Research Article

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A note on maximal operators related to Laplace-Bessel differential operators on variable exponent Lebesgue spaces

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Abstract: In this paper, we consider the maximal operator related to the Laplace-Bessel differential operator (B-maximal operator) on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ variable exponent Lebesgue spaces. We will give a necessary condition for the boundedness of the B-maximal operator on variable exponent Lebesgue spaces. Moreover, we will obtain that the B-maximal operator is not bounded on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ variable exponent Lebesgue spaces in the case of $p_- = 1$. We will also prove the boundedness of the fractional maximal function associated with the Laplace-Bessel differential operator (fractional B-maximal function) on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ variable exponent Lebesgue spaces.

Keywords: B-maximal operator, generalized translation operator, singular integrals, variable exponent Lebesgue spaces

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1 Introduction

This paper is associated with the maximal function generated by the Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \qquad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad 1 \le k \le n,$$

which is known as an important differential operator in harmonic analysis. The maximal operator, fractional maximal operator, and singular integrals generated by the Laplace-Bessel differential operator have been studied by many mathematicians such as Stempak, Kipriyanov, Klyuchantsev, Lyakhov, Gadjiev, Aliev, Guliyev, Ekincioglu, Serbetci, Kaya, and others [1–14].

In recent years, there have been important developments in the theory of variable exponent Lebesgue spaces. On variable exponent Lebesgue spaces, the boundedness of some operators in harmonic analysis, such as maximal operator, and singular integral operator, has an important role. On variable exponent Lebesgue spaces, the boundedness of the classical maximal operator and singular integral operators has been studied by Diening [15,16], Cruz-Uribe et al. [17], Cruz-Uribe et al. [18], and Adamowicz et al. [19].

The purpose of this study is to extend the theory of variable exponent Lebesgue spaces. The boundedness of the B-maximal operator plays an important role in obtaining the boundedness of the convolution-type singular integral operators related to the Laplace-Bessel differential operator. We will obtain that a necessary condition for the boundedness of the B-maximal operator on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ variable exponent Lebesgue spaces, using $\mathbb{R}_+$ translation according to n-variables. Moreover, we will obtain that the B-maximal

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operator is not bounded on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ variable exponent Lebesgue spaces in the case of $p_- = 1$. This article is organized as follows. In Section 2, we give some results that are useful for us. In Section 3, we obtain a necessary condition for the boundedness of the B-maximal operator on variable exponent Lebesgue spaces. Moreover, we prove that the B-maximal operator is not bounded on variable exponent Lebesgue spaces for $p_- = 1$. In Section 4, we prove the boundedness of the fractional maximal function associated with the Laplace-Bessel differential operator (fractional B-maximal function) on variable exponent Lebesgue spaces.

2 Notations and preliminaries

Let $x = (x', x'')$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, and $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$. Denote $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0, 1 \le k \le n\}$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \dots + \gamma_k$. $B_+(x, r)$ denote the open ball of radius r centered at x, i.e., $B_+(x, r) = \{y \in \mathbb{R}^n_{k,+} : |x - y| < r\}$. Given a measurable set $B_+(0, r) \subset \mathbb{R}^n_{k,+}$, then

$$|B_+(0, r)|_\gamma = \int\limits_{B_+(0,r)} (x')^\gamma \mathrm{d}x = \omega(n, k, \gamma) r^{n+|\gamma|},$$

where $\omega(n, k, \gamma) = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\left(\frac{\gamma_i}{2}\right)}$, and $\Gamma(\gamma) = \int_0^\pi e^{-x} x^{\gamma-1} \mathrm{d}x$.

The generalized translation operator is defined by

$$T^y f(x) := C_{\gamma,k} \int\limits_0^\pi \dots \int\limits_0^\pi f\left[(x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k}, x'' - y''\right] \mathrm{d}\gamma(\alpha),$$

where $C_{\gamma,k} = \pi^{-\frac{k}{2}} \Gamma\left(\frac{\gamma_i+1}{2}\right) \left[\Gamma\left(\frac{\gamma_i}{2}\right)\right]^{-1}$, $(x_i, y_i)_{\alpha_i} = (x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $1 \le k \le n$, and $\mathrm{d}\gamma(\alpha) = \prod_{i=1}^k \sin^{\gamma_i - 1} \alpha_i \mathrm{d}\alpha_i$ [8,20]. It is well known that the generalized translation operator is closely connected with Δ_B-Laplace-Bessel differential operator.

