Czechoslovak Mathematical Journal, 67 (142) (2017), 329–337

A NOTE ON MODEL STRUCTURES ON ARBITRARY FROBENIUS CATEGORIES

Zhi-Wei Li, Xuzhou

Received October 26, 2015. First published March 1, 2017.

Abstract. We show that there is a model structure in the sense of Quillen on an arbitrary Frobenius category $\mathcal{F}$ such that the homotopy category of this model structure is equivalent to the stable category $\underline{\mathcal{F}}$ as triangulated categories. This seems to be well-accepted by experts but we were unable to find a complete proof for it in the literature. When $\mathcal{F}$ is a weakly idempotent complete (i.e., every split monomorphism is an inflation) Frobenius category, the model structure we constructed is an exact (closed) model structure in the sense of Gillespie (2011).

Keywords: Frobenius categorie; triangulated categories; model structure

MSC 2010: 18E10, 18E30, 18E35

1. Introduction

It is well known that stable categories of Frobenius categories, see [8], as well as certain homotopy categories of Quillen model structures, see [14], are two important methods for constructing triangulated categories. This note is aimed at making it clear that the former method can always be recovered from the latter.

Recall that an *exact category* in the sense of Quillen in [15] is a pair $(\mathcal{F}, \mathcal{E})$ in which $\mathcal{F}$ is a full, extension-closed additive subcategory of an abelian category $\mathcal{A}$, and $\mathcal{E}$ is a class of all short exact sequences in $\mathcal{A}$ with terms in $\mathcal{F}$. There is an axiomatic description of an exact category in [12], Appendix A. Following in [12], in a short exact sequence $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ in $\mathcal{E}$, the morphism f is called an *inflation*, g is called a *deflation* and the short exact sequence itself is called a *conflation*. We will use $\operatorname{Inf}(\mathcal{F})$ and $\operatorname{Def}(\mathcal{F})$ to denote the class of inflations and deflations of $\mathcal{F}$, respectively.

This project was partly supported by National Natural Science Foundation of China (Nos. 11671174 and 11571329).

DOI: 10.21136/CMJ.2017.0582-15

An exact category $(\mathcal{F}, \mathcal{E})$ is called a *Frobenius category* if $\mathcal{F}$ has enough projective objects (relative to $\mathcal{E}$) and injective objects (relative to $\mathcal{E}$), and the projective objects coincide with the injective ones. We use $\mathcal{I}$ to denote the subcategory of injective-projective objects of $\mathcal{F}$. Note that $\mathcal{I}$ is closed under direct summands by [3], Corollary 11.7. Recall that given two morphisms $f, g\colon X \to Y$ in $\mathcal{F}$, f is said to be *stably equivalent* to g, written $f \sim g$, if $f - g$ factors through some object in $\mathcal{I}$. Denote by $\underline{\mathcal{F}}$ the stable category of $\mathcal{F}$ whose objects are objects in $\mathcal{F}$ and whose morphisms are stable equivalence classes of morphisms in $\mathcal{F}$. It has a well known triangulated structure as shown in [8], Theorem 2.6.

In a Frobenius category $\mathcal{F}$, a morphism f is called a *stable equivalence* if it is an isomorphism in the stable category $\underline{\mathcal{F}}$. We define the following three classes of morphisms in $\mathcal{F}$:

$$(*) \qquad \mathcal{C}of(\mathcal{F}) = \mathrm{Inf}(\mathcal{F}), \quad \mathcal{F}ib(\mathcal{F}) = \mathrm{Def}(\mathcal{F}), \quad \mathcal{W}e(\mathcal{F}) = \{\text{stable equivalences}\}.$$

The following result shows that $\mathcal{M}_{\mathcal{F}} := (\mathcal{C}of(\mathcal{F}), \mathcal{F}ib(\mathcal{F}), \mathcal{W}e(\mathcal{F}))$ is a classical model structure (that is, not necessarily a *closed model structure*) on $\mathcal{F}$ in the sense of [14], Section I.1, Page 1.1, Definition 1, (see Definition 2.2 for details). This is inspired by [10], Theorem 2.2.12, [11], Theorem 2.6, [6], Corollary 3.4, and [7], Proposition 4.1, for the cases when $\mathcal{F}$ is the module category of a Frobenius ring and a weakly idempotent complete (i.e. every split monomorphism is an inflation) exact category. More important, this result shows also that the associated homotopy category $\mathcal{H}o(\mathcal{M}_{\mathcal{F}})$ is equivalent to the stable category $\underline{\mathcal{F}}$ preserving the triangulated structures constructed by Quillen in [14], Section I.2, Page 2.9, Theorem 2, and Happel in [8], Theorem 2.6.

