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A Note on Lagrangians and Lovelock-Rund's Identities

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ABSTRACT

Lagrangians and Lovelock-Rund's identities are important derivations in theory of gravity which is generalization of Einstein's theory of general relativity. In this paper, we construct continuity equations in arbitrary Riemannian 4-spaces, which could be interpreted as conservation laws for the energy and momentum of the gravitational field. We put special attention in general relativity.

Keywords: Lagrangians in curved spaces, Energy and momentum in Riemannian spaces, Rund-Lovelock's identities

INTRODUCTION

For Lagrangians-based theories, we exploit from very beginning the transformation properties of fields. In this paper, we consider gravitational Lagrangians:

$$L = L(g_{ab}; g_{ab,c}; g_{ab,cd}), \tag{1}$$

where g_{ij} is the metric tensor and $_{,a} = \partial / \partial x^a$; L is a scalar density of weight one (Lovelock & Rund 1989) under an arbitrary coordinate transformation $\bar{x}^i = \bar{x}^i(x^j)$:

$$L = J\left(\frac{\bar{x}}{x}\right)\bar{L}\,. \tag{2}$$

Here we shall employ the property (2) to obtain (in general relativity) in natural manner the energy-momentum pseudo-tensors of Einstein (1916), (also see Trautman (1962), Adler *et al.* (1965), Davis (1974), Dirac (1975), Persides (1979), Palmer (1980), Landau and Lifshitz (1955), Anderson (1967), Misner *et al.* (1973), Synge (1976), Caltenco *et al.* (2005), Möller (1958), Laurent (1959), Florides (1962), Shah (1967), Goldberg (1958) and Stachel (1977)); and also the continuity equations of Komar (1959), Trautman (1964), Du Plessis (1969), and Moss (1972).

Rest of the paper is planned as follows: in the second Section, we indicate the notation and quantities to employ through the paper. The third Section is dedicated to the analysis of conservation laws originated from the equations (1) and (2), and the Hilbert's variational principle (Lovelock & Rund 1989, Misner *et al.* 1973, Rund 1966, Rund & Lovelock 1972):

$$\delta \int_{V_4} L \sqrt{-g}\, d^4 x = 0 \tag{3}$$

We also mention applications of L in the case of general relativity (Adler *et al.* 1965, Davis 1974, Dirac 1975, Anderson 1967, Schild 1967). The final Section concludes the paper.

RUND-LOVELOCK'S RELATIONS

Lovelock and Rund (1989), Rund (1966), Rund and Lovelock (1972), in their study of variational principles with Lagrangians verifying (1) and (2), showed the importance of the derivatives of L with respect to its arguments:

$$A^{ij} = A^{ji} \equiv \frac{\partial L}{\partial g_{ij}},\quad A^{ij,h} = A^{ji,h} \equiv \frac{\partial L}{\partial g_{ij,h}},$$

$$A^{ij,hk} = A^{ji,hk} = A^{ij,kh} \equiv \frac{\partial L}{\partial g_{ij,hk}} \tag{4}$$

and, in general, only $A^{ij,hk}$ has tensorial character. These quantities have the following properties:

$$A^{ij} = \frac{L}{2} g^{ij} + \frac{4}{3} A^{hk,jr} R_{k\ \mathrm{hr}}^{\ \mathrm{i}} - \Gamma^i{}_{rn} \Gamma^r{}_{km} A^{km,rh},$$

$$A^{ij,h} = \Gamma^i{}_{rk} A^{rk,jh} + \Gamma^j{}_{rk} A^{rk,ih} - \Gamma^h{}_{rk} A^{rk,ij},$$

$$A^{ij,hk} = A^{hk,ij},\ A^{ij,hk} + A^{ih,kj} + A^{ik,jh} = 0,\ A_r{}^{\mathrm{i,jh}}{}_{,\mathrm{hji}} = 0, \tag{5}$$

with the convention of Dedekind-Einstein of sum over repeated indices (Sinaceur 1990, Laugwitz 2008). On the other hand, the Euler-Lagrange expressions defined by:

$$L^{ij} = A^{ij} - A^{ij,h}{}_{,h} + A^{ij,hk}{}_{,hk}, \tag{6}$$

can be written using (5), in the form:

$$L^{ij} = \frac{L}{2} g^{ij} + \frac{2}{3} A^{hk,jr} R_{\mathrm{k}\ \mathrm{hr}}^{\ \mathrm{i}} + A^{ij,hk}{}_{;hk}, \tag{7}$$

where ;c represents the covariant derivative. Besides, we have the contracted Bianchi identities:

$$L^{ij}{}_{;j} = 0 \cdot \tag{8}$$

As an example, if L is the scalar density of weight one corresponding to general relativity (Adler *et al.* 1965, Davis 1974, Dirac 1975, Anderson 1967, Schild 1967):

$$L = \sqrt{-g}R, \qquad \mathrm{g} = \det(\mathrm{g}_{\mathrm{ab}}), \tag{9}$$

where $R \equiv R^{ij}{}_{ji}$ is the scalar curvature, then:

$$A^{ij,km} = \frac{1}{2}\sqrt{-g}\left(2g^{ij}g^{km} - g^{im}g^{jk} - g^{ik}g^{jm}\right) \tag{10}$$

Thus, (7) and (10) imply that the Euler-Lagrange relations are proportional to the Einstein tensor:

$$L^{ij} = -\sqrt{-g}G^{ij} \ , \tag{11}$$

which satisfies (8) because $G^{ab}{}_{;b} = 0$.