The B-convolution operator associated with the generalized translation operator is defined by

$$(f \otimes g)(x) = \int\limits_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^\gamma \mathrm{d}y.$$

We will denote by $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ the set of all measurable functions f on $\mathbb{R}^n_{k,+}$ such that the norm

$$\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma \mathrm{d}x \right)^{1/p}, \quad 1 \le p < \infty$$

is finite.

Now, we will introduce the space $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and state some fundamental properties of this space. For a measurable function $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$, let

$$p_- := \operatorname*{essinf}_{x \in \mathbb{R}^n_{k,+}} p(x), \quad p_+ := \operatorname*{esssup}_{x \in \mathbb{R}^n_{k,+}} p(x).$$

Then, we denote the set of variable exponent functions by $\mathcal{P}(\mathbb{R}^n_{k,+})$.

Definition 1. Given a function $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$, we say that $p(\cdot)$ is locally log-Hölder continuous, and denote this by $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n_{k,+})$, if there exists a constant $C_0 > 0$ such that

$$|p(x) - p(y)| \le \frac{C_0}{-\log|x - y|}, \quad |x - y| \le \frac{1}{2}, \; x, y \in \mathbb{R}^n_{k,+}.$$

We say that $p(\cdot)$ is log-Hölder continuous at infinity and denote this by $p(\cdot) \in \mathcal{P}_{\infty}^{\log}(\mathbb{R}_{k,+}^{n})$, if there exists a constant C_{∞} such that

$$|p(x) - p_{\infty}| \leq \frac{C_{\infty}}{\log(e + |x|)}, \quad x \in \mathbb{R}_{k,+}^{n},$$

where $p_{\infty} = \lim\limits_{x\to\infty} p(x) > 1$.

We define $L_{p(\cdot),\gamma}(\mathbb{R}_{k,+}^{n})$ to consist of measurable functions f such that the modular f on $\mathbb{R}_{k,+}^{n}$ satisfies

$$\varrho_{p(\cdot),\gamma} := \int\limits_{\mathbb{R}_{k,+}^{n}} |f(x)|^{p(x)} (x')^{\gamma} dx < \infty,$$

for a measurable function $p(\cdot) : \mathbb{R}_{k,+}^{n} \to [1, \infty]$. For $1 < p_{-} \leq p(x) \leq p_{+} < \infty$, the space $L_{p(\cdot),\gamma}(\mathbb{R}_{k,+}^{n})$ is a Banach space with respect to the norm

$$\|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}_{k,+}^{n})} = \inf\left\{\lambda > 0 : \varrho_{p(\cdot),\gamma}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

By $p'(x) = \frac{p(x)}{p(x)-1}$, $x \in \mathbb{R}_{k,+}^{n}$, we denote the conjugate exponent.

Proposition 1. *Let $p(\cdot) : \mathbb{R}_{k,+}^{n} \to [1, \infty]$ be Lebesgue measurable and $\mathbb{S}$ denote the set of simple functions on $\mathbb{R}_{k,+}^{n}$. If $p_{+} < \infty$, then $\mathbb{S}$ is a dense subset of $L_{p(\cdot),\gamma}(\mathbb{R}_{k,+}^{n})$.*

The space $L_{p(\cdot),\gamma}(\mathbb{R}_{k,+}^{n})$ coincides with the space

$$\left\{f(x) : \left|\int\limits_{\mathbb{R}_{k,+}^{n}} f(y)g(y)(y')^{\gamma} dy\right| < \infty \text{ for all } g \in L_{p'(\cdot),\gamma}(\mathbb{R}_{k,+}^{n})\right\}$$

up to the equivalence of the norms

$$\|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}_{k,+}^{n})} \approx \sup_{\|g\|_{L_{p'(\cdot),\gamma}} \leq 1} \left|\int\limits_{\mathbb{R}_{k,+}^{n}} f(y)g(y)(y')^{\gamma} dy\right|,$$

see [21, Proposition 2.2] and see also [22, Theorem 2.3] or [23, Theorem 3.5].

3 The B-maximal operator on $L_{p(\cdot),\gamma}(\mathbb{R}_{k,+}^{n})$

In this section, we obtain the boundedness of B-maximal operators, which play an important role in harmonic analysis, on variable exponent Lebesgue spaces. We also establish a necessary condition for the B-maximal operator to be bounded on $L_{p(\cdot),\gamma}(\mathbb{R}_{k,+}^{n})$ variable exponent Lebesgue spaces.