Theorem 1.1. *Let $\mathcal{F}$ be a Frobenius category. Then $\mathcal{M}_{\mathcal{F}}$ is a classical model structure on $\mathcal{F}$ and the associated homotopy category $\mathcal{H}o(\mathcal{M}_{\mathcal{F}})$ is equivalent to $\underline{\mathcal{F}}$ as triangulated categories.*

2. The proof of Theorem 1.2

2.1. The extension-lifting lemma. We need the following lemma which is a generalization of [2], Lemma VIII. 3.1, from abelian categories to exact categories. We refer the reader to [3], Definition 2.1, for an axiomatic description of an exact category which we will use freely. Recall that in an exact category $\mathcal{F}$, we can define the Yoneda Ext bifunctor $\mathrm{Ext}^1_{\mathcal{F}}(X, Y)$. It is the abelian group of equivalence classes of conflations $Y \rightarrowtail Z \twoheadrightarrow X$ in $\mathcal{F}$; see [13], Chapter XII.4, for details.

Lemma 2.1 (Extension-lifting lemma). *Let $\mathcal{F}$ be an exact category. Consider the commutative diagram of conflations*

$$\begin{array}{ccccc} & & A & \overset{i}{\rightarrowtail} & B & \overset{d}{\twoheadrightarrow} & X \\ & & {\scriptstyle\alpha}\downarrow & \overset{\lambda}{\swarrow} & \downarrow{\scriptstyle\beta} & & \\ Y & \underset{c}{\rightarrowtail} & C & \underset{p}{\twoheadrightarrow} & D. & & \end{array}$$

The following statements are equivalent:

(a) $\mathrm{Ext}^1_{\mathcal{F}}(X,Y)=0$;

(b) *there exists a lift $\lambda\colon B\to C$ such that $\lambda i=\alpha$ and $p\lambda=\beta$.*

Proof. (a) $\Rightarrow$ (b). We have the pullback diagram

$$\begin{array}{ccccc} Y & \overset{\varepsilon}{\rightarrowtail} & K & \overset{\zeta}{\twoheadrightarrow} & B \\ \| & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ Y & \overset{c}{\rightarrowtail} & C & \overset{p}{\twoheadrightarrow} & D. \end{array}$$

By [3], Proposition 1.12, the sequence $Y\overset{\varepsilon}{\rightarrowtail}K\overset{\zeta}{\twoheadrightarrow}B$ is a conflation in $\mathcal{F}$. Since $\beta i=p\alpha$, there exists a morphism $\varphi\colon A\to K$ such that $\zeta\varphi=i$, $\gamma\varphi=\alpha$. Embed φ into the abelian category $\mathcal{A}$. Let ξ be the cokernel of φ in $\mathcal{A}$. We have the following exact commutative diagram in $\mathcal{A}$:

$$\begin{array}{ccccc} & & Y & = & Y \\ & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\iota} \\ A & \overset{\varphi}{\rightarrowtail} & K & \overset{\xi}{\twoheadrightarrow} & L \\ \| & & \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\eta} \\ A & \overset{i}{\rightarrowtail} & B & \overset{d}{\twoheadrightarrow} & X. \end{array}$$

