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A Note on Naive Set Theory in LP

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Abstract Recently there has been much interest in naive set theory in nonstandard logics. This note continues this trend by considering a set theory with a general comprehension schema based on the paraconsistent logic **LP**. We demonstrate the nontriviality of the set theory so formulated, deduce some elementary properties of this system of sets, and also delineate some of the problems of this approach.

It has long been a desire among certain logicians that there be a generally satisfactory formalization of the naive theory of sets. Much work has gone into finding such a formalization, and this paper is another attempt to go some way in that direction. The paper is structured in three sections. First, I introduce the logic and the formalization of naive set theory that we will consider. Second, I give formal results concerning this theory—its nontriviality, its relationship to **ZF**, and the existence of empty and universal sets. Finally, I critically evaluate the theory and consider where a naive set theorist might go from here.

1 LP and naive set theory Any study of a theory must involve a choice concerning the logic in which the theory is embedded. The logic of choice for this exercise is the paraconsistent logic **LP**, which we introduce below.

We can define **LP** in various ways. It has the same semantics for connectives and quantifiers as Kleene's 3-valued logic, except that the middle value is designated. Also, it is the '$\rightarrow$' free fragment of the quasi-relevant logic **RM3**. It is also a simple revision of the classical predicate logic, in which a formula is allowed to be evaluated as both true and false. We will officially define it in this way.

Let $\mathcal{L}$ be a first order language with $\wedge$, $\neg$, and $\forall$ the primitive connectives and quantifier, $\in$ a dyadic predicate symbol, and $x, y, z, x_1, x_2 \ldots$ the variables. Then a pair $A = \langle D, I\rangle$ is said to be an **LP**-*model structure* if D is some nonempty domain of objects, and for each pair a, b of elements in D, we have $I(a \in b) \in \{\{1\},\{0\},\{0,1\}\}$. (We will define T, F, and B to be $\{1\}$, $\{0\}$ and $\{0,1\}$ respectively.)

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Let any function $S: \mathrm{Var}(\mathcal{L}) \to D$ from the variables of $\mathcal{L}$ to the domain be called an evaluation of the variables. Then given some evaluation S we can assign truth values to every formula in $\mathcal{L}$ by means of a function v_S defined inductively on the formulas as follows:

- $v_S(x_i \in x_j)) = I(S(x_i) \in S(x_j))$
- $1 \in v_S(\phi \wedge \psi)$ if and only if $1 \in v_S(\phi)$ and $1 \in v_S(\psi)$; and $0 \in v_S(\phi \wedge \psi)$ if and only if $0 \in v_S(\phi)$ or $0 \in v_S(\psi)$
- $1 \in v_S(\neg\phi)$ if and only if $0 \in v_S(\phi)$; and $0 \in v_S(\neg\phi)$ if and only if $1 \in v_S(\phi)$
- $1 \in v_S((\forall x)\phi(x))$ if and only if for each $d \in D$, $1 \in v_S(\phi(d))$; and $0 \in v_S((\forall x)\phi(x))$ if and only if for some $d \in D$, $0 \in v_S(\phi(d))$.

We tacitly take each $d \in D$ to function also as a name for a constant in the language $\mathcal{L}$, satisfying $S(d) = d$. This is primarily a labor-saving device, so that we can write $v_S(\phi(d))$ instead of $v_{S(x|d)}(\phi(x))$, and so on.

We can then define $\phi \vee \psi$ to be $\neg(\neg\phi \wedge \neg\psi)$, $\phi \supset \psi$ to be $\neg\phi \vee \psi$, $\phi \equiv \psi$ to be $(\phi \supset \psi) \wedge (\psi \supset \phi)$ and $(\exists x)\phi$ to be $\neg(\forall x)\neg\phi$. All that remains is to give a suitable definition of consequence, and this is done in the usual manner. For any set $\Sigma \cup \{\phi\}$ of formulas of $\mathcal{L}$, we say that ϕ is an **LP**-*consequence of* Σ, written $\Sigma \vDash_{\mathbf{LP}} \phi$, if and only if there is no **LP**-model structure A and no evaluation of variables S such that $1 \in v_S(\psi)$ for each $\psi \in \Sigma$ and $1 \notin v_S(\phi)$. In what follows, we will need various results concerning **LP**. The proofs of these facts are simple and are left to the reader.