Given a function $f \in L_{1}^{\mathrm{loc}}(\mathbb{R}^{n})$, then the Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} |B(x, r)|^{-1} \int\limits_{B(x,r)} |f(y)| dy,$$

for all $x \in \mathbb{R}^{n}$, where the supremum is taken over all balls (or cubes) $B \in \mathbb{R}^{n}$ which contain x. In this paper, we consider the Hardy-Littlewood maximal operator related to the Laplace-Bessel differential operator (B-maximal operator) (see [7]):

$$M_{\gamma} f(x) = \sup_{r>0} |B_{+}(0, r)|_{\gamma}^{-1} \int\limits_{B_{+}(0,r)} T^{y} |f(x)| (y')^{\gamma} dy.$$

Theorem 1. [7] *Let f be a function defined on $\mathbb{R}^n_+$.*
(*a*) *If $f \in L_{1,\gamma}(\mathbb{R}^n_+)$, then for every $t > 0$*

$$|\{x \in \mathbb{R}^n_+ : M_\gamma f(x) > t\}|_\gamma \leq \frac{C}{t}\|f\|_{L_{1,\gamma}(\mathbb{R}^n_+)},$$

where C is independent of x, r, t, and f.
(*b*) *If $f \in L_{p,\gamma}(\mathbb{R}^n_+)$, $1 < p \leq \infty$, then $M_\gamma f \in L_{p,\gamma}(\mathbb{R}^n_+)$ and*

$$\|M_\gamma f\|_{L_{p,\gamma}(\mathbb{R}^n_+)} \leq C\|f\|_{L_{p,\gamma}(\mathbb{R}^n_+)},$$

where C is independent of f.

The above theorem gives that the B-maximal operator is bounded on $L_{p,\gamma}(\mathbb{R}^n_+)$ Lebesgue spaces. We will now show that the B-maximal operator is bounded on variable exponent Lebesgue spaces. In order for the boundedness of the B-maximal operator on variable Lebesgue spaces, $p(x)$ must provide the log-Hölder continuous condition.

Definition 2. [3] Let $\mathcal{B}$ be a set of all Lebesgue measurable functions $p(\cdot) : \mathbb{R}^n_{k,+} \to (1, \infty)$. Then, the B-maximal operator M_γ is bounded on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.

Let $p'(\cdot) \in \mathcal{B}$, then we can write $p(\cdot) \in \mathcal{B}'$. From [15, Theorem 8.1], we get the characterization of $\mathcal{B}$.

Theorem 2. [24] *Let $p(\cdot) : \mathbb{R}^n_{k,+} \to (1, \infty)$ be Lebesgue measurable with $1 < p_- \leq p_+ < \infty$. Then, the following conditions are equivalent*:
(*i*) $p(\cdot) \in \mathcal{B}$;
(*ii*) $p'(\cdot) \in \mathcal{B}$;
(*iii*) $p(\cdot)/q \in \mathcal{B}$ *for some* $1 < q < p_-$;
(*iv*) $(p(\cdot)/q)' \in \mathcal{B}$ *for some* $1 < q < p_-$.

Proposition 2. [24] *Let $p(\cdot) \in \mathcal{B}$. We have a constant $C > 0$ so that for any $B_+ \in \mathbb{B}$,*

$$|B_+|_\gamma \leq \|\chi_{B_+}\|_{L_{p(\cdot),\gamma}}\|\chi_{B_+}\|_{L_{p'(\cdot),\gamma}} \leq C|B_+|_\gamma.$$

Theorem 3. *Let $p \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n_{k,+})$. Then,*

$$\|M_\gamma f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})} \leq C\|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})},$$

where C is independent of f.

Proof. To prove the boundedness of the B-maximal operator on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, we will use the maximal function on a space of homogeneous type. Therefore, we will first introduce the fractional maximal function on a space of homogeneous type. Given a pseudo-metric space (X, ρ) and a positive measure μ. Then, we say that (X, ρ, μ) is a homogeneous-type space, if the measure μ satisfies the doubling condition

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)), \tag{1}$$

where C is independent of x and $r > 0$, and $B(x, r) = \{y \in X : \rho(x, y) < r\}$.