Since $\mathcal{F}$ is extension-closed and $Y\overset{\iota}{\rightarrowtail}L\overset{\eta}{\twoheadrightarrow}X$ is a short exact sequence in $\mathcal{A}$, we know that $L\in\mathcal{F}$. Then both $Y\overset{\iota}{\rightarrowtail}L\overset{\eta}{\twoheadrightarrow}X$ and $A\overset{\varphi}{\rightarrowtail}K\overset{\xi}{\twoheadrightarrow}L$ are conflations in $\mathcal{F}$. The hypothesis implies that η is split, and thus its kernel $\iota\colon Y\to L$ is also split. By the above diagram, ι admits a factorization $\xi\varepsilon$. It follows that ε is split or equivalently $\zeta\colon K\to B$ is split. Then there is a morphism $\theta\colon B\to C$ such that $p\theta=\beta$. Then $p\theta i=\beta i=p\alpha$ implies that $p(\theta i-\alpha)=0$. Hence there exists a morphism $\mu\colon A\to Y$ such that $c\mu=\alpha-\theta i$. Since $\mathrm{Ext}^1_{\mathcal{F}}(X,Y)=0$, the pushout of the extension $A \overset{i}{\rightarrowtail} B \overset{d}{\twoheadrightarrow} X$ along μ splits. Hence there exists a morphism $v\colon B \to Y$ such that $\mu = vi$. Then $cvi + \theta i = c\mu + \theta i = \alpha$. Letting $\lambda = \theta + cv$ we have $\alpha = \lambda i$ and $p\lambda = p\theta + pcv = p\theta = \beta$. Hence $\lambda\colon B \to C$ is the desired lift.

(b) $\Rightarrow$ (a). Let $Y \overset{c}{\rightarrowtail} C \overset{p}{\twoheadrightarrow} X$ be an extension in $\mathcal{F}$. Then we have the commutative diagram of conflations

$$\begin{array}{ccccc} & & C & \xrightarrow{\binom{1}{0}} & C \oplus X & \xrightarrow{(0,1)} & X \\ & & \| & & \big\downarrow{\scriptstyle (p,1)} & & \\ Y & \xrightarrow{c} & C & \xrightarrow{p} & X. & & \end{array}$$

By assumption, there exists $(\kappa, \lambda)\colon C \oplus X \to C$ such that $p\kappa = 0$ and $p\lambda = \mathrm{Id}_X$. Hence p splits, thus $\mathrm{Ext}^1_{\mathcal{F}}(X, Y) = 0$. □

2.2. The classical model structures. We recall the definition of a model structure in the sense of [14], Section I.1, Page 1.1, Definition 1, which is called a *classical model structure* here. The reason is that the modern definition of a model structure often corressponds to what Quillen called a *closed model structure* [4], [10], [9].

Definition 2.2 ([14], Section I.1, Page 1.1, Definition 1)**.** A *classical Quillen model structure* on an exact category $\mathcal{F}$ consists of three classes of morphisms of $\mathcal{F}$ called cofibrations, fibrations and weak equivalences, denoted by $\mathcal{C}of$, $\mathcal{F}ib$ and $\mathcal{W}e$, respectively. It requires that cofibrations are inflations, fibrations are deflations and the following axioms.

(M0) (Lifting axiom.) Given a commutative diagram in $\mathcal{F}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y, \end{array}$$

if either i is a cofibration and p is a trivial fibration (i.e., a fibration which is a weak equivalence), or i is a trivial cofibration (i.e., a cofibration which is a weak equivalence) and p is a fibration, then there exists a morphism $h\colon B \to X$ such that $hi = f$ and $ph = g$.

(M1) Fibrations are closed under composition and pullback. Cofibrations are closed under composition and pushout. The pullback of a trivial fibration is a weak equivalence, the pushout of a trivial cofibration is a weak equivalence.

(M2) (Factorization axiom.) Any morphism f in $\mathcal{F}$ can be factored in two ways: (i) $f = pi$, where i is a cofibration and p is a trivial fibration, and (ii) $f = pi$, where i is a trivial cofibration and p is a fibration.

(M3) (Two out of three axiom.) If f, g are composable morphisms in $\mathcal{F}$ and if two of the three morphisms f, g and gf are weak equivalences, so is the third.

Note that by [14], Section I.5, Page 5.5, Proposition 2, a classical Quillen model structure is closed if and only if the classes of fibrations, cofibrations, and weak equivalences are each closed under retracts.

