A NOTE ON LOCAL GRADIENT ESTIMATE ON ALEXANDROV SPACES

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Abstract. In this note, we prove Cheng-Yau type local gradient estimate for harmonic functions on Alexandrov spaces with Ricci curvature bounded below. We adopt a refined version of Moser's iteration which is based on Zhang-Zhu's Bochner type formula in [24]. Our result improves the previous one of Zhang-Zhu [24] in the case of negative Ricci lower bound.

1. Introduction. In 1975, Yau [21] proved that complete Riemannian manifolds with nonnegative Ricci curvature have the Liouville property. Later, Cheng-Yau [4] proved the following local version of Yau's gradient estimate.

THEOREM A (Yau [21], Cheng-Yau [4]). *Let M^n be an n-dimensional complete non-compact Riemannian manifold with Ricci curvature bounded from below by $-K$ ($K \geq 0$). Then there exists a constant $C = C(n)$, depending only on n, such that every positive harmonic function u on geodesic ball $B_{2R} \subset M$ satisfies*

$$\frac{|\nabla u|}{u} \leq C \frac{1 + \sqrt{K}R}{R} \quad \text{in } B_R.$$

To prove the regularity of harmonic functions on Alexandrov spaces is a challenging problem because of the lack of the smoothness of the metrics; the Hölder continuity of harmonic functions is well-known (see e.g. Kuwae-Machigashira-Shioya [7]). In 1996, Petrunin [16] proved the Lipschitz continuity of harmonic functions on Alexandrov spaces. Recently, Zhang-Zhu [22] introduced a notion of Ricci curvature on Alexandrov spaces. Using a delicate argument initiated by Petrunin, Zhang-Zhu [24] proved the Bochner formula on Alexandrov spaces which gives a quantitative estimate, i.e., Yau's gradient estimate for harmonic functions.

THEOREM B (Zhang-Zhu [24]). *Let X be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ ($K \geq 0$), and let Ω be a bounded domain in X. Then there exists a constant $C = C(n, \sqrt{K}\text{diam}(\Omega))$, depending on n and $\sqrt{K}\text{diam}(\Omega)$, such that every positive harmonic function u on Ω satisfies*

$$(1) \qquad \frac{|\nabla u|}{u} \leq C \frac{1 + \sqrt{K}R}{R} \quad \text{in } B_R,$$

for any geodesic ball $B_{2R} \subset \Omega$. If $K = 0$, the constant C depends only on n.

For the case $K > 0$ Theorem B is not satisfactory, compared with Theorem A, since the constant C depends not only on the dimension n but also on $\sqrt{K}R$. In this note, we refine the argument of Zhang-Zhu [24] and derive a local gradient estimate analogous to the Riemannian case. Our main result is the following.

THEOREM 1.1. *Let X be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ ($K > 0$). Then there exists a constant $C = C(n)$, depending only on n, such that every positive harmonic function u on geodesic ball $B_{2R} \subset M$ satisfies*

$$\frac{|\nabla u|}{u} \leq C \frac{1 + \sqrt{K}R}{R} \quad \text{in } B_R.$$

Owing to the lack of regularity of harmonic functions on Alexandrov spaces, one cannot use the method of maximum principle that was used in Yau [21] and Cheng-Yau [4]. As in Zhang-Zhu [24], we start with the Bochner formula established in [24] and use Moser's iteration argument. For Alexandrov spaces with negative Ricci curvature lower bound, we refine a local uniform Sobolev inequality (see Theorem 3.1) and obtain an L^p estimate of $|\nabla u|^2$ for $p \sim 1 + \sqrt{K}R$; this estimate is a starting point of Moser's iteration. This is adapted from the idea of Wang-Zhang [19]. The similar idea has been successfully applied to Finsler manifolds by the second author [20].

The rest of the paper is organized as follows: In Section 2, we recall some basic and known results on Alexandrov spaces. In Section 3, we prepare the analytic tool , i.e., Sobolev inequality for Moser's iteration. Section 4 is devoted to the proof of Theorem 1.1.

2. Preliminaries on Alexandrov spaces. A metric space (X, d) is called an Alexandrov space if it is a complete locally compact geodesic space with sectional curvature bounded below locally in the sense of Alexandrov, i.e., locally satisfying Toponogov's triangle comparison and of finite Hausdorff dimension. We refer to [1, 2] for the basic facts of Alexandrov spaces. It is well-known that the Bishop-Gromov volume comparison holds on Alexandrov spaces.