Let (X, ρ, μ) be a homogeneous-type space and the maximal function defined as follows

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)| \mathrm{d}\mu(y).$$

It is well known that the maximal operator M_μ is bounded on $L_{p(\cdot)}(X,\mu)$ [19, Corollary 1.6]. We shall use this result in the case $X=\mathbb{R}^n_{k,+}$, $\rho(x,y)=|x-y|$, $\mathrm{d}\mu(x)=(x')^\gamma \mathrm{d}x$. It is clear that this measure satisfies the doubling condition (1) (see [14]). Also in [3,7,24], it was proved that

$$M_\gamma f(x)\lesssim M_\mu f(x). \tag{2}$$

From (5) and the boundedness of the maximal operator M_μ on $L_{p(\cdot)}(X,\mu)$, we obtain the boundedness of the B-maximal operator on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, i.e.,

$$\|M_\gamma f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})}\lesssim \|M_\mu f\|_{L_{p(\cdot)}(X,\mu)}\lesssim \|f\|_{L_{p(\cdot)}(X,\mu)}\le \|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})}.$$

Thus, the proof of theorem is completed. □

It is well known that the B-maximal operator is not bounded on $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ spaces. Therefore, the B-maximal operator is not bounded on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ the variable exponent Lebesgue spaces in the case of $p_-=1$, too.

Theorem 4. *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n_{k,+})$. If $p_-=1$, then B-maximal operator is not bounded on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ variable exponent Lebesgue spaces.*

Proof. For $m\in\mathbb{N}$, take s_m such that

$$1<s_m<(n+k+|\gamma|)\left(n+k+|\gamma|-\frac{1}{m+1}\right)^{-1}.$$

For all m, since $p_-=1$, the set

$$E_m=\{x\in\mathbb{R}^n_{k,+}: p(x)<s_m\}$$

has positive measure. From Lebesgue differentiation theorem, for each χ_{E_m},

$$\lim_{r\to 0^+}\frac{|B_+(0,r)\cap E_m|_\gamma}{|B_+(0,r)|_\gamma}=1.$$

Particularly, if $0<r\le R_m$, then there exists $0<R_m<1$ such that

$$\frac{|B_+(0,r)\cap E_m|_\gamma}{|B_+(0,r)|_\gamma}>1-2^{-(n+k+|\gamma|)(m+1)}. \tag{3}$$

Let $B_m=B_+(0,R_m)$ and define

$$f_m(x)=|x|^{-n-k-|\gamma|+\frac{1}{m+1}}\chi_{B_+(0,R_m)\cap E_m}(x).$$

Now, we will prove $f_m\in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|M_\gamma f_m\|_{p(\cdot),\gamma}\ge C(m+1)\|f_m\|_{p(\cdot),\gamma}.$$

First, note that since $R_m<1$ and $-n-k-|\gamma|+\frac{1}{m+1}<0$,

$$\varrho_\gamma(f_m)=\int_{B_m\cap E_m}|x|^{\left(-n-k-|\gamma|+\frac{1}{m+1}\right)p(x)}(x')^\gamma \mathrm{d}x\le\int_{B_m\cap E_m}|x|^{\left(-n-k-|\gamma|+\frac{1}{m+1}\right)s_m}(x')^\gamma \mathrm{d}x<\infty.$$

Then, we will use the equivalent definition of the B-maximal operator and averages over balls. Let $x\in B_m\cap E_m$ and take $r=|x|\le R_m$. Then, we get

$$M_\gamma f_m(x)\ge\frac{1}{|B_+(0,r)|_\gamma}\int_{B_+(0,r)\cap E_m}T^y|x|^{-n-k-|\gamma|+\frac{1}{m+1}}(y')^\gamma \mathrm{d}y.$$

Let $\delta_m=2^{-(m+1)}$. Then, we have

$$|\{y:\delta_m r<|y|<r\}|=(1-2^{-(n+k+|\gamma|)(m+1)})|B_+(0,r)|_\gamma.$$

Therefore, since $|x|^{-n-k-|\gamma|+\frac{1}{m+1}}$ is radially decreasing and from (3), since $|B_+(0,r)\cap E_m|_\gamma \ge (1-2^{-(n+k+|\gamma|)(m+1)})$ $|B_+(0,r)|_\gamma$, we get

$$\begin{aligned} M_\gamma f_m(x) &\ge \frac{1}{|B_+(0,r)|_\gamma}\int\limits_{B_+(0,r)\cap E_m} T^y|x|^{-n-k-|\gamma|+\frac{1}{m+1}}(y')^\gamma dy \\ &\ge Cr^{-n-k-|\gamma|}\int\limits_{\{\delta_m r<|y|<r\}} T^y|x|^{-n-k-|\gamma|+\frac{1}{m+1}}(y')^\gamma dy \\ &\ge C(m+1)\left(1-\delta_m^{\frac{1}{m+1}}\right)|x|^{-n-k-|\gamma|+\frac{1}{m+1}} \ge C(m+1)f_m(x). \end{aligned}$$

If $x\notin B_m\cap E_m$, then this inequality is trivial. Therefore, $\|M_\gamma f_m\|_{p(\cdot),\gamma} \ge C(m+1)\|f_m\|_{p(\cdot),\gamma}$. Thus, when $p_-=1$, we obtain that B-maximal operator is not bounded on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ variable exponent Lebesgue spaces. □

Now, let us give an example which B-maximal operator is not bounded on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ variable exponent Lebesgue spaces.

