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Research Article

A Note on Locally Inverse Semigroup Algebras

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Let R be a commutative ring and S a finite locally inverse semigroup. It is proved that the semi-group algebra R[S] is isomorphic to the direct product of Munn algebras \mathcal{M} ($R[G_J]$, m_J , n_J ; P_J) with $J \in S/\mathcal{D}$, where m_J is the number of \mathcal{R} -classes in J, n_J the number of \mathcal{L} -classes in J, and G_J a maximum subgroup of J. As applications, we obtain the sufficient and necessary conditions for the semigroup algebra of a finite locally inverse semigroup to be semisimple.

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1. Main results

A regular semigroup S is called a *locally inverse semigroup* if for all idempotent $e \in S$, the local submonoid eSe is an inverse semigroup under the multiplication of S. Inverse semigroups are locally inverse semigroups. Inverse semigroup algebras are a class of semigroup algebras which is widely investigated. One of fundamentally important results is that a finite inverse semigroup algebra is the direct product of full matrix algebras over group algebras of the maximum subgroups of this finite inverse semigroup. Consider that all local submonoids of a locally inverse semigroup are inverse semigroups, it is a very natural problem whether a finite locally inverse semigroup algebra has a similar representation to inverse semigroup algebras. This is the main topic of this note.

Let \mathcal{A} be an R-algebra. Let m and n be positive integers, and let P be a fixed $n \times m$ matrix over \mathcal{A} . Let $\mathcal{M} := \mathcal{M}(\mathcal{A}; m, n; P)$ be the vector space of all $m \times n$ matrices over \mathcal{A} . Define a product \circ in \mathcal{M} by

$$A \circ B = APB \quad (A, B \in \mathcal{M}), \tag{1.1}$$

where APB is the usual matrix product of A, P, and B. Then \mathcal{M} is an algebra over R. Following [1], we call \mathcal{M} the Munn $m \times n$ matrix algebra over \mathcal{A} with sandwich matrix P.

By a *semisimple semigroup*, we mean a semigroup each of whose principal factor is either a completely 0-simple semigroup or a completely simple semigroup. It is well known that a finite regular semigroup is semisimple. The Rees theorem tells us that any completely 0-simple semigroup (completely simple semigroup) is isomorphic to some Rees matrix semigroup $\mathcal{M}^0(G,I,\Lambda;P)$ ($\mathcal{M}(G,I,\Lambda;P)$), and vice versa (for Rees matrix semigroups, refer to [1]). In what follows, by the phrase "Let $S = \bigcup_{J \in S/2} \mathcal{M}^0(G_J;I_J,\Lambda_J;P_J)$ be a finite regular semigroup," we mean that S is a finite regular semigroup in which the principal factor of S determined by the \mathcal{Q} -class J is isomorphic to the Rees matrix semigroup $\mathcal{M}^0(G_J;I_J,\Lambda_J;P_J)$ or $\mathcal{M}(G_J;I_J,\Lambda_J;P_J)$ for any $J \in S/\mathcal{Q}$.

The following is the main result of this paper.

Theorem 1.1. Let $S = \bigcup_{J \in S/\mathcal{J}} \mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$ be a finite locally inverse semigroup. Then the semi-group algebra R[S] is isomorphic to the direct product of $\mathcal{M}(R[G_J]; |I_J|, |\Lambda_J|; P_J)$ with $J \in S/\mathcal{J}$.

Based on Theorem 1.1 and [1, Lemma 5.17, page 162, and Lemma 5.18, page 163], the following corollary is straightforward.

Corollary 1.2. Let $S = \bigcup_{J \in S/2} \mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$ be a finite locally inverse semigroup. Then the semi-group algebra R[S] has an identity if and only if $|I_J| = |\Lambda_J|$ and P_J is invertible in the full matrix algebra $M_{|I_J|}(R[G_J])$ for all $J \in S/2$.

Reference [1, Lemma 5.18, page 163] told us that $\mathcal{M}(R[G_J], m_J, n_J; P_J)$ is isomorphic to the full matrix algebra $M_{n_J}(R[G_J])$ if $\mathcal{M}(R[G_J], m_J, n_J; P_J)$ has an identity. Now, we have the following.

Corollary 1.3. Let $S = \bigcup_{J \in S/\mathcal{D}} \mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$ be a finite locally inverse semigroup. If R[S] has an identity, then R[S] is isomorphic to the direct product of the full matrix algebras $M_{|I_J|}(R[G_J])$ with $J \in S/\mathcal{D}$.