Kuwae-Shioya [8, 9, 10, 11] introduced and investigated a notion of infinitesimal Bishop-Gromov volume comparison, $BG(n, \kappa)$. Let (X, d) be an n-dimensional Alexandrov space. Set for any $\kappa \in \mathbf{R}$,

$$\operatorname{sn}_\kappa(t) = \begin{cases} \dfrac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}, & \kappa > 0, \\ t, & \kappa = 0, \\ \dfrac{\sinh(\sqrt{|\kappa|}t)}{\sqrt{|\kappa|}}, & \kappa < 0. \end{cases}$$

For any $p \in X$, denote by $\mathrm{r}_p(x) := d(x, p)$ the distance function from p. For $p \in X$ and $0 < t \leq 1$, we define a subset $W_{p,t} \subset X$ and a map $\Phi_{p,t} : W_{p,t} \to X$ as follows: $x \in W_{p,t}$ if and only if there exists some $y \in X$ such that $x \in py$ and $\mathrm{r}_p(x) : \mathrm{r}_p(y) = t : 1$, where py is a minimal geodesic (shortest path) from p to y. For any $x \in W_{p,t}$, such y is unique

(by Toponogov's triangle comparison) and we define $\Phi_{p,t}(x) = y$. Let $\mathcal{H}^n$ denote the n-dimensional Hausdorff measure on (X, d). The infinitesimal Bishop-Gromov condition for X with the measure $\mathcal{H}^n$, $BG(n, \kappa)$, is defined as follows: For any $p \in X$ and $t \in (0, 1]$, we have

$$(2) \qquad d((\Phi_{p,t})_* \mathcal{H}^n)(x) \geq \frac{t \operatorname{sn}_\kappa (t r_p(x))^{n-1}}{\operatorname{sn}_\kappa (r_p(x))^{n-1}} d\mathcal{H}^n(x)$$

for any $x \in X$ ($r_p(x) < \pi/\sqrt{\kappa}$ if $\kappa > 0$), where $(\Phi_{p,t})_* \mathcal{H}^n$ is the push-forward of the measure $\mathcal{H}^n$ by $\Phi_{p,t}$. In Riemannian case, the condition $BG(n, \kappa)$ is equivalent to $\mathrm{Ric} \geq (n-1)\kappa$. This definition roots in a key property of the Ricci curvature on Riemannian manifolds, i.e., the infinitesimal volume comparison. There exists another notion of volume comparison called $MCP(n, \kappa)$, which was defined by Sturm [18] and Ohta [15]. $MCP(n, \kappa)$ is equivalent to $BG(n, \kappa)$ for Alexandrov spaces (see [15, 10]).

The infinitesimal version of Bishop-Gromov volume comparison implies the global version of Bishop-Gromov volume comparison, which is also called relative volume comparison (see [10]). Denote by $B_R(p) := \{x \in X; d(x, p) < R\}$ the geodesic ball of radius R centered at p, by $|B_R(p)| := \mathcal{H}^n(B_R(p))$ the volume of the ball $B_R(p)$ and by $V_r^{n,\kappa}$ the volume of a geodesic ball of radius r in the space form $\Pi^{n,\kappa}$, i.e., the complete simply connected n-dimensional Riemannian manifold with constant sectional curvature κ.

THEOREM 2.1 ([10]). *Let X be an Alexandrov space satisfying $BG(n, \kappa)$, $\kappa \leq 0$. Then for any $p \in X$ and $0 < r < R$, we have*

$$(3) \qquad \frac{|B_R(p)|}{|B_r(p)|} \leq \frac{V_R^{n,\kappa}}{V_r^{n,\kappa}} \leq e^{2\sqrt{-(n-1)\kappa}R} \left(\frac{R}{r} \right)^n.$$

Zhang-Zhu [22, 23] introduced a geometric version of lower Ricci curvature bound on Alexandrov spaces, denoted by $\mathrm{Ric} \geq -K$ ($K \geq 0$). This definition reveals another important feature of Ricci curvature in Riemannian case, the Bochner formula. For details, we refer to [24]. It was proved in [22] that $\mathrm{Ric} \geq -K$ for an n-dimensional Alexandrov space implies Lott-Villani-Sturm's curvature dimension condition $CD(n, K)$ (for definition, see [12, 13, 18]) and Kuwae-Shioya's infinitesimal Bishop-Gromov comparison $BG(n, -K/(n-1))$.