CONTINUITY EQUATIONS IN RIEMANNIAN SPACES

The variational principle (3) is invariant under general transformations $\bar{x}^i = \bar{x}^i(x^j)$, in particular, we can use infinitesimal coordinate changes:

$$\bar{x}^i = x^i + \varepsilon^i \xi^i\left(x^i\right) \ , \tag{12}$$

without sum over i, and ε^i denoting small constant parameters. The Noether's theorem (Noether 1918) establishes that each continuous symmetry transformation which leaves the corresponding variational principle invariant, implies a conservation law, and hence a constant of motion. For further detail, we refer to Weyl (1935), Rund (1966), Trautman (1967), Kimberling and Byers (1996), Byers (1996), Lam *et al.* (2014), Kosman-Schwarzbach (2011) and Neuenschwander (2011). Here we employ the Noether's theorem via the approach of Lanczos (1969), Lanczos (1970), Lam *et al.* (2014) and Lanczo (1973); and we apply the equation (12) to equation (3) but now considering that ε^i are new variational variables, then the Lagrange equations for ε^i give the continuity equations of Noether. Thus, it is possible to deduce the following important relations not found explicitly in the literature on gravitational energy-momentum pseudo-tensors (Synge 1967, Bak *et al.* 1994, Chang *et al.* 1999, Babak & Grishchuk 2000):

$$\left(B_r{}^i\xi^r - U_r{}^{ij}\xi^r{}_{,j} - \frac{1}{2}A_r{}^{i,jh}\xi^r{}_{,jh}\right)_{,i} = 0, \tag{13}$$

where

$$U_r{}^{ij} = A_r{}^{i,jh}{}_{;h} + \Gamma_{krh}A^{kj,ih} - \frac{1}{2}\Gamma^i{}_{hk}A_r{}^{j,hk} \ , \tag{14}$$

$$B_r^i = U_r^{ij}{}_{,j} = -\frac{L}{2}\delta_r^i + L_r{}^i + \frac{1}{2}\left(A^{jh,i} - A^{jh,ki}{}_{,k}\right)g_{jh,r} + \frac{1}{2}A^{hk,ij}g_{kh,jr} \tag{15}$$

Besides, with (8) and (15), it is easy to obtain the conservation law:

$$B_r{}^i{}_{,i} \equiv L_r{}^i{}_{;i} = 0 \,. \tag{16}$$

In the case of general relativity theory, L given by (9), and a complete study of (13) when ξ^r is a vectorial field, leads to results of Komar (1959) and Du Plessis (1969), and if ξ^r is a Killing vector, then it is also possible to deduce the relations of Trautman (1964) and Moss (1972). On the other hand, (16) allows to construct the energy-momentum pseudo-tensors of Landau and Lifshitz (1955), Caltenco *et al.* (2005), Misner *et al.* (1973), Synge (1976), Anderson (1967), Möller (1958), Goldberg (1958) and Stachel (1977).

Sometimes in the Einstein theory, we use the Lagrangian (Adler *et al.* 1965, Misner *et al.* 1973, Anderson 1967, Dirac 1975):

$$\bar{L} = \sqrt{-g}\,g^{ab}\left(\Gamma^i{}_{ab}\Gamma^j{}_{ij} - \Gamma^i{}_{ja}\Gamma^j{}_{ib}\right), \tag{17}$$

such that $\sqrt{-g}R = \bar{L} + (ordinary \ \ divergence)$, then the empty field equations are the same for (9) and (17), besides $\partial\bar{L}/\partial g_{ij,hk} = 0$. However, $\bar{L}$ satisfies (2) when ξ^r are constants, therefore (13) is equivalent to $B_r{}^i{}_{,i} = 0$ and from (15):

$$B_r{}^i = 8\pi\sqrt{-g}t_r{}^i = \frac{1}{2}\left(\frac{\partial\bar{L}}{\partial g_{jh,i}}g_{jh,r} - \bar{L}\delta_r^i\right), \tag{18}$$

where $t_r{}^i$ is the canonical energy-momentum pseudo-tensor of Einstein (1916), (also see Trautman (1962), Adler *et al.* (1965), Davis (1974), Dirac (1975), Persides (1979), Palmer (1980), Caltenco *et al.* (2010)). Thus, the conservation law $t_r{}^i{}_{,i} = 0$ is implied by the translational invariance of $\bar{L}$.

CONCLUSION

It is thus, the Lanczos technique for the Noether theorem gives the continuity equations (13) and (16) which have total information on energy-momentum quantities for gravitational theories in Riemannian spaces. Thus, the energy-momentum can be regarded as the most fundamental conserved quantity being associated with a symmetry of the space time geometry.

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