Example 1. Let

$$p(x)=\begin{cases}2, & 0<x\le 1;\\ 4, & x>1,\end{cases}$$

and $f(x)=|x|^{-2/5}\chi_{(0,1)}(x)$. Then, since $|x|^{-4/5}\chi_{(0,1)}(x)\in L_{1,\gamma}(\mathbb{R}_+)$, $f\in L_{p(\cdot),\gamma}(\mathbb{R}_+)$. Moreover, $M_\gamma f\notin L_{p(\cdot),\gamma}(\mathbb{R}_+)$. If $x>1$, then we have

$$\begin{aligned} \sup_{r>0}\frac{1}{|B_+(0,r)|_\gamma}\int\limits_{B_+(0,r)} T^y|f(x)|y^\gamma dy &= \frac{1}{\omega(1,\gamma)r^{1+\gamma}}\int\limits_{B_+(0,r)} T^y|x|^{-2/5}y^\gamma dy \\ &= \frac{1}{\omega(1,\gamma)r^{1+\gamma}}\int\limits_{B_+(0,r)} |x-y|^{-2/5}y^\gamma dy \\ &\ge \frac{1}{\omega(1,\gamma)r^{1+\gamma}}\int\limits_{B_+(0,r)} |y|^{-2/5+\gamma} dy \\ &= \frac{1}{\omega(1,\gamma)r^{1+\gamma}}\int\limits_{B_+(0,r)} r^{-2/5+\gamma} dr \\ &= \frac{1}{\omega(1,\gamma)} r^{-2/5}\notin L_{4,\gamma}((0,1)). \end{aligned}$$

Therefore, $\varrho_\gamma(M_\gamma f)=\infty$, and thus, $M_\gamma f\notin L_{p(\cdot),\gamma}(\mathbb{R}_+)$.

4 The fractional B-maximal function on variable exponent Lebesgue spaces

The fractional maximal function associated with the Laplace-Bessel differential operator (fractional B-maximal function) is defined as follows

$$M_\gamma^\alpha f(x)=\sup_{r>0}|B_+(0,r)|_\gamma^{\frac{\alpha}{n+|\gamma|}-1}\int\limits_{B_+(0,r)} T^y|f(x)|(y')^\gamma dy, \quad 0\le\alpha<n+|\gamma|.$$

Note that we get the B-maximal function $M_\gamma^0 f=M_\gamma f$ for $\alpha=0$ (see [6]).

Theorem 5. *Let* $0 \le \alpha < n + |\gamma|$, $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, *and* $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$ *be such that* $1 < p_- \le p_+ < \frac{n+|\gamma|}{\alpha}$ *and* $p(\cdot) \in LH(\mathbb{R}^n_{k,+})$. *Define* $q(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$ *by* $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n+|\gamma|}$. *Then,*

$$\|M^\alpha_\gamma f\|_{L_{q(\cdot),\gamma}(\mathbb{R}^n_{k,+})} \le C\|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})},$$

where C *is a constant independent of* f.

Proof. We use the fractional maximal function on a space of homogeneous type. Therefore, we will firstly introduce the fractional maximal function on a space of homogeneous type. Let (X, ρ) be a pseudo-metric space, (X, ρ, μ) be a homogeneous type space, and $B(x, r)$ be as above. Then, the fractional maximal function is defined as:

$$M^\beta_\mu f(x) = \sup_{r>0} \mu(B(x, r))^{\beta-1} \int_{B(x,r)} |f(y)| d\mu(y), \quad 0 \le \beta < 1.$$

The fractional maximal operator $M^\beta_\mu f$ satisfies the strong type inequality for $1 < p_- < \frac{1}{\beta}$, $\frac{1}{p(x)} - \frac{1}{q(x)} = \beta$ (see [25,26]), i.e.,

$$\left(\int_X |M^\beta_\mu f(x)|^{q(x)} d\mu(x)\right)^{\frac{1}{q(x)}} \le C\left(\int_X |f(x)|^{p(x)} d\mu(x)\right)^{\frac{1}{p(x)}}. \tag{4}$$

To complete the proof of Theorem 5, we will use the following statements. In the case $X = \mathbb{R}^n_{k,+}$, $\rho(x, y) = |x - y|$, $\beta = \frac{\alpha}{n+|\gamma|}$, $0 \le \alpha < n + |\gamma|$, and $d\mu(x) = (x')^\gamma dx$. It is clear that this measure satisfies the doubling condition (1).