We recall some basic results on Alexandrov spaces. For a domain $\Omega \subset X$, we denote by $\mathrm{Lip}(\Omega)$ ($\mathrm{Lip}_0(\Omega)$) the space of (compact supported) Lipschitz functions on Ω. It can be shown that every Lipschitz function is differentiable $\mathcal{H}^n$-almost everywhere and has bounded gradient (see Cheeger [3]). For a precompact domain $\Omega' \subset\subset X$ and $u \in \mathrm{Lip}(\Omega')$, the $W^{1,2}$ norm of u is defined as

$$\|u\|_{W^{1,2}(\Omega')}^2 = \int_{\Omega'} u^2 + \int_{\Omega'} |\nabla u|^2.$$

Here, each integral above means the integration with respect to $\mathcal{H}^n$. The space $W^{1,2}(\Omega')$ (resp. $W_0^{1,2}(\Omega')$) is the completion of $\mathrm{Lip}(\Omega')$ (resp. $\mathrm{Lip}_0(\Omega')$) with respect to the $W^{1,2}$ norm defined above. For a domain $\Omega \subset X$, the local $W^{1,2}$ space of Ω, $W_{\mathrm{loc}}^{1,2}(\Omega)$, consists of functions u with $u|_{\Omega'} \in W^{1,2}(\Omega')$ for any $\Omega' \subset\subset \Omega$. For $u \in W_{\mathrm{loc}}^{1,2}(\Omega)$, we define a linear

functional $\mathcal{L}_u$ on Ω (corresponding to Δu in the smooth setting) by

$$\mathcal{L}_u(\eta) = -\int_\Omega \langle \nabla u, \nabla \eta \rangle \quad \text{for } \eta \in \text{Lip}_0(\Omega)\,.$$

In general, $\mathcal{L}_u$ is a signed Radon measure on Ω. Let $f \in L^2(\Omega)$. We then say a function $u \in W^{1,2}_{\text{loc}}(\Omega)$ solves the Poisson equation $\mathcal{L}_u = f \cdot \mathcal{H}^n$ on Ω if

$$\mathcal{L}_u(\eta) = \int_\Omega f \cdot \eta \quad \text{for } \eta \in \text{Lip}_0(\Omega)\,.$$

Zhang-Zhu [24] established the following Bochner formula on Alexandrov spaces. For simplicity, we only formulate it in the following special case, where we choose $f(x, s) = -s$ in [24, Theorem 1.2], since it is sufficient for our purpose.

THEOREM 2.2 (Zhang-Zhu [24] Theorem 1.2). *Let Ω be a domain in an n-dimensional Alexandrov space X with* $\text{Ric} \geq -K$. *let $u \in \text{Lip}(\Omega)$ solve the Poisson equation*

$$\mathcal{L}_u = -|\nabla u|^2 \cdot \mathcal{H}^n\,.$$

Then $|\nabla u|^2 \in W^{1,2}_{\text{loc}}(\Omega)$ and

$$(4) \qquad \frac{1}{2}\mathcal{L}_{|\nabla u|^2} \geq \left(\frac{1}{n}|\nabla u|^4 - \langle \nabla u, \nabla |\nabla u|^2 \rangle - K|\nabla u|^2\right) \cdot \mathcal{H}^n\,.$$

3. Local uniform Poincaré inequality and Sobolev inequality. The Sobolev inequality is necessary to carry out Moser's iteration. In view of the standard theory for general metric measure spaces, one needs the volume doubling condition and the local uniform Poincaré inequality to prove the local Sobolev inequality. In fact, for an Alexandrov space with $\text{Ric} \geq -K$ there are stronger volume growth properties, i.e., infinitesimal Bishop-Gromov volume comparison $BG(n, -K/(n-1))$ and the global version of (3). These volume growth properties of Alexandrov spaces with Ricci curvature bounded below are sufficient to prove the Poincaré inequality.

Since the argument for proving the Poincaré inequality is now standard, we give only a sketch here (see e.g. [17, Theorem 5.6.5], [7, Theorem 7.2], [6, Theorem 3.1]). By the infinitesimal Bishop-Gromov volume comparison and the change of variables, one can show the so-called weak Poincaré inequality, i.e., for any $u \in W^{1,2}_{\text{loc}}(B_{2R})$,

$$\fint_{B_R} |u - \bar{u}|^2 \leq Ce^{C\sqrt{K}R}R^2 \fint_{B_{2R}} |\nabla u|^2\,,$$

where $C = C(n)$ and $\bar{u} = \frac{1}{|B_R|}\int_{B_R} u$. The only difference of the weak Poincaré inequality from the desired one (5) below is that the integration in the right-hand side is over B_{2R} instead of B_R. A Whitney-type argument, which relies on the doubling property of the measure, yields the Poincaré inequality from the weak one. None of these arguments involve the smoothness assumptions of the metric, hence they are adaptable to our setting. We remark that the precise constant, $e^{C\sqrt{K}R}$, in the Poincaré inequality is crucial for this paper.