- If $v_S(P) = \mathrm{B}$ then $1 \in v_S(P \equiv Q)$ for any formula Q.
- $P, P \supset Q \vDash_{\mathbf{LP}} Q \vee (P \wedge \neg P)$ and $P \supset Q, Q \supset R \vDash_{\mathbf{LP}} (P \supset R) \vee (Q \wedge \neg Q)$, but modus ponens and transitivity for '$\supset$' both fail.
- In the same way, transitivity of '$\equiv$' fails in general, but we have $P \equiv Q$, $Q \equiv R \vDash_{\mathbf{LP}} (P \equiv R) \vee (Q \wedge \neg Q)$.

For further detail on **LP**, the interested reader is referred to Priest [4], the paper in which **LP** was introduced.

Any candidate for a naive set theory has to have an axiom schema of comprehension, which states that for each predicate there is a set containing exactly the elements that satisfy that predicate, and an axiom of extensionality, which states that sets with the same elements are equal. For this formulation of naive set theory, we will translate these axioms as follows. The comprehension schema is translated as:

$$(\exists x)(\forall y)(y \in x \equiv \phi(y))$$

for each ϕ in which x is not free. And for extensionality we require that:

$$(\forall x)(\forall y)((\forall z)(z \in x \equiv z \in y) \supset x = y).$$

We then let $\mathcal{N}$ be the set of all instances of the comprehension schema along with the axiom of extensionality. $\mathcal{N}$ is then what we mean by 'Naive Set Theory'.

The observant reader will notice that we have not yet defined the meaning of '='. We set $x = y =_{df} (\forall z)(x \in z \equiv y \in z)$. This has been used in other nonstandard set theories (see, for example, Brady [1]). For each of these definitions, other choices are available. We will take up some of the issues raised by these choices in the final section. Now is the time to see what we can do with $\mathcal{N}$.

2 ***Formal results*** First, we will demonstrate that $\mathcal{N}$ is in fact an inconsistent theory. More formally, we give the following result.

Theorem 1 *In any model-structure $\langle D, I\rangle$ that models $\mathcal{N}$, there is an object $r \in D$ such that $I(r \in r) = \mathrm{B}$. (And so, $\mathcal{N} \models_{\mathbf{LP}} (\exists x)(x \in x \wedge x \notin x)$ as one would expect.)*

Proof: Let S be any assignment of the variables and v_S the corresponding evaluation. By the comprehension schema $1 \in v_S((\exists x)(\forall y)(y \in x \equiv y \notin y))$, and so there is an $r \in D$ where $1 \in v_S((\forall y)(y \in r \equiv y \notin y))$, and thus $1 \in v_S(r \in r \equiv r \notin r)$. But it is easily checked that $1 \in v_S(p \equiv \neg p)$ only when $v_S(p) = \mathrm{B}$, and so $v_S(r \in r) = I(r \in r) = \mathrm{B}$ as desired.

By its nature, it is impossible to check (in a finite time) whether or not a model-structure satisfies every instance of the comprehension schema. Instead, we provide a sufficient condition for a model to validate the schema that takes $o(n^2 2^n)$ time to check for a model of size n. To do this we need to examine more closely the model-structures in question.

A model-structure $A = \langle D, I\rangle$, where $D = \{a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots\}$ (but D need not be denumerable) and $I(a_i \in a_j) = e_{ij}$, can be illustrated in a diagram as follows:

$x \in y$	a_1	a_2	$\cdots$	a_j	$\cdots$
a_1	e_{11}	e_{12}	$\cdots$	e_{1j}	$\cdots$
a_2	e_{21}	e_{22}	$\cdots$	e_{2j}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
a_i	e_{i1}	e_{i2}	$\cdots$	e_{ij}	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

This is the said to be *incidence matrix* of the model-structure A.

The column vector $\langle d_1, d_2 \ldots\rangle$, (written here as a row to save space) where each $d_i \in \{\mathrm{T,B,F}\}$ is said to *subsume* a vector $\langle c_1, c_2 \ldots\rangle$ of the same height if and only if $c_i \sqsubseteq d_i$ for each i. A matrix is said to *cover* a vector if and only if one of its columns subsumes that vector.

