreover, for Λ in an open dense subset of all Legendrian submanifolds with C^∞ topology, the self-intersection of $\Pi(\Lambda)$ consists of a finite number of transverse double points. Legendrian submanifolds which satisfy this property are called

MSC2010: primary 53D12; secondary 53D42.

Keywords: Legendrian submanifold, Lagrangian cobordism, Thurston–Bennequin number, Legendrian contact homology.

chord generic. A *Reeb chord* of Λ is a path along the flow of the Reeb vector field which begins and ends on Λ . Since $R_\alpha = \partial_z$, there is a one-to-one correspondence between Reeb chords of Λ and double points of $\Pi(\Lambda)$. From now on we assume that all Legendrian submanifolds of \mathbb{R}^{2n+1} are connected and chord-generic.

The *symplectization* of \mathbb{R}^{2n+1} is the symplectic manifold $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha))$, where t is a coordinate on \mathbb{R} .

Definition 1.1. Let Λ_- and Λ_+ be two Legendrian submanifolds of \mathbb{R}^{2n+1} . We say that Λ_- is cobordant to Λ_+ if there exists a smooth cobordism $(L; \Lambda_-, \Lambda_+)$, and an embedding from L to $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha))$ such that

$$\begin{aligned} L|_{(-\infty, -T_L] \times \mathbb{R}^{2n+1}} &= (-\infty, -T_L] \times \Lambda_-, \\ L|_{[T_L, \infty) \times \mathbb{R}^{2n+1}} &= [T_L, \infty) \times \Lambda_+ \end{aligned}$$

for some $T_L \gg 0$ and $L^c := L|_{[-T_L-1, T_L+1] \times \mathbb{R}^{2n+1}}$ is compact. In the case of a Lagrangian (exact Lagrangian) embedding, we say that Λ_- is Lagrangian (exact Lagrangian) cobordant to Λ_+ . We will in general not distinguish between L and L^c and call both L .

From now on we assume that all embedded cobordisms in the symplectization of \mathbb{R}^{2n+1} are orientable.

We next define some notations. If L is an embedded, embedded Lagrangian, or embedded exact Lagrangian cobordism from Λ_- to Λ_+ , we write

$$\Lambda_- \prec_L \Lambda_+, \quad \Lambda_- \prec_L^{\text{lag}} \Lambda_+, \quad \text{or} \quad \Lambda_- \prec_L^{\text{ex}} \Lambda_+,$$

respectively. If L_Λ is a filling, Lagrangian filling, or exact Lagrangian filling of Λ in the symplectization of \mathbb{R}^{2n+1} , that is, L_Λ is an embedded, embedded Lagrangian, or embedded exact Lagrangian cobordism with empty $-\infty$ -boundary and $+\infty$ -boundary Λ , then we write $\emptyset \prec_{L_\Lambda} \Lambda$, $\emptyset \prec_{L_\Lambda}^{\text{lag}} \Lambda$ or $\emptyset \prec_{L_\Lambda}^{\text{ex}} \Lambda$, respectively.

For the discussion about Lagrangian cobordisms between Legendrian knots, we refer to [Chantraine 2010; Ekholm et al. \geq 2013], and for the obstructions to the existence of Lagrangian cobordisms defined using the theory of generating families, we refer to [Sabloff and Traynor 2010; Sabloff and Traynor 2011].

Legendrian contact homology. Legendrian contact homology was independently introduced by Eliashberg, Givental, and Hofer [Eliashberg et al. 2000] and, for Legendrian knots in \mathbb{R}^3 , by Chekanov [2002]. We now briefly remind the reader of the definition of the linearized Legendrian contact homology complex of a closed, orientable, chord-generic Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$; for more details see [Ekholm et al. 2005a].

Let \mathcal{C} be the set of Reeb chords of Λ . Since Λ is generic, \mathcal{C} is a finite set. Let A_Λ be the vector space over \mathbb{Z}_2 generated by the elements of \mathcal{C} and \mathcal{A}_Λ the unital

tensor algebra over A_Λ , that is,

$$\mathcal{A}_\Lambda = \bigotimes_{k=0}^{\infty} A_\Lambda^{\otimes k}.$$

\mathcal{A}_Λ is a differential graded algebra whose grading is denoted by $|\cdot|$ and whose differential is denoted by ∂_Λ . \mathcal{A}_Λ is called a Legendrian contact homology differential graded algebra of Λ . For the definitions of $|\cdot|$ and ∂_Λ we refer to Section 2 of [Ekholm et al. 2005b].

Note that it is difficult to use Legendrian contact homology in practical applications, as it is the homology of an infinite dimensional noncommutative algebra with a nonlinear differential. One of the ways to extract useful information from the Legendrian contact homology differential graded algebra is to follow Chekanov's [2002] linearization method, which uses an augmentation $\varepsilon : \mathcal{A}_\Lambda \rightarrow \mathbb{Z}_2$ to produce a finite-dimensional chain complex $\text{LC}^\varepsilon(\Lambda)$ whose homology is denoted by $\text{LCH}^\varepsilon(\Lambda)$. More precisely, ε is a graded algebra map $\varepsilon : \mathcal{A}_\Lambda \rightarrow \mathbb{Z}_2$ that satisfy the following two conditions:

- (1) $\varepsilon(1) = 1$;
- (2) $\varepsilon \circ \partial_\Lambda = 0$.

Consider the graded isomorphism $\varphi^\varepsilon : \mathcal{A}_\Lambda \rightarrow \mathcal{A}_\Lambda$ defined by $\varphi^\varepsilon(c) = c + \varepsilon(c)$. This map defines a new differential $\partial^\varepsilon(c) := \varphi^\varepsilon \circ \partial_\Lambda \circ (\varphi^\varepsilon)^{-1}(c)$ and $\text{LC}^\varepsilon(\Lambda) := (A_\Lambda, \partial_1^\varepsilon)$, where $\partial_1^\varepsilon : A_\Lambda \rightarrow A_\Lambda$ is a 1-component of ∂^ε . We let $\text{LCH}_\varepsilon(\Lambda)$ be the homology of the dual complex $\text{LC}_\varepsilon(\Lambda) := \text{Hom}(\text{LC}^\varepsilon(\Lambda), \mathbb{Z}_2)$.

Following Ekholm [2008], we observe that exact Lagrangian cobordism between two Legendrian submanifolds can be used to define a map between the Legendrian contact homology algebras.

In this paper, we establish the following two long exact sequences.