Let $\mathcal{M} = (\mathcal{C}of, \mathcal{F}ib, \mathcal{W}e)$ be a classical model structure on an exact category $\mathcal{F}$. As described in the beginning of [6], Section 4, we can construct Quillen's homotopy category of $\mathcal{M}$ without the full assumption that all limits and colimits of $\mathcal{F}$ are finite. Recall that an object $A \in \mathcal{F}$ is called *cofibrant* if $0 \to A \in \mathcal{C}of$, it is called *fibrant* if $A \to 0 \in \mathcal{F}ib$, and it is called *trivial* if $0 \to A \in \mathcal{W}e$. Each object of $\mathcal{F}$ which is both cofibrant and fibrant is called *bifibrant*; we use $\mathcal{M}_{cf}$ to denote the subcategory of $\mathcal{F}$ consisting of bifibrant objects. A *path object* for an object $A \in \mathcal{F}$ is an object A^I of $\mathcal{F}$ together with a factorization of the diagonal map $A \xrightarrow{(1_A, 1_A)} A \oplus A$: $A \xrightarrow{s} A^I \xrightarrow{(p_0, p_1)} A \oplus A$ where s is a trivial cofibration and (p_0, p_1) a fibration such that $p_0 s = p_1 s = 1_A$. Two morphisms $f, g\colon A \to B$ in $\mathcal{F}$ are called *right homotopic* if there exists a path object B^I for B and a morphism $H\colon A \to B^I$ such that $f = p_0 H$ and $g = p_1 H$. In this case, H is called a *right homotopy* from f to g. If f and g are right homotopic, we denote it by $f \overset{r}{\sim} g$. Dually, one can define the notions of a *cylinder object*, *left homotopic* $\overset{l}{\sim}$, *left homotopy*, respectively. In the subcategory of bifibrant objects, the relations $\overset{r}{\sim}$ and $\overset{l}{\sim}$ coincide and yield an equivalence relation $\overset{h}{\sim}$. The homotopy category $\mathcal{H}o(\mathcal{M})$ of $\mathcal{M}$ is the Gabriel-Zisman localization [5] of $\mathcal{F}$ with respect to the class of weak equivalences; it is equivalent to the quotient category $\mathcal{M}_{cf}/\overset{h}{\sim}$ by Quillen's homotopy category theorem [14], Section I.1, Page 1.13, Theorem 1.

If $\mathcal{F}$ is a Frobenius category, recall that we have defined the classes of cofibrations, fibrations and weak equivalences of $\mathcal{F}$ in $(*)$. The following lemma characterizes trivial cofibrations and trivial fibrations.

Lemma 2.3. *If $\mathcal{F}$ is a Frobenius category, then*

$$\mathcal{C}of(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F}) = \{i \in \operatorname{Inf}(\mathcal{F})\colon \operatorname{coker} i \in \mathcal{I}\},$$
$$\mathcal{F}ib(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F}) = \{p \in \operatorname{Def}(\mathcal{F})\colon \ker p \in \mathcal{I}\}.$$

Proof. We only prove the first statement since the proof of the other one is similar. If $i\colon X \to Y$ is a trivial cofibration, then i has a stable inverse $j\colon Y \to X$.

By definition, there is an injective object I and morphisms $t\colon\ X \to I$ and $s\colon\ I \to X$ such that $\mathrm{Id}_X - ji = st$. By [3], Proposition 2.12, the morphism $\binom{i}{t}\colon\ X \to Y \oplus I$ is an inflation, so it has a cokernel J. Consider the following commutative diagram of conflations:

$$\begin{array}{ccccc}
X & \overset{i}{\rightarrowtail} & Y & \twoheadrightarrow & \operatorname{coker} i \\
\Vert & & \uparrow{\scriptstyle (1,0)} & & \uparrow{\scriptstyle \eta} \\
X & \overset{\binom{i}{t}}{\rightarrowtail} & Y \oplus I & \overset{h}{\twoheadrightarrow} & J.
\end{array}$$

By the dual of [3], Proposition 2.12, the right square of the above diagram is a pullback. Thus $\eta\colon\ J \to \operatorname{coker} i$ has a kernel I, so it is a deflation by [3], Proposition 2.15. Since the morphisms i and $(1,0)$ are stable equivalences and any stable equivalence satisfies two out of three properties, we know that $\binom{i}{t}$ is a stable equivalence, and then h factors through an injective object. Since $(j,s)\binom{i}{t} = \mathrm{Id}_X$, we know that the second short exact sequence in the above diagram is split. Thus there is a morphism $m\colon\ J \to Y \oplus I$ such that $\mathrm{Id}_J = hm$, and then Id_J factors through an injective object, from which we can show that J is an injective object. By the conflation

$$I \rightarrowtail J \overset{\eta}{\twoheadrightarrow} \operatorname{coker} i$$

we know that $\operatorname{coker} i$ is a direct summand of J, and so it is an injective object.