Example 1

The matrix

$x \in y$	a	b
a	B	B
b	T	F

covers $\begin{pmatrix}\mathrm{B}\\ \mathrm{T}\end{pmatrix}$ and $\begin{pmatrix}\mathrm{T}\\ \mathrm{F}\end{pmatrix}$ but not $\begin{pmatrix}\mathrm{F}\\ \mathrm{B}\end{pmatrix}$.

A column vector is said to be *classical* if and only if its entries are each either T or F. These are enough definitions to enable us to state our next result.

Theorem 2 *An incidence matrix that covers every classical column (of the appropriate size) is a model of the comprehension schema.*

Proof: Let $A = \langle D, I\rangle$ be the model from such a matrix, and let S be an arbitrary assignment of the variables. (Hence it can be noted that $v_S(a \in b) = e_{ab}$, the (a, b) entry in the incidence matrix.) Let $\phi(y)$ be an arbitrary formula in which x is not free. It is enough to show that $1 \in v_S((\exists x)(\forall y)(y \in x \equiv \phi(y)))$.

Consider the column vector $\langle v_S(\phi(a))\rangle_{a\in D}$. It subsumes at least one classical column so we may select one, say $\langle d_a\rangle_{a\in D}$. As this is classical, each d_a is either T or F. By our assumption, there is at least one column of the matrix that subsumes this vector, let one such be $\langle e_{ab}\rangle_{a\in D}$. This means that the vectors $\langle v_S(\phi(y))\rangle_{a\in D}$ and $\langle e_{ab}\rangle_{a\in D}$ both 'share' a classical column and hence differ in entries when one (at least) has the value B.

We noted before that $\langle e_{ab}\rangle_{a\in D} = \langle v_S(a \in b)\rangle_{a\in D}$. So for each $a \in D$, $v_S(a \in b)$ and $v_S(\phi(a))$ differ only when one (at least) is B. This ensures that for each $a \in D$

$$1 \in v_S(a \in b \equiv \phi(a)).$$

Then we have $1 \in v_S((\forall y)(y \in b \equiv \phi(y)))$, and because x is not free in $\phi(y)$ this is enough to ensure that

$$1 \in v_S((\exists x)(\forall y)(y \in x \equiv \phi(y))),$$

as we set out to show. As there are 2^n classical columns of size n, and at most n^2 entries to check for each column, this can be done in $o(n^2 2^n)$ time.

Example 2 The matrix:

$x \in y$	a	b
a	B	B
b	T	F

gives a model of the comprehension schema. This is an immediate corollary of Theorem 2.

One question worth asking about $\mathcal{N}$ is whether or not it has any models. We will show that it does (and it has many) by examining how it compares to the classical set theory **ZF**. In the remainder of this section we will prove that each of the **ZF** axioms is a theorem of $\mathcal{N}$, apart from the axiom of foundation, but that not every classical consequence of **ZF** is a theorem of $\mathcal{N}$. First we will need some lemmas.

Lemma 3

$x \in y$	V	r
V	B	F
r	B	T

gives an **LP** *model for* $\mathcal{N}$.

Proof: Comprehension is assured by Theorem 2, and as the matrix for equality is

$x = y$	V	r
V	B	F
r	F	B

it is easy to see that extensionality is also satisfied.

Lemma 4 $\mathcal{N} \not\models_{\mathbf{LP}} (\forall x)((\exists y)(y \in x) \supset (\exists y)(y \in x \wedge \neg(\exists z)(z \in y \wedge z \in x)))$. *That is, Foundation is not a theorem of* $\mathcal{N}$.

Proof: Let the sentence $(\forall x)((\exists y)(y \in x) \supset (\exists y)(y \in x \wedge \neg(\exists z)(z \in x \wedge z \in y)))$ be called $\mathcal{F}$. Consider the model of $\mathcal{N}$ given in the lemma above, and let S be an assignment of variables in which $S(x) = r$. It is easy to verify that $v_S((\exists y)(y \in x)) =$ T and also that $v_S((\exists y)(y \in x \wedge \neg(\exists z)(z \in y \wedge z \in x))) =$ F. Thus under any assignment S' of the variables, $v_{S'}(\mathcal{F}) =$ F, which is enough to give us the desired result as we have a model in which $\mathcal{F}$ fails.

