A Note on Maximal Triangle-Free Graphs

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Abstract

We show that a maximal triangle-free graph on n vertices with minimum degree δ contains an independent set of $3\delta - n$ vertices which have identical neighborhoods. This yields a simple proof that if the binding number of a graph is at least $3/2$ then it has a triangle. This was conjectured originally by Woodall.

We consider finite undirected graphs on n vertices with minimum degree δ. A *maximal triangle-free* graph is one which does not contain the triangle K_3 but the addition of any edge would create a triangle. Equivalently, it is a triangle-free graph of diameter two. We say that two (nonadjacent) vertices of a graph are *similar* if they have the same neighborhoods. Similarity is obviously an equivalence relation. In this paper we show that in a maximal triangle-free graph there is a similarity class of size at least $3\delta - n$. As a consequence we obtain a short proof that if the binding number of a graph is at least $3/2$ then the graph contains a triangle.

We denote the set of neighbors of a vertex x by $N(x)$ and the degree of x by $\deg(x)$. For a set S of vertices, the neighborhood of S, denoted $N(S)$, is given by the set of all vertices which are adjacent to a vertex in S (i.e. $\bigcup_{v \in S} N(v)$). Then the binding number of the graph is the minimum of $|N(S)|/|S|$ taken over all nonempty sets S of vertices such that $N(S)$ is not the whole graph. Further, we denote the number of vertices in the similarity class of vertex x by $s(x)$.

Theorem 1 *Let G be a maximal triangle-free graph on n vertices with minimum degree δ. Then there is a vertex v such that*

$$s(v) \geq \delta + 2\deg(v) - n.$$

In particular, if G has no pair of similar vertices then $\delta \leq (n+1)/3$.

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PROOF. If every two nonadjacent vertices are similar then G is a complete multipartite graph. Indeed G is a complete bipartite graph, and the conclusion of the theorem holds for any vertex v of minimum degree.

Otherwise there exist vertices a and b that are nonadjacent and dissimilar. Let a and b be such a pair for which the overlap $|N(a) \cap N(b)|$ is maximized. Since a and b are dissimilar, there is a vertex x in $N(a) - N(b)$ say. Observe that $N(x) \cap N(b)$ is nonempty; otherwise the edge xb may be added to G without producing a triangle.

There are two cases:

1. *There are vertices y_1 and y_2 in $N(x) \cap N(b)$ such that y_1 and y_2 are dissimilar.* Since G is triangle-free, the two sets $N(x) \cup N(b)$ and $N(y_1) \cup N(y_2)$ are disjoint. Likewise, the two sets $N(a) \cap N(b)$ and $N(x) \cap N(b)$ are disjoint. By our choice of the pair $\{a, b\}$ it holds that $|N(y_1) \cap N(y_2)| \leq |N(a) \cap N(b)|$. Hence

$$\begin{aligned} n &\geq |N(x) \cup N(b)| + |N(y_1) \cup N(y_2)| \\ &= \deg(x) + \deg(b) - |N(x) \cap N(b)| + \deg(y_1) + \deg(y_2) - |N(y_1) \cap N(y_2)| \\ &\geq \deg(x) + \deg(y_1) + \deg(y_2) + (\deg(b) - |N(x) \cap N(b)| - |N(a) \cap N(b)|) \\ &\geq \deg(x) + \deg(y_1) + \deg(y_2). \end{aligned}$$

   Thus $\delta \leq n/3$, and the conclusion of the theorem holds for any vertex v of minimum degree.

2. *All the vertices in $N(x) \cap N(b)$ are similar.* Let y be a vertex in the set $Y = N(x) \cap N(b)$. Note that $s(y) = |Y|$, and that $y \notin N(a)$. We may assume that the vertices in $X = N(y) \cap N(a)$ are similar, otherwise we are back in Case 1. Note that $x \in X$.

   Since G is triangle-free, the two sets $N(x) \cup N(b)$ and $N(y)$ are disjoint. Thus

$$n \geq \deg(x) + \deg(b) - |Y| + \deg(y).$$

   Similarly, $n \geq \deg(y) + \deg(a) - |X| + \deg(x)$. Addition of these two inequalities yields:

$$s(x) + s(y) + 2n \geq 2\deg(x) + 2\deg(y) + 2\delta.$$

   Thus the statement of the theorem holds either for $v = x$ or for $v = y$. QED

If G is an r-regular maximal triangle-free graph on n vertices, then Theorem 1 shows there is a similarity class in G of size at least $3r - n$. This is sharp for a number of graphs including: the complete bipartite graph $K(b,b)$; the expansion of the 5-cycle $C_5 \otimes K_s$ which has $5s$ vertices, is $2s$-regular and has similarity classes of size s; and the complement $\overline{C_{3r-1}^{r-1}}$ of the $(r-1)$st power of the cycle on $3r-1$ vertices, which is r-regular and has no pair of similar vertices.

As a consequence of Theorem 1 we obtain another proof that the binding number at least 3/2 guarantees the existence of a triangle. This result, along with Woodall's more general conjecture [3] that binding number at least 3/2 guarantees cycles of all lengths, was first established by Shi [1, 2]. This proof is much simpler than Shi's proof of the triangle part of Woodall's conjecture.

Theorem 2 *Let G be a graph on n vertices. If for every set S of vertices it holds that $|N(S)| \geq \min(3|S|/2, n)$, then G has a triangle.*

PROOF. Let G be triangle-free. We must find an S with $|N(S)| < \min(3|S|/2, n)$. Clearly we may assume that G is maximal triangle-free.

We claim that there is a vertex v for which $s(v) \geq 2\deg(v) - 2n/3$. If $\delta < n/3$ this is obvious; if $\delta \geq n/3$ then use Theorem 1. So let S denote the set of vertices not in $N(v)$. Then $|N(S)| \leq n - s(v)$. Further, $|S| = n - \deg v \geq 2n/3 - s(v)/2$. Thus $|N(S)| \leq n - s(v) < n - 3s(v)/4 \leq 3|S|/2$. QED

References

[1] R. Shi, The binding number of a graph and its triangle, *Acta Math. Appl. Sinica (English Series)* **2** (1985), 79–86.

[2] R. Shi, The binding number of a graph and its pancyclism, *Acta Math. Appl. Sinica (English Series)* **3** (1987), 257–269.

[3] D. Woodall, The binding number of a graph and its Anderson number, *J. Combin. Theory B* **15** (1973), 225–255.