Theorem 1.2. *Let Λ_- and Λ_+ be two closed, orientable Legendrian submanifolds of \mathbb{R}^{2n+1} such that $\emptyset \prec_{L_\Lambda}^{\text{ex}} \Lambda_-$. Then from the condition $\Lambda_- \prec_L^{\text{ex}} \Lambda_+$ it follows that there is an exact sequence*

$$(1-1) \quad \begin{aligned} \rightarrow H_i(\Lambda_-) &\rightarrow H_i(L) \oplus \text{LCH}_{\varepsilon_-}^{n-i+2}(\Lambda_-) \\ &\rightarrow \text{LCH}_{\varepsilon_+}^{n-i+2}(\Lambda_+) \rightarrow H_{i-1}(\Lambda_-) \rightarrow . \end{aligned}$$

In addition, $\Lambda_- \prec_L^{\text{ex}} \Lambda_+$ implies that there is an exact sequence

$$(1-2) \quad \begin{aligned} \rightarrow \text{LCH}_{\varepsilon_-}^{n-i+2}(\Lambda_-) &\rightarrow \text{LCH}_{\varepsilon_+}^{n-i+2}(\Lambda_+) \\ &\rightarrow H_i(L, \Lambda_-) \rightarrow \text{LCH}_{\varepsilon_-}^{n-i+3}(\Lambda_-) \rightarrow . \end{aligned}$$

Here $\text{LCH}_{\varepsilon_{\pm}}^i(\Lambda_{\pm})$ is the linearized Legendrian contact cohomology of Λ_{\pm} over \mathbb{Z}_2 , linearized with respect to the augmentation ε_{\pm} . ε_{-} is the augmentation induced by $L_{\Lambda_{-}}$, and ε_{+} is the augmentation induced by L and ε_{-} .

We thank Joshua Sabloff and Lisa Traynor for pointing out how to get the second long exact sequence in [Theorem 1.2](#).

The Thurston–Bennequin invariant. The Thurston–Bennequin invariant (number) of a closed, orientable, connected Legendrian submanifold Λ of \mathbb{R}^{2n+1} was independently defined for $n = 1$ by Bennequin [1983] and by Thurston, and was generalized to the case when $n \geq 1$ by Tabachnikov [1988].

Pick an orientation on $\Lambda \subset \mathbb{R}^{2n+1}$. Push Λ slightly off of itself along $R_{\alpha} = \partial_z$ to get another oriented submanifold Λ' disjoint from Λ . The Thurston–Bennequin invariant of Λ is the linking number

$$\text{tb}(\Lambda) = \text{lk}(\Lambda, \Lambda').$$

Note that $\text{tb}(\Lambda)$ is independent of the choice of orientation on Λ , since changing it also changes the orientation of Λ' .

Our goal is to prove the following theorem.

Theorem 1.3. *Let Λ_{-} and Λ_{+} be two closed, orientable Legendrian submanifolds of \mathbb{R}^{2n+1} .*

(1) *If n is even and $\Lambda_{-} \prec_L \Lambda_{+}$,*

$$\text{tb}(\Lambda_{+}) + \text{tb}(\Lambda_{-}) = (-1)^{n/2+1} \chi(L).$$

(2) *If n is odd, $\emptyset \prec_{L_{\Lambda_{-}}}^{\text{ex}} \Lambda_{-}$, and $\Lambda_{-} \prec_L^{\text{ex}} \Lambda_{+}$,*

$$\text{tb}(\Lambda_{+}) - \text{tb}(\Lambda_{-}) = (-1)^{((n-2)(n-1))/2+1} \chi(L).$$

Constructions and examples. Chantraine [2010] described the way to construct Lagrangian cobordisms from Legendrian isotopies of Legendrian knots. We show that the construction of Chantraine works in high dimensions. More precisely, we prove the following:

Proposition 1.4. *Let Λ_{-} , Λ_{+} be two closed, orientable Legendrian submanifolds of \mathbb{R}^{2n+1} that are Legendrian isotopic. Then there exists an exact Lagrangian cobordism L such that*

$$\Lambda_{-} \prec_L^{\text{ex}} \Lambda_{+}.$$

Front spinning is a procedure invented by Ekholm, Etnyre, and Sullivan [Ekholm et al. 2005b] to construct a closed, orientable Legendrian submanifold $\Sigma \Lambda \subset \mathbb{R}^{2n+3}$ from a closed, orientable Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$. We will provide a detailed description of this procedure in [Section 4](#), and prove the following property of it.

Proposition 1.5. *Let Λ_-, Λ_+ be two closed, orientable Legendrian submanifolds of \mathbb{R}^{2n+1} . If $\Lambda_- \prec_L^{\text{lag}} \Lambda_+$, there exists a Lagrangian cobordism ΣL such that*

$$\Sigma \Lambda_- \prec_{\Sigma L}^{\text{lag}} \Sigma \Lambda_+.$$

In addition, if $\Lambda_- \prec_L^{\text{ex}} \Lambda_+$, there exists an exact Lagrangian cobordism ΣL such that $\Sigma \Lambda_- \prec_{\Sigma L}^{\text{ex}} \Sigma \Lambda_+$.

Finally, we apply [Proposition 1.5](#) to the exact Lagrangian cobordisms from [\[Ekholm et al. \$\geq\$ 2013\]](#) and construct exact Lagrangian cobordisms between the nonisotopic Legendrian tori described in [\[Ekholm et al. 2005b\]](#).

Proposition 1.6. *There are infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n -tori in \mathbb{R}^{2n+1} .*

2. Proof of [Theorem 1.2](#)

Proof. In this section, we prove the existence of the long exact sequences described in [Theorem 1.2](#). We first construct an exact Lagrangian filling of Λ_+ .

Since Λ_- is connected, and L, L_{Λ_-} are exact Lagrangian cobordisms in the symplectization of \mathbb{R}^{2n+1} such that the $(-\infty)$ -boundary of L , which is Λ_- , agrees with the $(+\infty)$ -boundary of L_{Λ_-} , L and L_{Λ_-} can be joined to the exact Lagrangian cobordism L_{Λ_+} in the symplectization of \mathbb{R}^{2n+1} , where L_{Λ_+} is obtained by gluing the positive end of L_{Λ_-} to the negative end of L . Since the $-\infty$ -boundary of L_{Λ_-} is empty, the $-\infty$ -boundary of L_{Λ_+} is also empty.