It is well known that

$$\mu(B_+(x, r)) = |B_+(x, r)|_\gamma \le Cr^{n+|\gamma|} \prod_{i=1}^{k} \max\left\{1, (x_i/r)^{\gamma_i}\right\}$$

and

$$T^y \chi_{B_+(0,r)}(x) \le C \prod_{i=1}^{k} \min\left\{1, \left(\frac{r}{x_i}\right)^{\gamma_i}\right\},$$

for $1 \le k \le n$ and $1 \le i \le k$ (see [7]).

Now we will show that

$$M^\alpha_\gamma f(x) \lesssim M^\beta_\mu f(x). \tag{5}$$

Consider the definition of the B-fractional maximal function

$$M^\alpha_\gamma f(x) = |B_+(0, r)|_\gamma^{\frac{\alpha}{n+|\gamma|}-1} \int_{B_+(0,r)} T^y |f(x)| (y')^\gamma dy.$$

Then, we can write

$$M^\alpha_\gamma f(x) = |B_+(0, r)|_\gamma^{\frac{\alpha}{n+|\gamma|}-1} \int_{\mathbb{R}^n_{k,+}} T^y |f(x)| \chi_{B_+(0,r)}(y)(y')^\gamma dy = |B_+(0, r)|_\gamma^{\frac{\alpha}{n+|\gamma|}-1} \int_{\mathbb{R}^n_{k,+}} |f(y)| T^y \chi_{B_+(0,r)}(x)(y')^\gamma dy,$$

where

$$T^y \chi_{B_+(0,r)}(x) = C_\gamma \int_0^\pi \ldots \int_0^\pi \chi_{B_+(0,r)} [(x_i, y_i)_{\alpha_i}, x'' - y''] d\gamma(\alpha),$$

and $\chi_{B_+(0,r)}$ is the characteristic function of $\chi_{B_+(0,r)} \subset \mathbb{R}^n_{k,+}$. Note that $T^y\chi_{B_+(0,r)}(x) = 0$ for every $y \in \mathbb{R}^n_{k,+}\setminus B_+(x,r)$, i.e., $T^y\chi_{B_+(0,r)}(x)$ is supported in the ball $B_+(x,r)$. Then, we have

$$M^\alpha_{\gamma,r}f(x) = |B_+(0,r)|_\gamma^{\frac{\alpha}{n+|\gamma|}-1} \int\limits_{B_+(x,r)} |f(y)|T^y\chi_{B_+(0,r)}(x)(y')^\gamma \mathrm{d}y.$$

Write

$$M^\alpha_\gamma f(x) \le M^\alpha_{\gamma,0}f(x) + M^\alpha_{\gamma,n}f(x),$$

where

$$M^\alpha_{\gamma,0}f(x) = \sup_{\substack{r\le x_i\\ 1\le i\le k}} |B_+(0,r)|_\gamma^{\frac{\alpha}{n+|\gamma|}-1} \int\limits_{B_+(x,r)} |f(y)|T^y\chi_{B_+(0,r)}(x)(y')^\gamma \mathrm{d}y,$$

$$M^\alpha_{\gamma,n}f(x) = \sup_{\substack{r> x_i\\ 1\le i\le k}} |B_+(0,r)|_\gamma^{\frac{\alpha}{n+|\gamma|}-1} \int\limits_{B_+(x,r)} |f(y)|T^y\chi_{B_+(0,r)}(x)(y')^\gamma \mathrm{d}y.$$

In the case $x_i < r$, by taking into account that $\mu(B_+(x,r)) \le (2r)^{n+|\gamma|}$, $|B_+(0,r)|_\gamma = \omega(n,k,\gamma)r^{n+|\gamma|}$, and $T^y\chi_{B_+(0,r)}(x) \le 1$, we get

$$\begin{aligned} M^\alpha_{\gamma,0}f(x) &= \sup_{\substack{r\le x_i\\ 1\le i\le k}} |B_+(0,r)|_\gamma^{\frac{\alpha}{n+|\gamma|}-1} \int\limits_{B_+(x,r)} |f(y)|T^y\chi_{B_+(0,r)}(x)(y')^\gamma \mathrm{d}y \\ &\le \omega(n,k,\gamma)^{\frac{\alpha}{n+|\gamma|}-1} 2^{n+|\gamma|-\alpha} \sup_{r>0} \mu(B(x,r))^{\beta-1} \int\limits_{B(x,r)} |f(y)|\mathrm{d}\mu(y) \\ &\le \omega(n,k,\gamma)^{\frac{\alpha}{n+|\gamma|}-1} 2^{n+|\gamma|-\alpha} M^\beta_\mu f(x). \end{aligned}$$