The following corollary is a consequence of Corollary 1.3.

Corollary 1.4. Let $S = \bigcup_{J \in S/2} \mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$ be a finite locally inverse semigroup. Then the semi-group algebra R[S] is semisimple if and only if for all $J \in S/2$,

- (1) $|I_I| = |\Lambda_I|$;
- (2) P_I is invertible in the full matrix algebra $M_{|I_I|}(R[G_I])$;
- (3) $R[G_I]$ is semisimple.

2. Proof of Theorem 1.1

For our purpose, we have the Möbius inversion theorem [2].

Lemma 2.1. Let (P, \leq) be a locally finite partially ordered set (i.e., intervals are finite) in which each principal ideal has a maximum and G be an Abelian group. Suppose that $f: P \to G$ is a function and define $g: P \to G$ by $g(x) = \sum_{y \leq x} f(y)$. Then $f(x) = \sum_{y \leq x} g(y) \mu(x, y)$, where μ is a Möbius function.

Now assume that *S* is a regular semigroup and $a, b \in S$. Define

$$a \le b \iff$$
 there exist e , $f \in E(S)$ such that $a = eb = bf$. (2.1)

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Then \leq is a partial order on S. Following [3], we call \leq the *natural partial order* on S. Equivalently, $a \leq b$ if and only if for every (for some) $f \in E(R_b)$ ($f \in E(L_b)$), there exists $e \in E(R_a)$ ($e \in E(L_a)$) such that $e \leq f$ and a = eb (a = be). Moreover, Nambooripad [3, 4] proved that S is a locally inverse semigroup if and only if the natural partial order \leq is compatible with respect to the multiplication of S.

Lemma 2.2. Let S be a locally inverse semigroup and $a, b \in S$. Then for any $u \le ab$, there exist $x \le a$ and $y \le b$ such that u = xy, $x \in R_u$ and $y \in L_u$.

Proof. For any $e \in E(R_a)$, we have ea = a and eab = ab. Let z be an inverse of ab. Clearly, $abz \in E(R_{ab})$. Note that eabz = abz. It is easy to check that $abze \in E(S)$, $abze \le e$, and abzRabze. Hence abzeRab and there exists $g \in E(S)$ such that u = gab and $g \le abze (\le e)$. Thus $ga \le a$. On the other hand, since R is a left congruence and since abzeRab, we have u = gabRgabze = g; while since aRe, we have gaRge = g. These imply that uRga. Dually, we have $h \in E(S)$ such that u = abh, $bh \le b$ and uLbh. Since u = gab = abh = uh = (ga) (bh), we know that ga and ga and ga are the required elements ga and ga.

Define a multiplication \otimes on $S^0 = S \cup \{0\}$ by

$$x \otimes y = \begin{cases} xy & \text{if } x \neq 0, y \neq 0, \text{ and } y, xy \in J_x; \\ 0 & \text{otherwise,} \end{cases}$$
 (2.2)

where xy is the product of x and y in S. By the arguments of [4, page 9], (S^0, \otimes) is a semigroup. We denote by S^{\otimes} the semigroup (S^0, \otimes) . For any $J \in S/\mathcal{J}$, we denote $J^0 = J \cup \{0\}$. It is easy to check that (J^0, \otimes) is a subsemigroup of S^{\otimes} , which is isomorphic to the principal factor of S determined by J. We will denote the semigroup (J^0, \otimes) by J^{\otimes} . By the definition of \otimes , it is easy to see that in the semigroup S^{\otimes} ,

- (i) $J_x^{\otimes} \otimes J_x^{\otimes} \subseteq J_x^{\otimes}$ for all $x \in S$;
- (ii) $J_x^{\otimes} \otimes J_y^{\otimes} = 0$ for all $x, y \in S$ such that $x \notin J_y$.

Thus $R_0[S^{\otimes}]$ is the direct sum of the contracted semigroup algebras $R_0[J^{\otimes}]$ with $J \in S/\mathcal{Q}$. Note that J^{\otimes} is isomorphic to some principal factor of S. We observe that J^{\otimes} is a completely 0-simple semigroup since S is a semisimple semigroup, and thus J^{\otimes} is isomorphic to some Rees matrix semigroup $\mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$. By a result of [1], $R_0[\mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)]$ is isomorphic to $\mathcal{M}(R[G_J], |I_J|, |\Lambda_J|; P_J)$. Consequently, to verify Theorem 1.1, we need only to prove that R[S] is isomorphic to $R_0[S^{\otimes}]$.

