119. A Note on M-Space and Topologically Complete Space

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(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1969)

In the previous paper [4] we have proved that every paracompact M-space with weight $|A|$ (=the cardinality of the set A) is the perfect image of a closed subset of $D(A)$ and a subset of $N(A)$, where $D(A)$ is Cantor discontinum (=the product of two points discrete spaces D_α, $\alpha \in A$), and $N(A)$ is Baire's 0-dimensional space (=the product of countably many copies of the discrete space A), and also stated the following theorem without proof. (Throughout this paper we assume that A is an infinite set and that spaces are Hausdorff. As for terminologies and symbols in the present paper, see J. Nagata [3] and [4].)

Theorem 1. *A space X with weight $|A|$ is a paracompact M-space iff (=if and only if) it is homeomorphic to a closed subset of $S\times P(A)$, where S is a subspace of generalized Hilbert space $H(A)$, and $P(A)$ is the product of the copies I_α, $\alpha \in A$ of the unit interval* $[0, 1]$.

The purpose of the present paper is to give a proof of Theorem 1 and extend our study to paracompact, topologically complete spaces (in the sense of E. Čech), which form an important subclass of paracompact M-spaces.

Proof of Theorem 1. Since the sufficiency of the condition is obvious, we shall prove only the necessity. There is a perfect map (=mapping) from X onto a metric space Y with weight $\leqq |A|$. Let $\{f_\lambda \mid \lambda \in A\}$ be a collection of continuous functions $|X\to[0, 1]$ such that for each point x of X and each nbd (=neighborhood) M of x, there is $\lambda \in A$ for which $f_\lambda(x)=1$, $f_\lambda(X-M)=0$.
Then we define a map $h\,|\,X\to Y\times P(A)$ by

$$h(X)=\varphi(x)\times(f_\lambda(x)\mid\lambda\in A),\ x\in X.$$

It is obvious that h is one-to-one and continuous. It is also easy to show that h^{-1} is continuous. Hence h is a topological map. To show that $h(X)$ is closed in $Y\times P(A)$, let $z=y\times(q_\lambda\mid\lambda\in A)\in Y\times P(A)-h(X)$. Then $\varphi^{-1}(y)\cap[\bigcap_{\lambda\in A} f_\lambda^{-1}(q_\lambda)]=\phi$, because otherwise for every point x in the non-empty intersection $h(x)=z$ holds, and thus $z\in h(X)$. Since each $f_\lambda^{-1}(q_\lambda)$ can be expressed as $f_\lambda^{-1}(q_\lambda)=\bigcap_{n=1}^{\infty} f_\lambda^{-1}\left(\left[q_\lambda-\frac{1}{n},\ q_\lambda+\frac{1}{n}\right]\right)$ ([ ] denotes a closed interval.) and since $\varphi^{-1}(y)$ is compact, there are $\lambda_1, \cdots, \lambda_k \in A$ and a natural number n such that $\varphi^{-1}(y)\cap H=\phi$ in X if we put $H=\bigcap_{i=1}^{k} f_{\lambda_i}^{-1}\left(\left[q_{\lambda_i}-\frac{1}{n}, q_{\lambda_i}+\frac{1}{n}\right]\right)$. Now, recall that φ is a closed map, and hence $V=Y-\varphi(H)$ is an open nbd of y in Y. Thus $V\times\{(p_\lambda \mid \lambda\in A)\in P(A) q_{\lambda_i}-\frac{1}{n}<P_{\lambda_i}<q_{\lambda_i}+\frac{1}{n}, \tau=1, \cdots, k\}$ is a nbd of z in $Y\times P(A)$ which is disjoint from $h(X)$. Therefore $h(X)$ is closed in $Y\times P(A)$. By C.H. Dowkers's theorem (See J. Nagata [3]) Y is homeomorphic to a subspace S of $H(A)$, and thus the theorem is proved.

Now we can specialize the above theorem in case that X is topologically complete in the sense of E. Čech. (We are indebted to Professor K. Nagami for calling our attention to the special case.) Let us begin with a lemma.

Lemma. *Every complete metric space X with weight $|A|$ is homeomorphic to a closed of $H(A)$.*

Proof. By C.H. Dowker's theorem X is homeomorphic to a subset S of $H(A)$. Since X is topologically complete, S is a G_δ-set in $H(A)$ (See J. Nagata [3]), i.e. $S=\bigcap_{n=1}^{\infty} U_n$ for open sets U_n, $n=1, 2, \cdots$ in $H(A)$. For each natural number n let us define a continuous function f_n on U_n by

$$f_n(x)=\frac{1}{\rho(x, X-U_n)}, \quad x\in U_n, \quad \text{where } \rho \text{ denotes the metric in } H(A).$$

Then $f(x)=(x, f_1(x), f_2(x), \cdots)$, $x\in S$ is a continuous map from S into $H(A)\times E^\infty$, where E^∞ is the product of countably many copies of the 1-dimensional Euclidean space. We can easily show that f is a topological map and that $f(S)$ is closed in $H(A)\times E^\infty$. The proof is just a copy of the proof of Kuratowski's theorem in separable case (see J. Nagata [3], p. 210). On the other hand E^∞ is homeomorphic to separable Hilbert space by R.D. Anderson's theorem [1]. Thus $H(A)\times E^\infty$ is homeomorphic to $H(A)$.