In the case $x_i \ge r$, since $\mu(B_+(x,r)) \le (2r)^{n+|\gamma|}\max\left\{\left(\frac{x_i}{r}\right)^{\gamma_i}\right\}$, $T^y\chi_{B_+(0,r)}(x) \le C\left(\frac{r}{x_i}\right)^{\gamma_i}$, and $|B_+(0,r)|_\gamma = \omega(n,k,\gamma)r^{n+|\gamma|}$, we get

$$\begin{aligned} M^\alpha_{\gamma,n}f(x) &= \sup_{r\le x_1} |B_+(0,r)|_\gamma^{\frac{\alpha}{n+|\gamma|}-1} \int\limits_{B_+(x,r)} |f(y)|T^y\chi_{B_+(0,r)}(x)(y')^\gamma \mathrm{d}y \\ &= C\omega(n,k,\gamma)^{\frac{\alpha}{n+|\gamma|}-1} 2^{n+|\gamma|-\alpha} M^\beta_\mu f(x). \end{aligned}$$

Therefore,

$$M^\alpha_\gamma f(x) \le M^\alpha_{\gamma,0}f(x) + M^\alpha_{\gamma,n}f(x) \le CM^\beta_\mu f(x).$$

Since the fractional maximal function $M^\beta_\mu f$ is bounded from $L_{p(\cdot)}(X,\rho,\mu)$ to $L_{q(\cdot)}(X,\rho,\mu)$ (see [25,26]) and by (5), the B-fractional maximal function $M^\alpha_\gamma f$ is bounded from $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, i.e.,

$$\|M^\alpha_\gamma f\|_{q(\cdot),\gamma} \lesssim \|M^\beta_\mu f\|_{q(\cdot),\gamma} \lesssim \|f\|_{p(\cdot),\gamma}.$$

For $p_- = 1$ and $1 - \frac{1}{q(\cdot)} = \beta$, we have

$$|\{x \in \mathbb{R}^n_{k,+} : M^\alpha_\gamma f(x) > \tau\}|_\gamma \le \mu\left\{x \in \mathbb{R}^n_{k,+} : M^\beta_\mu f(x) > \frac{\tau}{C}\right\} \le \left(\frac{C}{\tau}\int\limits_{\mathbb{R}^n_{k,+}} |f(x)|\mathrm{d}\mu(x)\right)^{q(\cdot)},$$

where $C > 0$ is a positive constant. Thus, the proof is completed. □

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References

[1] I. A. Aliev and A. D. Gadjiev, *Weighted estimates of multidimensional singular integrals generated by the generalized shift operator*, Mat. Sb. **183** (1992), no. 9, 45–66; English, translated into Russian Acad. Sci. Sb. Math. **77** (1994), no. 1, 37–55, http://mi.mathnet.ru/eng/msb1071.

[2] I. Ekincioglu, *The boundedness of high order Riesz-Bessel transformations generated by the generalized shift operator in weighted $L_{p,\omega,\gamma}$-spaces with general weights*, Acta Appl. Math. **109** (2010), no. 2, 591–598, DOI: https://doi.org/10.1007/s10440-008-9334-z.

[3] I. Ekincioglu, V. S. Guliyev, and E. Kaya, *B_n-maximal operator and B_n-singular integral operators on variable exponent Lebesgue spaces*, Math. Slovaca **70** (2020), no. 4, 893–902, DOI: https://doi.org/10.1515/ms-2017-0401.

[4] I. Ekincioglu and E. Kaya, *Bessel type Kolmogorov inequalities on weighted Lebesgue spaces*, Applicable Anal. **100** (2019), no. 8, 1634–1643, DOI: https://doi.org/10.1080/00036811.2019.1659953.

[5] I. Ekincioglu and A. Serbetci, *On weighted estimates of high-order Riesz-Bessel transformations generated by the generalized shift operator*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 1, 53–64, DOI: https://doi.org/10.1007/s10114-004-0323-5.

[6] V. S. Guliev, *Sobolev theorems for B-Riesz potentials*, Dokl. RAN **358** (1998), no. 4, 450–451.