Remark It should be noted that by this theorem we have shown the nontriviality of $\mathcal{N}$, as we have some sentence that is not a provable consequence of $\mathcal{N}$. Whether or not $\neg\mathcal{F}$ is an **LP**-consequence of $\mathcal{N}$ is an open problem at the time of writing.

We now proceed to show that the axiom of infinity is an **LP**-consequence of $\mathcal{N}$. We do this by showing that $\mathcal{N}$ gives the existence of a universal set which must then satisfy any statement of the axiom of infinity.

Lemma 5 $\mathcal{N} \models_{\mathbf{LP}} (\exists x)(\forall y)(y \in x) \wedge (\exists x)(\forall y)(y \notin x)$, *that is* $\mathcal{N}$ *has both empty and universal sets.*

Proof: This is marginally harder to show than one might expect. The 'traditional' definition of an empty set as $\{x : x \neq x\}$ will not suffice here, because for some elements $x \neq x$ might receive the value B, so at some points the biconditional $x \in \varnothing \equiv x \neq x$ can be satisfied while $x \in \varnothing$ obtains the value T. A more devious approach is needed. Assume there are no empty sets. That is, for some model A and evaluation S, $v_S((\exists y)(\forall z)(z \notin y)) =$ F. Then the set of all empty sets must be empty, giving us the result.

Formally this proceeds as follows: under the above assumption, note that $v_S((\forall z)(z \notin b)) =$ F for each $b \in D$. Comprehension assures that $1 \in v_S((\exists x)(\forall y)(y \in x \equiv (\forall z)(z \notin y)))$, and hence that there is some $a \in D$ such that for each $b \in D$

$$1 \in v_S(b \in a \equiv (\forall z)(z \notin b)).$$

But for each $b \in D$, $v_S((\forall z)(z \notin b)) =$ F, so for each b (and this fixed a) $0 \in v_S(b \in a)$. But in terms of quantifiers, this just means that

$$1 \in v_S((\exists x)(\forall y)(y \notin x)),$$

which is the desired result. So we have an empty set. An analogous argument will ensure the existence of a universal set—its details are left to the interested reader. (*Hint*: If there are no universal sets, the set of all nonuniversal sets must be universal.)

This result is enough to ensure the truth of the axiom of infinity for the following reason. On one standard interpretation, the axiom goes as follows:

$$(\exists x)(\varnothing \in x \wedge (\forall y)(y \in x \supset \{y\} \cup y \in x)),$$

and without even inquiring as to what $\{y\} \cup y$ might mean in $\mathcal{N}$, we can see that this is satisfied by a universal set since the consequent of the conditional will always be true.

The other **ZF** axioms are pared down versions of the comprehension schema, and extensionality is common to both set theories. In particular, we have that Pairing, Restricted Comprehension, Union, Replacement, and Power Set are each instances of the Comprehension schema, and so are theorems of $\mathcal{N}$.

Example 3 Pairing is $(\forall y)(\forall y')(\exists x)(\forall z)(z \in x \equiv (z = y \vee z = y'))$, which is simply an instance of the comprehension schema with ϕ set to be $(z = y \vee z = y')$.

We also want to consider whether classical consequence of $\mathbf{ZF}^-$ is an **LP**-consequence of $\mathcal{N}$. If so, there would be an immense recapture of classical set theory in this nonstandard system. This is, perhaps fortunately, not the case, for the following reason. $\mathbf{ZF}^- \vDash_{\mathbf{C}} \neg(\exists x)(\forall y)(y \in x)$, that is, it is a classical consequence of $\mathbf{ZF}^-$ that there is no universal set. This is not the case with $\mathcal{N}$, as the following matrix gives a model:

$x \in y$	a	b
a	B	T
b	B	T

(Comprehension follows from Theorem 2, and Extensionality is easily checked.) In this model, $\neg(\exists x)(\forall y)(y \in x)$ is evaluated as F, and hence it cannot be a theorem of $\mathcal{N}$. This is an example of how **LP**-consequence is strictly weaker than the classical consequence relation. Another is with $\mathcal{N}$ itself; every formula is a classical consequence of $\mathcal{N}$, but **LP** is less generous in its consequences.