We now use the Mayer–Vietoris long exact sequence for $L_{\Lambda_-}, L \subset L_{\Lambda_+}$. We extend L_{Λ_-} and L in such a way that $L_{\Lambda_-} \cap L$ is diffeomorphic to $\mathbb{R} \times \Lambda_-$. Hence the Mayer–Vietoris long exact sequence can be written as

$$\rightarrow H_i(\mathbb{R} \times \Lambda_-) \rightarrow H_i(L) \oplus H_i(L_{\Lambda_-}) \rightarrow H_i(L_{\Lambda_+}) \rightarrow H_{i-1}(\mathbb{R} \times \Lambda_-) \rightarrow .$$

Now we note that $H_i(\mathbb{R} \times \Lambda_-) \simeq H_i(\Lambda_-)$ for all i . Hence we can rewrite the Mayer–Vietoris long exact sequence as

$$(2-1) \quad \rightarrow H_i(\Lambda_-) \rightarrow H_i(L) \oplus H_i(L_{\Lambda_-}) \rightarrow H_i(L_{\Lambda_+}) \rightarrow H_{i-1}(\Lambda_-) \rightarrow .$$

We now remind the reader of the following fact, which comes from certain observations of Seidel in wrapped Floer homology [\[Abouzaid and Seidel 2010; Fukaya et al. 2009\]](#).

Fact 2.1 [\[Ekholm 2012\]](#). Let Λ be a closed, orientable, connected, chord-generic Legendrian submanifold of \mathbb{R}^{2n+1} and $\emptyset \prec_{L_\Lambda}^{\text{ex}} \Lambda$. Then

$$(2-2) \quad H_{n-i+2}(L_\Lambda) \simeq \text{LCH}_\varepsilon^i(\Lambda).$$

Here ε is the augmentation induced by L_Λ .

For the definition of the augmentation induced by a filling, we refer to Section 3 of [Ekholm 2008]. Also, [Ekholm 2012] provides a fairly complete sketch of a proof of Fact 2.1.

We change the indices in (2-2) and write it as

$$(2-3) \quad H_i(L_{\Lambda_{\pm}}) \simeq \text{LCH}_{\varepsilon_{\pm}}^{n-i+2}(\Lambda_{\pm}).$$

Using (2-3), we rewrite the Mayer–Vietoris long exact sequence (2-1) as

$$(2-4) \quad \begin{aligned} \rightarrow H_i(\Lambda_-) \rightarrow H_i(L) \oplus \text{LCH}_{\varepsilon_-}^{n-i+2}(\Lambda_-) \\ \rightarrow \text{LCH}_{\varepsilon_+}^{n-i+2}(\Lambda_+) \rightarrow H_{i-1}(\Lambda_-) \rightarrow . \end{aligned}$$

We now write the long exact sequence for the pair $(L_{\Lambda_-}, L_{\Lambda_+})$

$$(2-5) \quad \rightarrow H_i(L_{\Lambda_-}) \rightarrow H_i(L_{\Lambda_+}) \rightarrow H_i(L_{\Lambda_+}, L_{\Lambda_-}) \rightarrow H_{i-1}(L_{\Lambda_-}) \rightarrow .$$

Using (2-3) and the excision theorem for L_{Λ_+} , $L \subset L_{\Lambda_+}$, we write the long exact sequence (2-5) as

$$(2-6) \quad \begin{aligned} \rightarrow \text{LCH}_{\varepsilon_-}^{n-i+2}(\Lambda_-) \rightarrow \text{LCH}_{\varepsilon_+}^{n-i+2}(\Lambda_+) \\ \rightarrow H_i(L, \Lambda_-) \rightarrow \text{LCH}_{\varepsilon_-}^{n-i+3}(\Lambda_-) \rightarrow . \quad \square \end{aligned}$$

Remark 2.2. Under the conditions of Theorem 1.2, if $H_i(\Lambda_-) = H_{i-1}(\Lambda_-) = 0$ for some i , say when $\Lambda_- = S^n$ and $i, i-1 \neq 0, n$, then long exact sequence (2-4) implies that

$$\text{LCH}_{\varepsilon_+}^{n-i+2}(\Lambda_+) \simeq H_i(L) \oplus \text{LCH}_{\varepsilon_-}^{n-i+2}(\Lambda_-).$$

Hence, for such i , we get

$$H_i(L) \simeq \text{LCH}_{\varepsilon_+}^{n-i+2}(\Lambda_+) / \text{LCH}_{\varepsilon_-}^{n-i+2}(\Lambda_-).$$

Remark 2.3. We can rewrite the long exact sequences (2-4) and (2-6) using the relative symplectic field theory of $((\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha)), L_{\Lambda_{\pm}})$, since

$$(2-7) \quad E_1^i((\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha)), L_{\Lambda_{\pm}}) \simeq \text{LCH}_{\varepsilon_{\pm}}^i(\Lambda_{\pm})$$

over \mathbb{Z}_2 . For the definition of the relative symplectic field theory, we refer to [Ekholm 2008], and for the details about the isomorphism described in (2-7), we refer to [Ekholm 2012]. (We observe that since $L_{\Lambda_{\pm}}$ are connected, the associated spectral sequences have only one level.)

3. Proof of Theorem 1.3

Let n be even. We recall the following result:

Proposition 3.1 [Eliashberg 1990]. *Let Λ be a closed, orientable, connected, chord-generic Legendrian submanifold of \mathbb{R}^{2n+1} , where n is even. Then*

$$\text{tb}(\Lambda) = (-1)^{n/2+1} \frac{1}{2} \chi(\Lambda).$$

We now note that

$$(3-1) \quad \chi(\partial L) = 2\chi(L),$$

since the Euler characteristic of an even-dimensional boundary is twice the Euler characteristic of its bounded manifold; see Chapter 21 of [May 1999]. We now observe that $\partial L = \Lambda_+ \sqcup \Lambda_-$ and hence, from (3-1), we get that

$$(3-2) \quad 2\chi(L) = \chi(\partial L) = \chi(\Lambda_+) + \chi(\Lambda_-).$$

Then we use Proposition 3.1 and rewrite (3-2) as

$$(3-3) \quad 2\chi(L) = \chi(\Lambda_+) + \chi(\Lambda_-) = 2(-1)^{-n/2-1} (\text{tb}(\Lambda_+) + \text{tb}(\Lambda_-)).$$

From (3-3) it follows that

$$(3-4) \quad \text{tb}(\Lambda_+) + \text{tb}(\Lambda_-) = (-1)^{n/2+1} \chi(L).$$

This finishes the proof of Theorem 1.3 in the case when n is even.

We now prove case (2) of the theorem. First we provide an alternate definition of the Thurston–Bennequin number, found in [Ekholm et al. 2005a].