LEMMA 3.1 (local uniform Poincaré inequality).   *Let X be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ ($K > 0$). Then there exists $C = C(n)$ such that for $B_R \subset X$ and $u \in W^{1,2}_{loc}(B_R)$,*

$$(5) \qquad \int_{B_R} |u - \bar{u}|^2 \le C e^{C\sqrt{K}R} R^2 \int_{B_R} |\nabla u|^2 \,,$$

where $\bar{u} = \frac{1}{|B_R|} \int_{B_R} u$.

As long as the uniform local Poincaré inequality and the Bishop-Gromov volume comparison (3) are applicable, one can obtain the following local uniform Sobolev inequality by the same argument as in [14, Lemma 3.2] (see also [5]).

THEOREM 3.1 (local uniform Sobolev inequality).   *Let X be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ ($K > 0$). Then there exist two constants $\nu > 2$ and C, both depending only on n, such that for $B_R \subset X$ and $u \in W^{1,2}_{loc}(B_R)$,*

$$(6) \qquad \left(\int_{B_R} (u - \bar{u})^{\frac{2\nu}{\nu-2}} \right)^{\frac{\nu-2}{\nu}} \le e^{C(1+\sqrt{K}R)} R^2 |B_R|^{-\frac{2}{\nu}} \int_{B_R} |\nabla u|^2 \,,$$

where $\bar{u} = \frac{1}{|B_R|} \int_{B_R} u$. In particular,

$$(7) \qquad \left(\int_{B_R} u^{\frac{2\nu}{\nu-2}} \right)^{\frac{\nu-2}{\nu}} \le e^{C(1+\sqrt{K}R)} R^2 |B_R|^{-\frac{2}{\nu}} \int_{B_R} (|\nabla u|^2 + R^{-2} u^2) \,.$$

4.   Proof of Theorem 1.1.   Without loss of generality, we may assume that u is a positive harmonic function on B_{4R}. It was proved in [16] and [24] that u is locally Lipschitz continuous in B_{4R}, $|\nabla u|$ is lower semi-continuous in B_{4R} and $|\nabla u|^2 \in W^{1,2}_{loc}(B_{4R})$. Denote $v = \log u$. One can easily verify that

$$(8) \qquad \mathcal{L}_v = -|\nabla v|^2 \cdot \mathcal{H}^n \,.$$

Since $v \in \text{Lip}(B_{2R})$, by setting $f = |\nabla v|^2$, it follows from the Bochner formula (4) that for any $0 \le \eta \in \text{Lip}_0(B_{2R})$,

$$(9) \qquad \int_{B_{2R}} \langle \nabla \eta, \nabla f \rangle \le \int_{B_{2R}} \eta \left(2\langle \nabla v, \nabla f \rangle + 2Kf - \frac{2}{n} f^2 \right) \,.$$

In fact, by an approximation argument, (9) holds for any $0 \le \eta \in W^{1,2}_0(B_{2R}) \cap L^\infty(B_{2R})$. Let $\eta = \phi^2 f^\beta$, with $\phi \in \text{Lip}_0(B_{2R})$, $0 \le \phi \le 1$ and $\beta \ge 1$. Then η is an admissible test function for (9). Hence we have from (9) that

$$\int_{B_{2R}} \beta \phi^2 f^{\beta-1} |\nabla f|^2 + 2\phi f^\beta \langle \nabla f, \nabla \phi \rangle$$

$$\le \int_{B_{2R}} \phi^2 f^\beta \left(2\langle \nabla v, \nabla f \rangle + 2Kf - \frac{2}{n} f^2 \right) \,.$$

It follows that

$$\frac{4\beta}{(\beta+1)^2} \int_{B_{2R}} \phi^2 |\nabla f^{\frac{\beta+1}{2}}|^2 \leq \frac{4}{\beta+1} \int_{B_{2R}} \phi f^{\frac{\beta+1}{2}} |\nabla\phi| |\nabla f^{\frac{\beta+1}{2}}|$$

$$+ \frac{4}{\beta+1} \int_{B_{2R}} \phi^2 f^{\frac{\beta+2}{2}} |\nabla f^{\frac{\beta+1}{2}}|$$

$$- \int_{B_{2R}} \frac{2}{n} \phi^2 f^{\beta+2} + \int_{B_{2R}} 2K\phi^2 f^{\beta+1} .$$