Conversely, if i is an inflation with $\operatorname{coker} i \in \mathcal{I}$, then it is a trivial cofibration by construction. Thus we have

$$\mathcal{C}of(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F}) = \{i \in \mathrm{Inf}(\mathcal{F})\colon\ \operatorname{coker} i \in \mathcal{I}\}.$$

□

Lemma 2.4. *Let $\mathcal{F}$ be a Frobenius category. Let $(\mathcal{C}of(\mathcal{F}), \mathcal{F}ib(\mathcal{F}), \mathcal{W}e(\mathcal{F}))$ be defined as in $(*)$. Then the lifting axiom of Definition 2.2 holds.*

P r o o f. Consider the lifting problem

$$\begin{array}{ccc}
A & \longrightarrow & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\
B & \longrightarrow & Y
\end{array}$$

where $i \in \mathcal{C}of(\mathcal{F})$ and $p \in \mathcal{F}ib(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F})$. By Lemma 2.3, $\ker p \in \mathcal{I}$, thus $\mathrm{Ext}^1_{\mathcal{F}}(\operatorname{coker} i, \ker p) = 0$. So by Lemma 2.1, there exists the desired lift from B to X. Similarly, we can prove the other case. □

Lemma 2.5. *Let $\mathcal{F}$ be a Frobenius category. Let $(\mathcal{C}of(\mathcal{F}), \mathcal{F}ib(\mathcal{F}), \mathcal{W}e(\mathcal{F}))$ be defined as in $(*)$. Then every morphism f in $\mathcal{F}$ can be factorized as $f = pi = qj$, where $p \in \mathcal{F}ib(\mathcal{F})$, $i \in \mathcal{C}of(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F})$, $q \in \mathcal{F}ib(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F})$, $j \in \mathcal{C}of(\mathcal{F})$.*

Proof. If $f\colon X \to Y$ is an inflation, take $q = \mathrm{Id}_Y$, $j = f$. For the construction of p and i, note that since $\mathcal{F}$ has enough projective objects, there is a projective object P and a deflation $P \twoheadrightarrow \operatorname{coker} f$. Then we have the pullback diagram

$$\begin{array}{ccccc} X & \stackrel{i}{\rightarrowtail} & Y' & \twoheadrightarrow & P \\ \| & & \downarrow{\scriptstyle p} & & \downarrow \\ X & \underset{f}{\rightarrowtail} & Y & \twoheadrightarrow & \operatorname{coker} f. \end{array}$$

Since deflations are closed under taking pullbacks, we know that $p \in \mathcal{F}ib(\mathcal{F})$ and $i \in \mathcal{C}of(\mathcal{F})$. Since $P \in \mathcal{I}$, the morphism i is also a weak equivalence by Lemma 2.3. This gives the factorization of $f = pi$.

Dually, we can prove that the claim holds if $f\colon X \to Y$ is a deflation. Now suppose that $f\colon X \to Y$ is an arbitrary morphism in $\mathcal{F}$. It can be factorized as

$$X \overset{\binom{\mathrm{Id}_X}{0}}{\rightarrowtail} X \oplus Y \overset{(f, \mathrm{Id}_Y)}{\twoheadrightarrow} Y$$

where $\binom{\mathrm{Id}_X}{0}$ is an inflation by Lemma 2.7 of [3] and (f, Id_Y) is a deflation by the dual of [3], Proposition 1.12. Write $\binom{\mathrm{Id}_X}{0} = p'i$ with $p' \in \mathcal{F}ib(\mathcal{F})$ and $i \in \mathcal{C}of(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F})$. Then $p = (f, \mathrm{Id}_Y)p'$ is in $\mathcal{F}ib(\mathcal{F})$ and satisfies $pi = (f, \mathrm{Id}_Y)p'i = (f, \mathrm{Id}_Y)\binom{\mathrm{Id}_X}{0} = f$. Similarly, write $(f, \mathrm{Id}_Y) = qj'$ with $q \in \mathcal{F}ib(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F})$ and $j' \in \mathcal{C}of(\mathcal{F})$. Let $j = j'\binom{\mathrm{Id}_X}{0}$. Then $f = qj$ with $q \in \mathcal{F}ib(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F})$ and $j \in \mathcal{C}of(\mathcal{F})$. □