For the convenience of description, we introduce the semigroup \overline{S} . Put $\overline{S} = \{\overline{x} \mid x \in S\} \cup \{0\}$. Define a multiplication on \overline{S} as follows:

$$\overline{x}*\overline{y} = \overline{x \otimes y},\tag{2.3}$$

where we will identify $\overline{0}$ with 0. It is easy to see that \overline{S} is isomorphic to S^{\otimes} . Hence the contracted semigroup algebra $R_0[\overline{S}]$ is isomorphic to the contracted semigroup algebra $R_0[S^{\otimes}]$. For $J \in S/\mathcal{J}$, we denote $\overline{J} = \{\overline{x} \mid x \in J\} \cup \{0\}$. It is easy to check that $(\overline{J}, *)$ is a subsemigroup of \overline{S} isomorphic to the semigroup J^{\otimes} . So, for any $J, K \in S/\mathcal{J}$, we have

$$\overline{J}*\overline{K} \begin{cases} \subseteq \overline{J} & \text{if } K = J, \\ = 0 & \text{otherwise.} \end{cases}$$
(2.4)

For Theorem 1.1, it remains to prove the following lemma.

Lemma 2.3. $R[S] \cong R_0[\overline{S}].$

Proof. We consider the mapping $\varphi: R[S] \to R_0[\overline{S}]$ given on the basis by $\varphi(s) = \sum_{t \leq s} \overline{t}$ $(s \in S)$. Clearly, φ is well defined. Of course, φ and $\overline{\bullet}$ may be regarded as the mappings of the ordered set (S, \leq) into the additive group of $R_0[\overline{S}]$. Now, by applying the Möbius inversion theorem to the mappings φ and $\overline{\bullet}$, we have

$$\overline{s} = \sum_{t \le s} \varphi(t)\mu(t,s) = \varphi\left(\sum_{t \le s} t\mu(t,s)\right),\tag{2.5}$$

where μ is the Möbius function for (S, \leq) . Hence φ is surjective.

We will prove that φ is injective. For $\alpha_0 = \sum_{x \in S} p_x^0 x \in R[S]$, we denote by $\sup(\alpha_0)$ the set $\{x \in S \mid p_x^0 \neq 0\}$ and by $M(\alpha_0)$ the set of maximal elements in the set $\sup(\alpha_0)$ with respect to the partial order \leq . In recurrence, we define $\alpha_n = \alpha_{n-1} - \sum_{x \in M(\alpha_{n-1})} p_x^{n-1} x$, where $\alpha_n = \sum_{x \in \sup(\alpha_n)} p_x^n x$. Let $\beta_n = \sum_{x \in \sup(\beta_n)} q_x^n x$ with $n = 0, 1, 2, \ldots$ If $\varphi(\alpha_n) = \varphi(\beta_n)$, then by the definition of φ , $\sum_{x \in M(\alpha_n)} p_x \overline{x} + \Gamma_{\alpha_n} = \varphi(\alpha_n) = \varphi(\beta_n) = \sum_{y \in M(\beta_n)} q_y^n \overline{y} + \Gamma_{\beta_n}$, where $\Gamma_{\alpha_n} = \sum_{x \in M(\alpha_n)} \sum_{y \in S, y < x} p_y^n \overline{y}$ and $\Gamma_{\beta_n} = \sum_{x \in M(\beta_n)} \sum_{y \in S, y < x} q_y^n \overline{y}$, and hence $\sum_{x \in M(\alpha_n)} p_x^n \overline{x} = \sum_{x \in M(\beta_n)} q_x^n \overline{x}$, thus $M(\alpha_n) = M(\beta_n)$ and $p_x^n = q_x^n$ for any $x \in M(\alpha_n)$. This can imply the following.

Fact 2.4. If $\varphi(\alpha_n) = \varphi(\beta_n)$, then $M(\alpha_n) = M(\beta_n)$ and by the definition of φ , $\varphi(\alpha_{n+1}) = \varphi(\beta_{n+1})$.

By the definition of φ , the following facts are immediate.