[7] V. S. Guliev, *On maximal function and fractional integral, associated with the Bessel differential operator*, Math. Inequal. Appl. **6** (2003), no. 2, 317–330, DOI: https://doi.org/10.7153/mia-06-30.

[8] M. I. Klyuchantsev, *On singular integrals generated by the generalized shift operator I*, Sibirsk. Math. Zh. **11** (1970), no. 4, 810–821; translation in Siberian Math. J. **11** (1970), no. 4, 612–620, DOI: https://doi.org/10.1007/BF00969676.

[9] I. A. Kipriyanov and M. I. Klyuchantsev, *On singular integrals generated by the generalized shift operator. II*, Sibirsk. Mat. Zh. **11** (1970), no. 5, 1060–1083; translation in Siberian Math. J. **11** (1970), no. 5, 787–804, DOI: https://doi.org/10.1007/BF00967838.

[10] L. N. Lyakhov, *On a class of spherical functions and singular pseudodifferential operators*, Dokl. Akad. Nauk. **272** (1983), no. 4, 781–784.

[11] L. N. Lyakhov, *Multipliers of the mixed Fourier-Bessel transformation*, Proc. Steklov Inst. Math. **214** (1996), 227–242.

[12] R. P. Agarwal, S. Gala, and M. A. Ragusa, *A regularity criterion of the 3D MHD equations involving one velocity and one current density component in Lorentz space*, Z. Angew. Math. Phys. **71** (2020), art. 95, DOI: https://doi.org/10.1007/s00033-020-01318-4.

[13] M. A. Ragusa and V. B. Shakhmurov, *A Navier-Stokes-type problem with high-order elliptic operator and applications*, Mathematics **8** (2020), no. 12, 2256, DOI: https://doi.org/10.3390/math8122256.

[14] K. Stempak, *Almost everywhere summability of Laguerre series*, Studia Math. **100** (1991), no. 2, 129–147, DOI: https://doi.org/10.4064/sm-100-2-129-147.

[15] L. Diening, *Maximal function on Orlicz-Musielak spaces and generalized Lebesgue space*, Bull. Sci. Math. **129** (2005), no. 8, 657–700, DOI: https://doi.org/10.1016/j.bulsci.2003.10.003.

[16] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(x)}$*, Math. Inequal. Appl. **7** (2004), no. 2, 245–253, DOI: https://doi.org/10.7153/mia-07-27.

[17] D. Cruz-Uribe, A. Fiorenza, and C. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238.

[18] D. Cruz-Uribe, A. Fiorenza, J. Martell, and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239–264.

[19] T. Adamowicz, P. Harjulehto, and P. Hästö, *Maximal operator in variable exponent Lebesgue spaces on unbounded quasimetric measure spaces*, Math. Scand. **116** (2015), no. 1, 5–22, DOI: https://doi.org/10.7146/math.scand.a-20448.

[20] B. M. Levitan, *Bessel function expansions in series and Fourier integrals*, Uspekhi Mat. Nauk **6** (1951), no. 2(42), 102–143, http://mi.mathnet.ru/eng/umn6834.

[21] H. Kwok-Pun, *Singular integral operators, John-Nirenberg inequalities and Triebel-Lizorkin type spaces on weighted Lebesgue spaces with variable exponents*, Rev. Un. Mat. Argentina **57** (2016), no. 1, 85–101.

[22] O. Kovacik and J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), no. 4, 592–618.

[23] S. G. Samko, *Differentiation and integration of variable order and the spaces $L^{p(x)}$*, Proceedings of the International Conference on Operator Theory and Complex and Hypercomplex Analysis, 12–17 December, 1994, Mexico City, Mexico, Contemp. Math. **212** (1998), pp. 203–219, DOI: https://doi.org/10.1090/conm/212/02884.

[24] E. Kaya, *Maximal operators related to Laplace-Bessel operators on variable exponent Lebesgue spaces*, Ph.D. thesis, Kutahya Dumlupinar University, Kutahya, Turkey, 2018.
[25] C. Capone, D. Cruz-Uribe, and A. Fiorenza, *The fractional maximal operator and fractional integrals on variable L^p spaces*, Rev. Mat. Iberoam. **23** (2007), no. 3, 743–770, DOI: https://doi.org/10.4171/rmi/511.
[26] D. Cruz-Uribe and P. Shukla, *The boundedness of fractional maximal operators on variable Lebesgue spaces over spaces of homogeneous type*, arXiv preprint: http://arXiv.org/abs/arXiv:1511.09456 (2015).