So, we have the last of our results resolved, and we can report the following theorem.

Theorem 6 *Each axiom of* $\mathbf{ZF}^-$ *is a theorem of* $\mathcal{N}$, *but* $\mathcal{N} \nvDash_{\mathbf{LP}} \mathfrak{T}$. *Furthermore,* $\mathcal{N} \vDash_{\mathbf{LP}} (\exists x)(\forall y)(y \in x)$, *which is not a classical consequence of* **ZF**.

As a final technical result, we will show that it is a consequence of $\mathcal{N}$ that there are sets x and y such that $x \neq y$.

Theorem 7 $\mathcal{N} \vDash_{\mathbf{LP}} (\exists x)(\exists y)(x \neq y)$.

Proof: Note that $\mathcal{N}$ contains a Russell set, that is for each model-structure $A = \langle D, I \rangle$ and each evaluation of the variables S, there is some $r \in D$ such that $1 \in v_S((\forall y)(y \in r \equiv y \notin y))$, and so, $v_S(r \in r \equiv r \notin r) = \mathrm{B}$. So, $v_S(\neg(r \in r \equiv r \in r)) = \mathrm{B}$, and this ensures that $1 \in v_S(\neg(\forall z)(r \in z \equiv r \in z))$, which means that $1 \in v_S((\exists x)\neg(\forall z)(x \in z \equiv x \in z)) = v_S((\exists x)(x \neq x))$. Thus, as the model and evaluation were arbitrary, we have $\mathcal{N} \vDash_{\mathbf{LP}} (\exists x)(x \neq x)$, which is more than enough for the intended result.

3 What this might mean The choice of **LP** as the logic in which to embed a naive set theory is not without justification. As we have noticed, it is easy to work in since models are quite easy to construct. Secondly, it is perhaps the most natural paraconsistent expansion of classical predicate logic. It leaves all things in predicate logic as they are, except to allow that sentences could be both true and false. In particular, in any consistent fragment of its domain, **LP** acts identically to the classical predicate calculus.

The results we have found are encouraging, and there is much scope for further work on $\mathcal{N}$, but this approach is not without its problems and limitations. This final section lists a few of these.

A possible objection to our formulation of $\mathcal{N}$ is that the axiom of infinity cannot mean what we wish it to mean because the theory has finite models. This is not a decisive objection, because the models we use are 'pathological', in the same sense as the paraconsistent finite models of arithmetic studied by Meyer and Mortensen (see [3]). These models are not intended to be in any sense canonical but are rather a useful construct for proving theorems. The fact that $\mathcal{N}$ has finite models makes strictly finitary nontriviality proofs possible and so should not be sneezed at. The reason the axiom of infinity still holds is that a finite model is very 'coarse'—it identifies in its domain objects that are distinguished in larger models. For example, in the one-element model, the empty set and the universal set are identified, whereas there are larger models in which they are distinguished. The reason this identification is possible is the fact that the logic is paraconsistent. Sets with incompatible properties can be identified in a model with a set that has both these properties (and hence is paradoxical). This parallels the finite models of arithmetic given by Meyer and Mortensen. For example, they use a model with two elements, 0 and 1, which take the place of each even and odd number respectively. The formula $0 = 0$ is evaluated as both true and false in this model, as 0 is overworked in the model, and not every even number is equal to every other even number. On the other hand, $0 = 1$ is simply false, as no even number is equal to any odd number. As this is a nontrivial model of relevant arithmetic, we have a finitary proof of the nontriviality of relevant arithmetic. By using similar means, we have a finitary proof of the nontriviality of naive set theory in **LP**.