Using the Hölder inequality, we obtain

$$\int_{B_{2R}} \phi^2 |\nabla f^{\frac{\beta+1}{2}}|^2 \leq C_1 \int_{B_{2R}} |\nabla\phi|^2 f^{\beta+1} + C_2 \int_{B_{2R}} \phi^2 f^{\beta+2}$$

$$-C_3\beta \int_{B_{2R}} \phi^2 f^{\beta+2} + C_4\beta K \int_{B_{2R}} \phi^2 f^{\beta+1} .$$

We remark that from now on, constants $C_1, C_2, \ldots$, depend only on n.

For $\beta \geq 2C_2/C_3$, we have

$$\int_{B_{2R}} |\nabla(\phi f^{\frac{\beta+1}{2}})|^2 + \frac{1}{2}C_3\beta \int_{B_{2R}} \phi^2 f^{\beta+2}$$

(10)

$$\leq 2C_1 \int_{B_{2R}} |\nabla\phi|^2 f^{\beta+1} + C_4\beta K \int_{B_{2R}} \phi^2 f^{\beta+1} .$$

Using the Sobolev inequality (7), we obtain

(11)
$$\left(\int_{B_{2R}} \phi^{2\chi} f^{(\beta+1)\chi} \right)^{\frac{1}{\chi}} \leq e^{C_5(1+\sqrt{K}R)} R^2 |B_{2R}|^{-\frac{2}{\nu}} \left(C_6 \int_{B_{2R}} |\nabla\phi|^2 f^{\beta+1} \right.$$

$$\left. + (C_7\beta K + C_8 R^{-2}) \int_{B_{2R}} \phi^2 f^{\beta+1} - \beta \int_{B_{2R}} \phi^2 f^{\beta+2} \right),$$

where $\chi = \nu/(\nu - 2)$.

We first use (11) to prove the following:

LEMMA 4.1. *There exist constants C_9 and C_{10} such that for $\beta_0 = C_{10}(1 + \sqrt{K}R)$ and $\beta_1 = (\beta_0 + 1)\chi$, we have $f \in L^{\beta_1}(B_{\frac{3}{2}R})$ and*

(12)
$$\|f\|_{L^{\beta_1}\left(B_{\frac{3}{2}R}\right)} \leq C_9 \frac{(1 + \sqrt{K}R)^2}{R^2} |B_{2R}|^{\frac{1}{\beta_1}} .$$

PROOF. Let $C_{10} \geq 2C_2/C_3$ such that $\beta_0 = C_{10}(1 + \sqrt{K}R)$ satisfies (10) and (11). We rewrite (11) for $\beta = \beta_0$ as

(13)
$$\left(\int_{B_{2R}} \phi^{2\chi} f^{(\beta_0+1)\chi} \right)^{\frac{1}{\chi}} \leq e^{C_{11}\beta_0} |B_{2R}|^{-\frac{2}{\nu}} \left(C_6 R^2 \int_{B_{2R}} |\nabla\phi|^2 f^{\beta_0+1} \right.$$

$$\left. + C_{12}\beta_0^3 \int_{B_{2R}} \phi^2 f^{\beta_0+1} - \beta_0 R^2 \int_{B_{2R}} \phi^2 f^{\beta_0+2} \right).$$

We estimate the second term in the right-hand side of (13) as follows:

$$
C_{12}\beta_0^3 \int_{B_{2R}} \phi^2 f^{\beta_0+1} = C_{12}\beta_0^3 \left(\int_{\{f \geq 2C_{12}\beta_0^2 R^{-2}\}} \phi^2 f^{\beta_0+1} + \int_{\{f < 2C_{12}\beta_0^2 R^{-2}\}} \phi^2 f^{\beta_0+1} \right)
$$

(14)

$$
\leq \frac{1}{2}\beta_0 R^2 \int_{B_{2R}} \phi^2 f^{\beta_0+2} + C_{13}^{\beta_0+1} \beta_0^3 \left(\frac{\beta_0}{R} \right)^{2(\beta_0+1)} |B_{2R}| .
$$

Set $\phi = \psi^{\beta_0+2}$ with $\psi \in \mathrm{Lip}_0(B_{2R})$ satisfying

$$
0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ in } B_{\frac{3}{2}R}, \ |\nabla\psi| \leq \frac{C_{14}}{R}.
$$