Let Λ be a closed, orientable, connected, chord-generic Legendrian submanifold of \mathbb{R}^{2n+1} and let c be a Reeb chord of Λ with end points a and b such that $z(a) > z(b)$. We define $V_a := d\Pi(T_a\Lambda)$ and $V_b := d\Pi(T_b\Lambda)$. Given an orientation on Λ , V_a and V_b are oriented n -dimensional transverse subspaces of \mathbb{R}^{2n} . If the orientation of $V_a \oplus V_b$ agrees with that of \mathbb{R}^{2n} , we say that the sign of c , denoted by $\text{sign}(c)$, is $+1$, otherwise we say that it is -1 . Then

$$(3-5) \quad \text{tb}(\Lambda) = \sum_c \text{sign}(c),$$

where the sum is taken over all Reeb chords c of Λ .

The following proposition was proven using (3-5):

Proposition 3.2 [Ekholm et al. 2005b]. *If $\Lambda \subset \mathbb{R}^{2n+1}$ is a closed, orientable, connected, chord generic Legendrian submanifold,*

$$\text{tb}(\Lambda) = (-1)^{((n-2)(n-1))/2} \sum_{c \in \mathcal{C}} (-1)^{|c|}.$$

We now construct an exact Lagrangian filling of Λ_+ . We do it the same way as in the proof of Theorem 1.2, namely L_{Λ_+} is obtained by gluing the positive end of L_{Λ_-} to the negative end of L in the symplectization of \mathbb{R}^{2n+1} .

By using [Proposition 3.2](#) and taking Euler characteristics of the long exact sequence (1-2), we get

$$(3-6) \quad \text{tb}(\Lambda_+) - \text{tb}(\Lambda_-) = (-1)^{((n-2)(n-1))/2+1} \chi(L).$$

This finishes the proof of [Theorem 1.3](#) when n is odd. □

Remark 3.3. When $n = 1$ we can write (3-6) as

$$\text{tb}(\Lambda_+) - \text{tb}(\Lambda_-) = -\chi(L),$$

which coincides with the formula from [Theorem 1.2](#) of [\[Chantraine 2010\]](#).

Remark 3.4. Observe that the condition of [Theorem 1.3](#) in the case when n is odd is much stronger than the condition of [Theorem 1.3](#) in the case when n is even. If n is even, $\emptyset \prec_{L_{\Lambda_-}}^{\text{ex}} \Lambda_-$ and $\Lambda_- \prec_L^{\text{ex}} \Lambda_+$, then, taking Euler characteristics of the long exact sequence (1-2) and using [Proposition 3.2](#), we get that

$$\text{tb}(\Lambda_+) + \text{tb}(\Lambda_-) = (-1)^{n/2+1} \chi(L).$$

The proof of [Theorem 1.3](#) can be easily modified to become a proof of the following remark.

Remark 3.5. Let Λ be a closed, orientable Legendrian submanifold of \mathbb{R}^{2n+1} .

(1) If n is even and $\emptyset \prec_{L_{\Lambda}} \Lambda$,

$$\text{tb}(\Lambda) = (-1)^{n/2+1} \chi(L_{\Lambda}).$$

(2) If n is odd and $\emptyset \prec_{L_{\Lambda}}^{\text{ex}} \Lambda$,

$$\text{tb}(\Lambda) = (-1)^{((n-2)(n-1))/2+1} \chi(L_{\Lambda}).$$

4. Examples

In this section, we describe a few examples of Lagrangian cobordisms. These examples are based on [\[Chantraine 2010; Ekholm et al. 2005b\]](#) and the work of Ekholm, Honda, and Kálmán [\[Ekholm et al. ≥ 2013\]](#). For the constructions of Lagrangian cobordisms based on the generating families technique, we refer to [\[Bourgeois et al. ≥ 2013\]](#).

Example 4.1. *Proof of [Proposition 1.4](#).* Let Λ_- and $\Lambda_+ \subset \mathbb{R}^{2n+1}$ be two closed, orientable Legendrian submanifolds which are Legendrian isotopic. Then there is a smooth isotopy of a closed manifold Λ to \mathbb{R}^{2n+1} given by $\varphi : \Lambda \times [0, 1] \rightarrow \mathbb{R}^{2n+1}$ such that $\Lambda_\nu := \varphi(\Lambda, \nu)$ is Legendrian for all $\nu \in [0, 1]$, $\Lambda_- = \Lambda_0$ and $\Lambda_+ = \Lambda_1$. We now construct L such that $\Lambda_- \prec_L^{\text{ex}} \Lambda_+$. Observe that in the construction below one can omit the assumption that Λ_-, Λ_+, L are connected. In the case of Legendrian knots in \mathbb{R}^3 , the construction of L was described in [\[Chantraine 2010\]](#),

Theorem 1.1]. In our case, the construction of Chantraine can be described in the following way.

- (1) Note that $\mathbb{R} \times \Lambda_-$ is an exact Lagrangian submanifold of $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha))$.
- (2) Theorem 2.6.2 of [Geiges 2008] implies that there is a compactly supported one-parameter family of contactomorphisms f_ν which realizes the isotopy $(\Lambda_\nu)_{\nu \in [0,1]}$.
- (3) Proposition 2.2 from [Chantraine 2010] implies that a contactomorphism of \mathbb{R}^{2n+1} lifts to a Hamiltonian diffeomorphism of the symplectization

$$(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha)).$$

- (4) Let H be a Hamiltonian on $\mathbb{R} \times \mathbb{R}^{2n+1}$ whose flow realizes the lifts of f_ν s. The existence of H follows from (3). Following Chantraine, we construct

$$H' : \mathbb{R} \times \mathbb{R}^{2n+1} \times [0, 1] \rightarrow \mathbb{R}$$

such that

$$H'(t, x, \nu) = \begin{cases} H(t, x, \nu) & \text{for } t > T; \\ 0 & \text{for } t < -T. \end{cases}$$

Here $T \gg 0$.

- (5) Let ϕ^ν be the Hamiltonian flow of H' . We now observe that $\phi^1(\mathbb{R} \times \Lambda_-)$ coincides with $\mathbb{R} \times \Lambda_-$ near $-\infty$ and with $\mathbb{R} \times \Lambda_+$ near ∞ .
- (6) Since $\mathbb{R} \times \Lambda_-$ is exact and ϕ^1 a Hamiltonian diffeomorphism, $L := \phi^1(\mathbb{R} \times \Lambda_-)$ is exact. \square

Remark 4.2. Eliashberg and Gromov [1998] provided another proof of the fact that Legendrian isotopy implies Lagrangian cobordism.