2.3. The proof of Theorem 1.2. We first prove that $\mathcal{M}_{\mathcal{F}}$ is a classical model structure on $\mathcal{F}$ by verifying the axioms (M0)–(M3) of Definition 2.2 one by one. By construction, all the three classes of morphisms $\mathcal{C}of(\mathcal{F}), \mathcal{F}ib(\mathcal{F})$ and $\mathcal{W}e(\mathcal{F})$ contain isomorphisms. By Lemma 2.4, we have (M0). (M2) follows from Lemma 2.5. Since stable equivalence satisfies two out of three properties, we know that (M3) holds. For the proof of (M1), note that fibrations are deflations which are stable under composition and pullback by the axiomatic description of an exact category. Similarly, we know that cofibrations are stable under composition and pushout. Suppose that we have a pullback diagram

$$\begin{array}{ccc} Z' & \longrightarrow & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \longrightarrow & Y \end{array}$$

with $p \in \mathcal{F}ib(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F})$. By Lemma 2.3, $\ker p \in \mathcal{I}$. By the dual of [3], Proposition 2.12, q is a deflation with $\ker q = \ker p \in \mathcal{I}$. Thus by Lemma 2.3 again, $q \in \mathcal{F}ib(\mathcal{F}) \cap \mathcal{W}e(\mathcal{F})$ and in particular, it is a weak equivalence. Similarly, we can prove that the pushout of a morphism which is both a cofibration and a weak equivalence is a weak equivalence. Thus $\mathcal{M}_{\mathcal{F}}$ is a model structure on $\mathcal{F}$.

With this model structure, every object in $\mathcal{F}$ is bifibrant. For each object X in $\mathcal{F}$, we can choose a conflation $\Omega(X) \overset{\iota_X}{\rightarrowtail} P(X) \overset{p_X}{\twoheadrightarrow} X$ with $P(X) \in \mathcal{I}$ since $\mathcal{F}$ is a Frobenius category. Then $X \oplus P(X)$ is a path object of X:

$$X \xrightarrow{\begin{pmatrix} 1_X \\ 0 \end{pmatrix}} X \oplus P(X) \xrightarrow{\begin{pmatrix} 1_X & p_X \\ 1_X & 0 \end{pmatrix}} X \oplus X.$$

It is straightforward to verify that two morphisms $f, g\colon X \to Y$ are homotopic if and only if they are stably equivalent. Thus $\mathcal{H}o(\mathcal{M}_{\mathcal{F}})$ is equivalent to $\underline{\mathcal{F}}$ by Quillen's homotopy category theorem [14], Section I.1, Page 1.13, Theorem 1. Now we have to prove that the triangle structure on $\underline{\mathcal{F}}$ constructed by Quillen in [14], Section I.2, Page 2.9, Theorem 2, coincides with the one constructed by Happel in [8], Theorem 2.6. In fact, for each morphism $f\colon X \to Y$, since $P(X) \in \mathcal{I}$, there is a morphism x_f such that $fp_X = p_Y x_f$, and then there exists a commutative diagram of conflations

$$\begin{array}{ccccc} \Omega(X) & \xrightarrow{\begin{pmatrix} \iota_X \\ 0 \end{pmatrix}} & X \oplus P(X) & \xrightarrow{\begin{pmatrix} 1_X & p_X \\ 1_X & 0 \end{pmatrix}} & X \oplus X \\ \downarrow{\scriptstyle \kappa_f} & & \downarrow{\scriptstyle \begin{pmatrix} f & 0 \\ 0 & x_f \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}} \\ \Omega(Y) & \xrightarrow[\begin{pmatrix} \iota_Y \\ 0 \end{pmatrix}]{} & Y \oplus P(Y) & \xrightarrow{\begin{pmatrix} 1_Y & p_Y \\ 1_Y & 0 \end{pmatrix}} & Y \oplus Y. \end{array}$$

Recall that the loop functor on $\underline{\mathcal{F}}$ defined by Quillen in [14], Section I.2, Page 2.9, Theorem 2, denoted by Ω, is defined by sending X to $\Omega(X)$ and $\underline{f}$ to $\underline{\kappa_f}$. It coincides with the one defined by Happel in [8], Theorem 2.6. This is an autoequivalence of $\underline{\mathcal{F}}$ by [8], Proposition 2.2.