One real problem with this formulation is that **LP** is weak. As we have noted, $A, A \supset B \models B$ and $A \equiv B, B \equiv C \models A \equiv C$ are among the inferences that are sacrificed. This raises the question of whether the axioms as stated actually mean what we have intended them to mean. In reply to this it could be said that they do in the case where our premises are classically valued. And as to when paradox is involved, one would not expect things to behave quite as naturally. As we have seen, the rules from classical logic that are not truth preserving in **LP** are truth preserving in the cases when the premises are classical. The approach would be somewhat vindicated if some classical subsection of the 'canonical model' of $\mathcal{N}$ could be found which was something resembling a model of classical set theory. (The appendix contains a result that shows that there are models which properly contain **ZF** models.) Then the bad behavior of $\mathcal{N}$ as a whole could be 'blamed' on the inconsistent objects such as $\{x : x \notin x\}$.

There is a logic **LPm** closely related to **LP** which is more generous than **LP** in its deductions. **LPm** counts as a model only the structures that are 'as consistent as possible'. (See Priest [5] for an account of **LPm**.) Even if naive set theory in **LP** is to be ultimately rejected, it is an important first step toward examining the theory in **LPm**.

The problems with the weakness of $\supset$ as an implication spread to the definition of equality. Recall that $x = y$ was taken to be a shorthand for $(\forall z)(x \in z \equiv y \in z)$. From $x = y$ it is classically possible to deduce $\phi(x) \equiv \phi(y)$ for any sentence $\phi(z)$, as we have some set a such that $x \in a \equiv \phi(x)$ and $y \in a \equiv \phi(y)$. As $x \in a \equiv y \in a$, by transitivity we have $\phi(x) \equiv \phi(y)$. This last step fails in **LP**, and it has a simple counterexample. The structure

$x \in y$	a	b	c
a	T	F	B
b	T	F	B
c	T	F	B

gives a model for $\mathcal{N}$, as can be easily shown, and $a = b$ comes out as true. However for $\phi(x)$ equal to $x \in x$, $\phi(a)$ is T but $\phi(b)$ is F.

The definition of equality has further strange properties, as is demonstrated by the proof of the existence of two distinct elements (Theorem 7). Whether or not this is a proof of the existence of two distinct objects is something worth thinking about, along with whether anything could be 'self-distinct'. Apart from anything else though, one thing this result can teach us is not to take sentences in **LP** at their face value, for they can mean something quite different from what they seem.

The last problem with this formulation is due to the fact that the language lacks term-forming operators. It is often said in introductory logic texts that 'this language is solely endowed with predicate symbols. Constants and functions can be defined in terms of these in the following manner . . .'. This can be done in **LP** in exactly the same way but with rather surprising results. As an example, consider the set $\{x : x \in x\}$. To express a statement concerning it in **LP** we need to state $(\exists y)(\forall x)(x \in y \equiv x \in x)$ (which is true, at least). So to say that $\{x : x \in x\} \in \{x : x \in x\}$, we need to say something like

$$(\alpha) \qquad (\exists y)((\forall x)(x \in y \equiv x \in x) \wedge y \in y).$$

So far so good, but if you want to inquire as to whether this is true, you get a surprising result. (This question arose in an effort to prove that $\mathcal{N}$ is incomplete.)

Note that for every model-structure $A = \langle D, I \rangle$ and every evaluation of the variables S, there is some element $r \in D$ such that $v_S(r \in r) = \text{B}$, namely, the object playing the part of the Russell set. Hence, $v_S(r \in a \equiv r \in r) = \text{B}$ for any element a. This means that $0 \in v_S((\forall x)(x \in a \equiv x \in x))$ for every a and so, $0 \in v_S((\exists y)(\forall x)(x \in y \equiv x \in x))$. But by comprehension, $1 \in v_S((\exists y)(\forall x)(x \in y \equiv x \in x))$, so this formula must receive B on any evaluation, and in any model. This means that (α) is always (at least) false, no matter what value the $y \in y$ part of the expression might take. (And in fact, models can be constructed in which it takes T, F and B.) So even if the set under consideration has properties that are not determined by $\mathcal{N}$, the fact that the sentence used to name it is both true and false means that the sentence describing the property must always come out as false. This is another shortcoming of this approach.