Fact 2.5. $\alpha_n = \beta_n$ if and only if $M(\alpha_n) = M(\beta_n)$ and $\alpha_{n+1} = \beta_{n+1}$.

Fact 2.6. If
$$\varphi(\alpha_n) = \varphi(\beta_n)$$
 and $M(\alpha_n) = \operatorname{supp}(\alpha_n)$, $M(\beta_n) = \operatorname{supp}(\beta_n)$, then $\alpha_n = \beta_n$.

Note that $|\operatorname{supp}(\alpha_0)| < \infty$ and $\operatorname{supp}(\alpha_{n+1}) \subseteq \operatorname{supp}(\alpha_n)$. We thus have a smallest integer k such that $M(\alpha_k) = \operatorname{supp}(\alpha_k)$. Clearly, $\alpha_{k+1} = 0$. This means that k is the smallest integer t such that $\alpha_{t+1} = 0$. Similarly, there exists the smallest integer l such that $\beta_{l+1} = 0$ and $M(\beta_l) = \operatorname{supp}(\beta_l)$. Now, assume $\varphi(\alpha_0) = \varphi(\beta_0)$. By using Fact 2.4 repeatedly,

$$\varphi(\alpha_1) = \varphi(\beta_1), \qquad \varphi(\alpha_2) = \varphi(\beta_2), \dots, \qquad \varphi(\alpha_{k+1}) = \varphi(\beta_{k+1}). \tag{2.6}$$

But $\varphi(\alpha_{k+1})=0$, we have $\varphi(\beta_{k+1})=0$ and by the definition of φ , $\beta_{k+1}=0$. Thus $k+1\geq l+1$ by the minimality of l, and $k\geq l$. Similarly, $l\geq k$. Therefore k=l. Since $\varphi(\alpha_k)=\varphi(\beta_k)$, by Fact 2.6, we have $\alpha_k=\beta_k$ since $M(\alpha_k)=\operatorname{supp}(\alpha_k)$ and $M(\beta_l)=\operatorname{supp}(\beta_l)$. Again by the hypothesis $\varphi(\alpha_0)=\varphi(\beta_0)$, and by Fact 2.4, $M(\alpha_0)=M(\beta_0)$; and by (2.6), $M(\alpha_1)=M(\beta_1)$, $M(\alpha_2)=M(\beta_2)$, ..., $M(\alpha_k)=M(\beta_k)$. By Fact 2.5, $M(\alpha_{k-1})=M(\beta_{k-1})$; and $\alpha_k=\beta_k$ imply $\alpha_{k-1}=\beta_{k-1}$; moreover, by using Fact 2.5 repeatedly, $\alpha_{k-2}=\beta_{k-2},\ldots,\alpha_1=\beta_1$ and $\alpha_0=\beta_0$. We have now proved that φ is injective.

Finally, for any $s, t \in S$, by (2.4), we have

$$\overline{s}*\overline{t} = \begin{cases} \overline{st} & \text{if } s, t \in J_{st}, \\ 0 & \text{otherwise,} \end{cases}$$
(2.7)

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and by Lemma 2.2,

$$\varphi(s)*\varphi(t) = \left(\sum_{x \leq s} \overline{x}\right) * \left(\sum_{y \leq t} \overline{y}\right) \\
= \sum_{x \in J_{st}, x \leq s} \sum_{y \in J_{st}, y \leq t} \overline{x} * \overline{y} \\
= \sum_{x \in J_{st}, x \leq s} \sum_{y \in J_{st}, y \leq t} \overline{x} \overline{y}.$$
(2.8)

Moreover, by Lemma 2.2, we have

$$\varphi(st) = \sum_{u \le st} \overline{u} = \sum_{x \in J_{st}, x \le s} \sum_{y \in J_{st}, y \le t} \overline{xy}$$

$$= \sum_{x \le s, x \in J_{st}} \sum_{y \le t, y \in J_{st}} \overline{x} * \overline{y} = \varphi(s) * \varphi(t).$$
(2.9)

Thus φ is a homomorphism of R[S] into $R_0[\overline{S}]$. Consequently, φ is an isomorphism of R[S] onto $R_0[\overline{S}]$.

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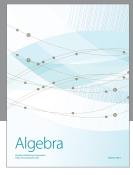
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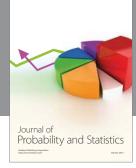
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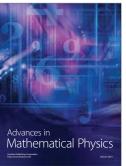






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