Example 4.3. *Proof of Proposition 1.5.* The following construction is based on the front spinning method invented in [Ekholm et al. 2005b].

First we recall the notion of the front projection. The *front projection* is a map Π_F from \mathbb{R}^{2n+1} to \mathbb{R}^{n+1} defined by

$$\Pi_F(x_1, y_1, \dots, x_n, y_n, z) = (x_1, x_2, \dots, x_n, z).$$

Let Λ be a closed, orientable Legendrian submanifold of \mathbb{R}^{2n+1} parametrized by $f_\Lambda : \Lambda \rightarrow \mathbb{R}^{2n+1}$. We write

$$f_\Lambda(p) = (x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p))$$

for $p \in \Lambda$. The front projection of Λ is parametrized by $\Pi_F \circ f_\Lambda$, and we have

$$\Pi_F \circ f_\Lambda(p) = (x_1(p), x_2(p), \dots, x_n(p), z(p)).$$

Without loss of generality we can assume that $x_1(p) > 0$ for all $p \in \Lambda$. We now embed \mathbb{R}^{n+1} to \mathbb{R}^{n+2} via

$$(x_1, \dots, x_n, z) \rightarrow (x_0 = 0, x_1, \dots, x_n, z)$$

and construct the suspension of Λ , denoted by $\Sigma \Lambda$, such that $\Pi_F(\Sigma \Lambda)$ is obtained from $\Pi_F(\Lambda)$ by rotating it around the subspace $x_0 = x_1 = 0$. $\Pi_F(\Sigma \Lambda)$ can be parametrized by $(x_1(p) \sin \theta, x_1(p) \cos \theta, x_2(p), \dots, x_n(p), z(p))$ with $\theta \in S^1$ and is the front projection of a Legendrian embedding $\Lambda \times S^1 \rightarrow \mathbb{R}^{2n+3}$. For the properties of $\Sigma \Lambda$ we refer to Lemma 4.16 of [Ekholm et al. 2005b].

Let Λ_- and Λ_+ be two closed, orientable Legendrian submanifolds of \mathbb{R}^{2n+1} such that

$$(4-1) \quad \Lambda_{\pm} \subset \{(x_1, y_1, \dots, x_n, y_n, z) \in \mathbb{R}^{2n+1} \mid x_1 > 0\}$$

and $\Lambda_- \prec_L^{\text{lag}} \Lambda_+$. Let L be parametrized by $f_L : L \rightarrow \mathbb{R}^{2n+2}$

$$f_L(p) = (t(p), x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p)).$$

Without loss of generality we assume that $x_1(p) > 0$ for all p . (Formula (4-1) implies that

$$\{f_L(p) \mid x_1(p) \leq 0\}$$

is compact and we can translate L so that $x_1(p) > 0$ for all p .) Then we construct a Lagrangian cobordism from $\Sigma \Lambda_-$ to $\Sigma \Lambda_+$ that we call ΣL . We define ΣL to be parametrized by

$$f_{\Sigma L} : L \times S^1 \rightarrow \mathbb{R} \times \mathbb{R}^{2n+3}$$

with

$$\begin{aligned} f_{\Sigma L}(p, \theta) \\ = (t(p), x_1(p) \sin \theta, y_1(p) \sin \theta, x_1(p) \cos \theta, y_1(p) \cos \theta, x_2(p), \dots, z(p)). \end{aligned}$$

Here $p \in L$ and $\theta \in S^1$.

We now show that ΣL is really a Lagrangian cobordism from $\Sigma \Lambda_-$ to $\Sigma \Lambda_+$. Let

$$\Lambda_+^{T_L} := \{(x_0, \dots, y_n, z) \mid (T_L, x_0, \dots, y_n, z) \in f_{\Sigma L}(\Sigma L) \cap (\{T_L\} \times \mathbb{R}^{2n+3})\},$$

$$\Lambda_-^{T_L} := \{(x_0, \dots, y_n, z) \mid (-T_L, x_0, \dots, y_n, z) \in f_{\Sigma L}(\Sigma L) \cap (\{-T_L\} \times \mathbb{R}^{2n+3})\}.$$

From the definition of T_L , it follows that

$$\begin{aligned} f_{\Sigma L}(\Sigma L) \cap ([T_L, \infty) \times \mathbb{R}^{2n+3}) &= [T_L, \infty) \times \Lambda_+^{T_L}, \\ f_{\Sigma L}(\Sigma L) \cap ((-\infty, -T_L] \times \mathbb{R}^{2n+3}) &= (-\infty, -T_L] \times \Lambda_-^{T_L}. \end{aligned}$$

In addition, we observe that $\Lambda_{\pm}^{T_L} \subset \mathbb{R}^{2n+3}$ can be parametrized by

$$f_{\Lambda_{\pm}^{T_L}} : \Lambda_{\pm} \times S^1 \rightarrow \mathbb{R}^{2n+3}$$

such that

$$f_{\Lambda_{\pm}^{T_L}}(p, \theta) = (x_1(p) \sin \theta, y_1(p) \sin \theta, x_1(p) \cos \theta, y_1(p) \cos \theta, x_2(p), \dots, z(p)).$$

Here $p \in \Lambda_{\pm} \subset \partial L$ and $\theta \in S^1$. We now prove that $\Lambda_{\pm}^{T_L}$ coincides with $\Sigma \Lambda_{\pm}$. It is clear that $\Pi_F(\Lambda_{\pm}^{T_L}) = \Pi_F(\Sigma \Lambda_{\pm})$. It remains to prove that $\Lambda_{\pm}^{T_L}$ is a Legendrian submanifold of \mathbb{R}^{2n+3} .