Then

$$
R^2 |\nabla\phi|^2 \leq C_{15}\beta_0^2 \phi^{\frac{2(\beta_0+1)}{\beta_0+2}} .
$$

By the Hölder inequality and the Young inequality, the first term in the right-hand side of (13) can be estimated as follows:

$$
C_6 R^2 \int_{B_{2R}} |\nabla\phi|^2 f^{\beta_0+1} \leq C_{16}\beta_0^2 \int_{B_{2R}} \phi^{\frac{2(\beta_0+1)}{\beta_0+2}} f^{\beta_0+1}
$$

(15)

$$
\leq C_{16}\beta_0^2 \left(\int_{B_{2R}} \phi^2 f^{\beta_0+2} \right)^{\frac{\beta_0+1}{\beta_0+2}} |B_{2R}|^{\frac{1}{\beta_0+2}}
$$

$$
\leq \frac{1}{2}\beta_0 R^2 \int_{B_{2R}} \phi^2 f^{\beta_0+2} + C_{17}\beta_0^{\beta_0+3} R^{-2(\beta_0+1)} |B_{2R}| .
$$

Substituting the estimates (14) and (15) into (13), we obtain

$$
\left(\int_{B_{2R}} \phi^{2\chi} f^{(\beta_0+1)\chi} \right)^{\frac{1}{\chi}} \leq 2e^{C_{11}\beta_0} C_{13}^{\beta_0+1} \beta_0^3 \left(\frac{\beta_0}{R} \right)^{2(\beta_0+1)} |B_{2R}|^{1-\frac{2}{\nu}} .
$$

Taking the $(\beta_0 + 1)$-st root on both sides, we get

$$
\|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})} \leq C_{18} \left(\frac{\beta_0}{R} \right)^2 |B_{2R}|^{\frac{1}{\beta_1}} .
$$

$\square$

Now we start from (11) and use Moser's iteration to prove Theorem 1.1.

Let $R_k = R + R/2^k$ and $\phi_k \in \mathrm{Lip}_0(B_{R_k})$ satisfy

$$
0 \leq \phi_k \leq 1, \quad \phi_k \equiv 1 \text{ in } B_{R_{k+1}}, \quad |\nabla\phi_k| \leq C\frac{2^{k+1}}{R} .
$$

Let β_0, β_1 be the numbers in Lemma 4.1 and $\beta_{k+1} = \beta_k \chi$ for $k \geq 1$. One can deduce from (11) with $\beta + 1 = \beta_k$ and $\phi = \phi_k$ that (we have dropped the last term in the right-hand side of (11) since it is negative)

$$
\|f\|_{L^{\beta_{k+1}}(B_{R_{k+1}})} \leq e^{C_{19}\frac{\beta_0}{\beta_k}} |B_{2R}|^{-\frac{2}{\nu}\frac{1}{\beta_k}} (4^k + \beta_0^2\beta_k)^{\frac{1}{\beta_k}} \|f\|_{L^{\beta_k}(B_{R_k})} .
$$

Hence by iteration we get

$$\|f\|_{L^\infty(B_R)} \le e^{C_{19}\beta_0 \sum_k \frac{1}{\beta_k}} |B_{2R}|^{-\frac{2}{\nu}\sum_k \frac{1}{\beta_k}} \prod_k (4^k + 2\beta_0^3 \chi^k)^{\frac{1}{\beta_k}} \|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})}.$$

Since $\sum_k \frac{1}{\beta_k} = \frac{\nu}{2}\frac{1}{\beta_1}$ and $\sum_k \frac{k}{\beta_k}$ converges, we have

$$\|f\|_{L^\infty(B_R)} \le C_{20} e^{C_{21}\frac{\beta_0}{\beta_1}} \beta_0^{\frac{3\nu}{2}\frac{1}{\beta_1}} |B_{2R}|^{-\frac{1}{\beta_1}} \|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})}$$

$$\le C_{22} |B_{2R}|^{-\frac{1}{\beta_1}} \|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})}.$$

Using Lemma 4.1, we conclude

$$\|f\|_{L^\infty(B_R)} \le C(n)\frac{(1+\sqrt{K}R)^2}{R^2},$$

which implies

$$\|\nabla \log u\|_{L^\infty(B_R)} \le C(n)\frac{1+\sqrt{K}R}{R}.$$

This proves Theorem 1.1.

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