Given any fibration $f\colon X \to Y$ in $\mathcal{F}$, we have a commutative diagram of conflations

$$\begin{array}{ccccc} \Omega(Y) & \xrightarrow{\iota_Y} & P(Y) & \xrightarrow{p_Y} & Y \\ \downarrow{\scriptstyle \xi_f} & & \downarrow{\scriptstyle \delta_f} & & \| \\ \ker f & \xrightarrow{\iota_f} & X & \xrightarrow{f} & Y. \end{array}$$

Since $\mathcal{F}$ is an additive category, $\Omega(Y)$ is a group object in $\underline{\mathcal{F}}$ and giving $\ker f$ a group action of $\Omega(Y)$ is equivalent to giving a morphism from $\Omega(Y)$ to $\ker f$. See also [1], Subsection 1.1. So the left triangles in $\underline{\mathcal{F}}$ constructed by Quillen are isomorphic to those of the form

$$\Omega(X) \xrightarrow{-\xi_f} \ker f \xrightarrow{\iota_f} X \xrightarrow{f} Y,$$

see [10], Theorem 6.2.1, Remark 7.1.3, for details. Then by [8], Lemma 2.7, we know that this triangle structure coincides with the one constructed by Happel. □

Remark 2.6. For a Frobenius category $\mathcal{F}$, by [6], Proposition 2.4, or [7], Proposition 4.1, the classical model structure $\mathcal{M}_{\mathcal{F}}$ constructed as in $(*)$ is closed if and only if the underlying category $\mathcal{F}$ is weakly idempotent complete. In this case, we get an *exact model structure* in the sense of [6].

Acknowledgement. The author would like to thank the referee for reading the paper carefully and for many suggestions on mathematics and English expressions.

References

[1] *H. Becker*: Models for singularity categories. Adv. Math. *254* (2014), 187–232. zbl MR doi
[2] *A. Beligiannis, I. Reiten*: Homological and homotopical aspects of torsion theories. Mem. Am. Math. Soc. *188* (2007), 207 pages. zbl MR doi
[3] *T. Bühler*: Exact categories. Expo. Math. *28* (2010), 1–69. zbl MR doi
[4] *W. G. Dwyer, J. Spalinski*: Homotopy theories and model categories. Handbook of Algebraic Topology. North-Holland, Amsterdam, 1995, pp. 73–126. zbl MR doi
[5] *P. Gabriel, M. Zisman*: Calculus of Fractions and Homotopy Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete 35, Springer, New York, 1967. zbl MR doi
[6] *J. Gillespie*: Model structures on exact categories. J. Pure Appl. Algebra *215* (2011), 2892–2902. zbl MR doi
[7] *J. Gillespie*: Exact model structures and recollements. J. Algebra *458* (2016), 265–306. zbl MR doi
[8] *D. Happel*: Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras. London Mathematical Society Lecture Note Series 119, Cambridge University Press, Cambridge, 1988. zbl MR doi
[9] *P. S. Hirschhorn*: Model Categories and Their Localizations. Mathematical Surveys and Monographs 99, American Mathematical Society, Providence, 2003. zbl MR
[10] *M. Hovey*: Model Categories. Mathematical Surveys and Monographs 63, American Mathematical Society, Providence, 1999. zbl MR
[11] *M. Hovey*: Cotorsion pairs, model category structures, and representation theory. Math. Z. *241* (2002), 553–592. zbl MR doi
[12] *B. Keller*: Chain complexes and stable categories. Manuscr. Math. *67* (1990), 379–417. zbl MR doi
[13] *S. Mac Lane*: Categories for the Working Mathematician. Graduate Texts in Mathematics 5, Springer, New York, 1998. zbl MR
[14] *D. G. Quillen*: Homotopical Algebra. Lecture Notes in Mathematics 43, Springer, Berlin, 1967. zbl MR doi
[15] *D. Quillen*: Higher algebraic K-theory. I. Algebraic K-Theory I.. Proc. Conf. Battelle Inst. 1972, Lecture Notes in Mathematics 341, Springer, Berlin, 1973, pp. 85–147. zbl MR doi

Author's address: Zhi-Wei Li, School of Mathematics and Statistics, Jiangsu Normal University, 101 Shanghai Road, Xuzhou 221116, Jiangsu, P. R. China, e-mail: `zhiweili@jsnu.edu.cn`.