It is easy to see that

$$(4-2) \quad f_{\Lambda_{\pm}^{T_L}}^* \left(dz - \sum_{i=0}^n y_i dx_i \right) = dz(p) - \sum_{i=2}^n y_i(p) dx_i(p) \\ - y_1(p)(\sin^2 \theta + \cos^2 \theta) dx_1(p) + (y_1(p)x_1(p) \sin \theta \cos \theta \\ - y_1(p)x_1(p) \sin \theta \cos \theta) d\theta.$$

Since Λ_{\pm} is a Legendrian submanifold of \mathbb{R}^{2n+1} and so $f_{\Lambda_{\pm}}^*(dz - \sum_{i=1}^n y_i dx_i) = 0$, we have

$$(4-3) \quad y_1(p) dx_1(p) = dz(p) - \sum_{i=2}^n y_i(p) dx_i(p).$$

Hence (4-2) and (4-3) imply that

$$(4-4) \quad f_{\Lambda_{\pm}^{T_L}}^* \left(dz - \sum_{i=0}^n y_i dx_i \right) = 0.$$

Since

$$f_{\Lambda_{\pm}}(p) := (x_1(p), \dots, y_n(p), z(p)),$$

where $p \in \Lambda_{\pm} \subset \partial L$ is a parametrization of an embedded submanifold of dimension n , and $x_1(p) > 0$ for $p \in \Lambda_{\pm} \subset \partial L$, one easily sees that

$$f_{\Lambda_{\pm}^{T_L}}(p) = (x_1(p) \sin \theta, y_1(p) \sin \theta, x_1(p) \cos \theta, y_1(p) \cos \theta, x_2(p), \dots, z(p)),$$

where $p \in \Lambda_{\pm}$, $\theta \in S^1$, is a parametrization of an embedded submanifold of dimension $n + 1$. Thus, using (4-4), we see that $\Lambda_{\pm}^{T_L}$ is an embedded Legendrian submanifold of \mathbb{R}^{2n+3} whose front projection coincides with $\Pi_F(\Sigma \Lambda_{\pm})$. Thus we get that $\Lambda_{\pm}^{T_L} = \Sigma \Lambda_{\pm}$.

We now note that

$$\begin{aligned}
(4-5) \quad f_{\Sigma L}^* \left(d \left(e^t \left(dz - \sum_{i=0}^n y_i dx_i \right) \right) \right) &= e^t (dt(p) \wedge dz(p) - \sum_{i=2}^n dy_i(p) \wedge dx_i(p)) \\
&- \sum_{i=2}^n y_i(p) dt(p) \wedge dx_i(p) - (y_1(p) (\sin^2 \theta + \cos^2 \theta) dt(p) \wedge dx_1(p) \\
&+ (\sin^2 \theta + \cos^2 \theta) dy_1(p) \wedge dx_1(p) + (\sin^2 \theta + \cos^2 \theta) x_1(p) y_1(p) d\theta \wedge d\theta \\
&+ (y_1(p) x_1(p) \sin \theta \cos \theta - y_1(p) x_1(p) \sin \theta \cos \theta) dt(p) \wedge d\theta \\
&+ (y_1(p) \sin \theta \cos \theta - y_1(p) \sin \theta \cos \theta) d\theta \wedge dx_1(p) \\
&+ (x_1(p) \sin \theta \cos \theta - x_1(p) \sin \theta \cos \theta) dy_1(p) \wedge d\theta).
\end{aligned}$$

In addition, observe that

$$\begin{aligned}
(4-6) \quad e^t (dt(p) \wedge dz(p) - \sum_{i=2}^n dy_i(p) \wedge dx_i(p) - \sum_{i=2}^n y_i(p) dt(p) \wedge dx_i(p)) \\
= e^t (y_1(p) dt(p) \wedge dx_1(p) + dy_1(p) \wedge dx_1(p)).
\end{aligned}$$

Hence (4-5) and (4-6) imply that

$$(4-7) \quad f_{\Sigma L}^* \left(d \left(e^t \left(dz - \sum_{i=0}^n y_i dx_i \right) \right) \right) = 0.$$

Since

$$f_L(p) = (t(p), x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p)),$$

where $p \in L$, is a parametrization of an embedded cobordism of dimension $n+1$ and $x_1(p) > 0$ for $p \in L$, one easily sees that

$$\begin{aligned}
f_{\Sigma L}(p, \theta) \\
= (t(p), x_1(p) \sin \theta, y_1(p) \sin \theta, x_1(p) \cos \theta, y_1(p) \cos \theta, x_2(p), \dots, z(p)),
\end{aligned}$$

where $p \in L$ and $\theta \in S^1$, is a parametrization of an embedded cobordism of dimension $n+2$. Hence we use (4-7) and see that ΣL is really an embedded Lagrangian cobordism from $\Sigma \Lambda_-$ to $\Sigma \Lambda_+$.

We now assume that $\Lambda_- \prec_L^{\text{ex}} \Lambda_+$. Then there is a function $h_L \in C^\infty(f_L(L), \mathbb{R})$ such that

$$dh_L = e^t \left(dz - \sum_{i=1}^n y_i dx_i \right).$$

From a calculation similar to (4-2) it follows that

$$(4-8) \quad f_{\Sigma L}^* \left(e^t \left(dz - \sum_{i=0}^n y_i dx_i \right) \right) = e^{t(p)} \left(dz(p) - \sum_{i=1}^n y_i(p) dx_i(p) \right).$$

Since $f_{\Sigma L}$ is an embedding, we can define $h_{\Sigma L} \in C^\infty(f_{\Sigma L}(\Sigma L), \mathbb{R})$ by setting

$$(f_{\Sigma L}^* h_{\Sigma L})(p, \theta) := (f_L^* h_L)(p).$$

Hence we use (4-8) and get

(4-9)

$$d(f_{\Sigma L}^* h_{\Sigma L}) = e^{t(p)} \left(dz(p) - \sum_{i=1}^n y_i(p) dx_i(p) \right) = f_{\Sigma L}^* \left(e^t \left(dz - \sum_{i=0}^n y_i dx_i \right) \right).$$

Therefore, since $f_{\Sigma L}$ is an embedding, (4-9) implies that

$$d(h_{\Sigma L}) = e^t \left(dz - \sum_{i=0}^n y_i dx_i \right).$$

Hence, ΣL is an exact Lagrangian cobordism. \square

Note that the proof of Proposition 1.5 can be easily modified to become a proof of the following remark.

Remark 4.4. Let Λ be a closed, orientable Legendrian submanifolds of \mathbb{R}^{2n+1} . If $\emptyset \prec_{L_\Lambda}^{\text{lag}} \Lambda$, there exists a Lagrangian filling $L_{\Sigma \Lambda}$ such that $\emptyset \prec_{L_{\Sigma \Lambda}}^{\text{lag}} \Sigma \Lambda$. In addition, if $\emptyset \prec_{L_\Lambda}^{\text{ex}} \Lambda$, there exists an exact Lagrangian filling $L_{\Sigma \Lambda}$ such that $\emptyset \prec_{L_{\Sigma \Lambda}}^{\text{ex}} \Sigma \Lambda$.

Before we discuss the next example, we briefly recall a few facts about exact Lagrangian cobordisms between Legendrian knots in \mathbb{R}^3 .