Theorem 2. *A space X with weight $|A|$ is a paracompact, topologically complete space iff it is homeomorphic to a closed subset of $H(A)\times P(A)$.*

Proof. The sufficiency of the condition is obvious, because $H(A)\times P(A)$ is paracompact and topologically complete by Z. Frolik's theorem [2]. To prove the necessity, let X be a paracompact, topologically complete space with weight $|A|$. Then by Theorem 1 X is homeomorphic to a closed set X' of $S\times P(A)$. By Frolik's another theorem [2] there is a perfect map from X onto a complete metric space Y. Since S and Y are homeomorphic, by Lemma S can be regarded as a closed subset of $H(A)$. Thus X' is a closed subset of $H(A)\times P(A)$.

Now, let us turn to specialize another theorem in [3] which was stated at the beginning of the present paper, too.

Theorem 3. *Every paracompact, topologically complete space Y with weight $|A|$ is the image of a closed subset of $D(A)\times N(A)$ by a perfect map.*

Proof. All we need is a slight modification on the proof of the general theorem given in [3], which is assumed to be known by the reader and will be called the 'previous proof.' Since Y is paracompact and topologically complete, by Frolik's theorem [2] there is a perfect map φ from Y onto a complete metric space Z. Let $\mathcal{W}_i$ be a locally finite open cover of Z such that $\mathcal{W}_1 > \mathcal{W}_2^* > \mathcal{W}_2 > \mathcal{W}_3^* > \cdots$ and such that mesh $\mathcal{W}_i \to 0$. Then we may assume $\mathcal{U}_i = f^{-1}(\mathcal{W}_i) = \{f^{-1}(W) \mid W \in \mathcal{W}_i\}$, $i=1, 2, \cdots$ in the previous proof, (where the word 'locally finite' was erroneously dropped to describe the properties of $\mathcal{U}_i$.) In the previous proof we put $S=\{(\alpha_1, \alpha_2, \cdots) \in N(A) \mid \bigcap_{k=1}^{\infty} F(\alpha_1, \cdots, \alpha_k) \neq \phi\}$ to prove that Y is the perfect image of a closed subset of $D(A)\times S$. Thus it suffices to prove that S is closed in $N(A)$ in the present case. Suppose $(\beta_1, \beta_2, \cdots) \in N(A)-S$. Then $\bigcap_{k=1}^{\infty} F(\beta_1, \cdots, \beta_k)=\phi$. Hence $F(\beta_1 \cdots \cdots \beta_k)=\phi$ for some k. Because otherwise we have here a decreasing sequence $\{F(\beta_1, \cdots, \beta_k) \mid k=1, 2, \cdots\}$ of non-empty closed sets. As implied by the construction of $F(\beta_1, \cdots, \beta_k)$, $F(\beta_1, \cdots, \beta_k) \subset U_k$ holds for some $U_k \in \mathcal{U}_k$. Hence $\{\varphi(F(\beta_1, \cdots, \beta_k)) \mid k=1, 2, \cdots\}$ is a Cauchy filter basis in the complete metric space Z. Therefore $\bigcap_{k=1}^{\infty} \varphi(F(\beta_1, \cdots \cdots, \beta_k)) \neq \phi$. Let $z \in \bigcap_{k=1}^{\infty} \varphi(F(\beta_1, \cdots, \beta_k))$. Then for each k we can choose $y_k \in F(\beta_1, \cdots, \beta_k) \cap \varphi^{-1}(z)$. Since $\varphi^{-1}(z)$ is compact, $\{y_k\}$ has a cluster point y in $\varphi^{-1}(z)$. Since $y \in F(\beta_1, \cdots, \beta_k)$, $k=1, 2, \cdots$, we contradict ourselves. Therefore $F(\beta_1, \cdots, \beta_k)=\phi$ for some k. Now $N(\beta_1, \cdots, \beta_k)=\{(\alpha_1, \alpha_2, \cdots) \in N(A) \mid \alpha_1=\beta_1, \cdots, \alpha_u=\beta_k\}$ is a nbd of $(\beta_1, \beta_2, \cdots)$ which does not intersect S. Thus S is a closed set in $N(A)$, and hence $D(A)\times S$ is closed in $D(A)\times N(A)$. In other words Y is the perfect image of a closed set of $D(A)\times N(A)$.

References

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