Theorem 4.5 [Ekholm et al. \geq 2013; Ekholm et al. 2007]. *There exists an exact Lagrangian cobordism for the following:*

- (1) Legendrian isotopy,
- (2) 0-resolution at a contractible crossing in the Lagrangian projection,
- (3) capping off a $\text{tb} = -1$ unknot with a disk.

See Figure 1 for the 0-resolution on the Lagrangian projection.

Following Ekholm, Honda, and Kálmán, we say that a *contractible crossing* of Λ is a crossing so that $z_1 - z_0$ can be shrunk to zero without affecting the other crossings. (Here z_1 is the z -coordinate on the upper strand and z_0 is the z -coordinate on the lower strand.)

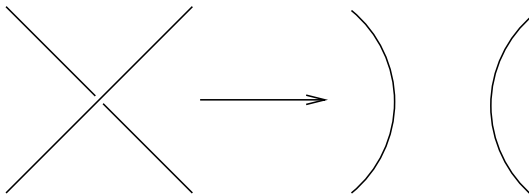


Figure 1. The 0-resolution on the Lagrangian projection.

Remark 4.6. Chantraine [2010] proved the first part of [Theorem 4.5](#).

Remark 4.7. Note that the second part of [Theorem 4.5](#) can be proven using the model from Section 3.3 of [Rizell 2012].

Conjecture 4.8 [Ekholm et al. \geq 2013; Ekholm et al. 2007]. If $\emptyset \prec_{L_\Lambda}^{\text{ex}} \Lambda$, then L_Λ is obtained by stacking exact Lagrangians cobordisms described in [Theorem 4.5](#).

Example 4.9. *Proof of Proposition 1.6.* We now use [Example 4.3](#) to get infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n -tori in \mathbb{R}^{2n+1} . We first recall that [Theorem 4.5](#) says that 0-resolution at a contractible crossing in the Lagrangian projection can be realized as an exact Lagrangian cobordism. Let T_{2k+1} be the Legendrian torus knot from [Example 4.18](#) of [Ekholm et al. 2005b]; see [Figure 2](#) for the Lagrangian projection of T_{2k+1} . One observes that all the crossings in the middle part of the Lagrangian projection are contractible (see [Ekholm et al. 2007] for the case of T_3) and hence one can get T_{2k-1} from T_{2k+1} by contracting c_{2k+1} and then c_{2k} . Let L_{2k}^{2k+1} be an exact Lagrangian cobordism which corresponds to the 0-resolution at c_{2k+1} and L_{2k-1}^{2k} an exact Lagrangian cobordism from T_{2k-1} to T_{2k} which corresponds to the resolution of c_{2k} . Then we stack L_{2k}^{2k+1} and L_{2k-1}^{2k} and get an exact Lagrangian cobordism that we call L_{2k-1}^{2k+1} such that

$$T_{2k-1} \prec_{L_{2k-1}^{2k+1}}^{\text{ex}} T_{2k+1}.$$

If we stack L_{2i-1}^{2i+1} s we get an exact Lagrangian cobordism L_{2j+1}^{2k+1} such that

$$T_{2j+1} \prec_{L_{2j+1}^{2k+1}}^{\text{ex}} T_{2k+1}$$

for $k > j$. We use the construction described in [Example 4.3](#) and get

$$\Sigma^n T_{2j+1} \prec_{\Sigma^n L_{2j+1}^{2k+1}}^{\text{ex}} \Sigma^n T_{2k+1}$$

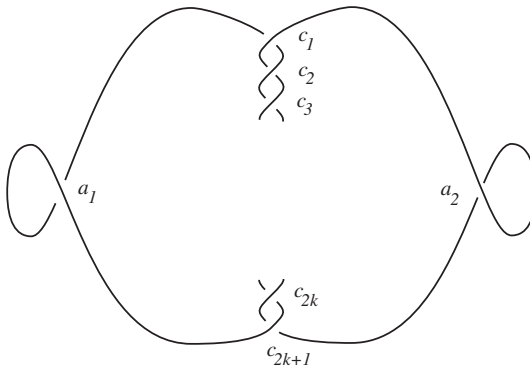


Figure 2. The knot T_{2k+1} ; cf. Figure 13 of [Ekholm et al. 2005b].

for $k > j$. We now recall that Ekholm, Etnyre, and Sullivan [Ekholm et al. 2005b, Theorem 4.19] proved that $\Sigma^n T_{2j+1}$ is not Legendrian isotopic to $\Sigma^n T_{2k+1}$ for $k > j + 1$ and $j \in \mathbb{N}$.

Hence we get infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n -tori in \mathbb{R}^{2n+1} . \square

Remark 4.10. Given $n \geq 1$, we observe that Theorem 4.19 of [Ekholm et al. 2005b] implies that all the Legendrian n -tori from Proposition 1.6 are not distinguished by the classical invariants.

Acknowledgements

The author is deeply grateful to Baptiste Chantraine, Vincent Colin, Olivier Collin, Octav Cornea, Tobias Ekholm, Yakov Eliashberg, John Etnyre, Paolo Ghiggini, Ko Honda, Clément Hyvrier, Georgios Dimitroglou Rizell, Joshua Sabloff, Lisa Traynor and Vera Vértési for helpful conversations and interest in his work. Specifically, we thank Joshua Sabloff and Lisa Traynor for pointing out the way to get the second long exact sequence in Theorem 1.2.

In addition, the author is grateful to the referee of an earlier version of this paper for many valuable comments and suggestions.

References

- [Abouzaid and Seidel 2010] M. Abouzaid and P. Seidel, “An open string analogue of Viterbo functoriality”, *Geom. Topol.* **14**:2 (2010), 627–718. MR 2011g:53190 Zbl 1195.53106
- [Bennequin 1983] D. Bennequin, “Entrelacements et équations de Pfaff”, pp. 87–161 in *Third Schnepfenried geometry conference, Vol. 1* (Schnepfenried, 1982), Astérisque **107**, Soc. Math. France, Paris, 1983. MR 86e:58070 Zbl 0573.58022
- [Bourgeois et al. \geq 2013] F. Bourgeois, J. Sabloff, and L. Traynor, “Lagrangian cobordisms via generating families with applications to Legendrian geography and botany”, in preparation.
- [Chantraine 2010] B. Chantraine, “Lagrangian concordance of Legendrian knots”, *Algebr. Geom. Topol.* **10**:1 (2010), 63–85. MR 2011f:57049 Zbl 1203.57010
- [Chekanov 2002] Y. Chekanov, “Differential algebra of Legendrian links”, *Invent. Math.* **150**:3 (2002), 441–483. MR 2003m:53153 Zbl 1029.57011
- [Ekholm 2008] T. Ekholm, “Rational symplectic field theory over \mathbb{Z}_2 for exact Lagrangian cobordisms”, *J. Eur. Math. Soc. (JEMS)* **10**:3 (2008), 641–704. MR 2009g:53130 Zbl 1154.57020
- [Ekholm 2012] T. Ekholm, “Rational SFT, linearized Legendrian contact homology, and Lagrangian Floer cohomology”, pp. 109–145 in *Perspectives in analysis, geometry, and topology*, edited by I. Itenberg et al., Progr. Math. **296**, Birkhäuser/Springer, New York, 2012. MR 2884034 Zbl 0608 2164
- [Ekholm et al. 2005a] T. Ekholm, J. Etnyre, and M. Sullivan, “The contact homology of Legendrian submanifolds in \mathbb{R}^{2n+1} ”, *J. Differential Geom.* **71**:2 (2005), 177–305. MR 2007f:53115 Zbl 1103.53048
- [Ekholm et al. 2005b] T. Ekholm, J. Etnyre, and M. Sullivan, “Non-isotopic Legendrian submanifolds in \mathbb{R}^{2n+1} ”, *J. Differential Geom.* **71**:1 (2005), 85–128. MR 2006i:53119 Zbl 1098.57013

- [Ekholm et al. 2007] T. Ekholm, K. Honda, and T. Kálmán, “Invariants of exact Lagrangian cobordisms”, lecture slides, 2007, <http://www.crm.umontreal.ca/Stanford2007/pdf/HondaSlides.pdf>.
- [Ekholm et al. \geq 2013] T. Ekholm, K. Honda, and T. Kálmán, “Invariants of exact Lagrangian cobordisms”, in preparation.
- [Eliashberg 1990] Y. Eliashberg, “Topological characterization of Stein manifolds of dimension > 2 ”, *Internat. J. Math.* **1**:1 (1990), 29–46. [MR 91k:32012](#) [Zbl 0699.58002](#)
- [Eliashberg and Gromov 1998] Y. Eliashberg and M. Gromov, “Lagrangian intersection theory: finite-dimensional approach”, pp. 27–118 in *Geometry of differential equations*, edited by A. Khovanskii et al., Amer. Math. Soc. Transl. Ser. 2 **186**, Amer. Math. Soc., Providence, RI, 1998. [MR 2002a:53102](#) [Zbl 0919.58015](#)
- [Eliashberg et al. 2000] Y. Eliashberg, A. Givental, and H. Hofer, “Introduction to symplectic field theory”, *Geom. Funct. Anal.* Special Volume, Part II (2000), 560–673. *GAF*A 2000 (Tel Aviv, 1999). [MR 2002e:53136](#) [Zbl 0989.81114](#)
- [Fukaya et al. 2009] K. Fukaya, P. Seidel, and I. Smith, “The symplectic geometry of cotangent bundles from a categorical viewpoint”, pp. 1–26 in *Homological mirror symmetry*, edited by A. Kapustin et al., Lecture Notes in Phys. **757**, Springer, Berlin, 2009. [MR 2011c:53213](#) [Zbl 1163.53344](#)
- [Geiges 2008] H. Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics **109**, Cambridge University Press, 2008. [MR 2008m:57064](#) [Zbl 1153.53002](#)
- [May 1999] J. P. May, *A concise course in algebraic topology*, University of Chicago Press, 1999. [MR 2000h:55002](#) [Zbl 0923.55001](#)
- [Rizell 2012] G. D. Rizell, “Legendrian ambient surgery and Legendrian contact homology”, preprint, 2012. [arXiv 1205.5544](#)
- [Sabloff and Traynor 2010] J. M. Sabloff and L. Traynor, “Obstructions to the existence and squeezing of Lagrangian cobordisms”, *J. Topol. Anal.* **2**:2 (2010), 203–232. [MR 2011g:53185](#) [Zbl 1210.57026](#)
- [Sabloff and Traynor 2011] J. Sabloff and L. Traynor, “Obstructions to Lagrangian cobordisms between Legendrian submanifolds”, preprint, 2011. [arXiv 1109.5660](#)
- [Tabachnikov 1988] S. L. Tabachnikov, “An invariant of a submanifold that is transversal to a distribution”, *Uspekhi Mat. Nauk* **43**:3(261) (1988), 193–194. In Russian; translated in *Russian Math. Surveys* **43**:3 (1998), 225–226. [MR 89m:58077](#)

Received December 6, 2011. Revised July 16, 2012.

ROMAN GOLOVKO
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ LIBRE DE BRUXELLES
CP 218, BOULEVARD DU TRIOMPHE
1050 BRUXELLES
BELGIUM
rgolovko@ulb.ac.be

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

V. S. Varadarajan (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Don Blasius
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Vijayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US \$400/year for the electronic version, and \$485/year for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2013 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 261 No. 1 January 2013

Hierarchies and compatibility on Courant algebroids	1
PAULO ANTUNES, CAMILLE LAURENT-GENGOUX and JOANA M. NUNES DA COSTA	
A new characterization of complete linear Weingarten hypersurfaces in real space forms	33
CÍCERO P. AQUINO, HENRIQUE F. DE LIMA and MARCO A. L. VELÁSQUEZ	
Calogero–Moser versus Kazhdan–Lusztig cells	45
CÉDRIC BONNAFÉ and RAPHAËL ROQUIER	
Coarse median spaces and groups	53
BRIAN H. BOWDITCH	
Geometrization of continuous characters of \mathbb{Z}_p^\times	95
CLIFTON CUNNINGHAM and MASOUD KAMGARPOUR	
A note on Lagrangian cobordisms between Legendrian submanifolds of \mathbb{R}^{2n+1}	101
ROMAN GOLOVKO	
On slope genera of knotted tori in 4-space	117
YI LIU, YI NI, HONGBIN SUN and SHICHENG WANG	
Formal groups of elliptic curves with potential good supersingular reduction	145
ÁLVARO LOZANO-ROBLEDO	
Codimension-one foliations calibrated by nondegenerate closed 2-forms	165
DAVID MARTÍNEZ TORRES	
The trace of Frobenius of elliptic curves and the p -adic gamma function	219
DERMOT MCCARTHY	
(DN) - (Ω) -type conditions for Fréchet operator spaces	237
KRZYSZTOF PISZCZEK	