One method that overcomes some of these deficiencies is to introduce terms $\{x:\phi(x)\}$ into the language, and replace the comprehension schema with the two-way rule:

Comprehension Rule

$$\frac{a \in \{x:\phi(x)\}}{\phi(x)}.$$

This approach and the previous one coincide in the classical case but diverge in the context of **LP**, and this new approach perhaps comes closer to our intuitions. On this reading $\{x:x \in x\} \in \{x:x \in x\}$ could be true or false (or both), independently of the status of sentences like (α). The Comprehension Rule can be proved to have nontrivial models by a modification of the argument in Brady [2], but the addition of extensionality makes things more difficult. (The comprehension schema in the infinite-valued logic of Łukasiewicz is consistent—see White [6]—but the addition of extensionality makes the theory trivial.) So more work remains to be done to see what we can make of naive set theory in **LP**.

4 Appendix We prove a result that shows that there are $\mathcal{N}$ model-structures such that a subdomain of the model is a model for **ZF** set theory. This result is largely due to Priest.

Theorem 8 *Given a classical* **ZF** *model-structure* $\mathcal{M} = \langle D, I \rangle$, *there is an* $\mathcal{N}$ *model-structure* $\mathcal{M}^+ = \langle D \cup \{a\}, I^+ \rangle$ *(where* $a \notin D$*), such that* I^+ *restricted to* D *is* I.

Proof: Let $\mathcal{L}$ be the language of $\mathcal{M}$ (in other words, it has a name for each element in D) and $\mathcal{L}^+$ the language of $\mathcal{M}^+$. We define I^+ on $D \cup \{a\}$ by determining that $I^+(a \in b) = I^+(b \in a) = \mathrm{B}$ for each $b \in D \cup \{a\}$, and I^+ to be equal to I on D. As the 'a' column of the incidence matrix simply contains B's, $\mathcal{M}^+$ is a model of comprehension. Extensionality is all we need to give us the result. We will prove a stronger result than this. We introduce a function ${}^-: \mathcal{L}^+ \to \mathcal{L}$, given by $\phi^- = \phi(a \mid a^-)$, where a^- is a distinguished member of D chosen for this purpose. We will show that $v(\phi^-) \subseteq v^+(\phi)$ for any sentence ϕ of $\mathcal{L}^+$, where v and v^+ are evaluations given by I and I^+ respectively (we drop the subscripted assignments of the variables, as we are concerned only with sentences). This result ensures that extensionality holds in $\mathcal{M}^+$, since it holds in $\mathcal{M}$. We first need to prove a lemma.

Lemma 9 *For each* $d \in D$, *and each sentence* $\phi(d)$ *in* $\mathcal{L}^+$, *if* $0 \in v^+(\phi(d))$ *then* $0 \in v^+(\phi(a))$.

Proof: It follows by induction on the complexity of $\phi(d)$ when put into prenex normal form. (It is to be noted that the prenex normal form of any formula (found in the classical manner) has exactly the *same* truth value as the original formula. This is a simple induction and is left to the reader as an exercise.)

- If $\phi(d) = \phi_1 \wedge \ldots \wedge \phi_n$, where each ϕ_i is of the form $b \in c$ or $b \notin c$ for $b, c \in D^+$. Either no ϕ_i contains a d, in which case the result is immediate, or one does, in which case when replaced by a the conjunct is evaluated as B, and the result also follows.

- $\phi(d) = \phi_1 \vee \ldots \vee \phi_n$ and the result holds for each ϕ_i. If $0 \in v^+(\phi(d))$ then $0 \in v^+(\phi_i)$ for each i, and by hypothesis $0 \in v^+(\phi_i(a \mid d))$, and so in the disjunction, giving our result.
- $\phi(d) = (\exists x)(\psi(x, d))$, and the result holds for each $\psi(e, d)$ for $e \in D \cup \{a\}$. Note that $v^+((\exists x)(\psi(x, d))) = \bigvee_{e \in D \cup \{a\}} v^+(\psi(e, d))$, and the result follows just as in the previous case.
- $\phi(d) = (\forall x)(\psi(x, d))$, and the result holds for each $\psi(e, d)$ for $e \in D \cup \{a\}$. Note that $v^+((\forall x)(\psi(x, d))) = \bigwedge_{e \in D \cup \{a\}} v^+(\psi(e, d))$, and the result follows as in the first case.

(Note, I have used $\bigvee$ and $\bigwedge$ as operations on infinite sets of truth values – they are (respectively) max and min on the order F < B < T, the obvious definition of infinitary disjunction and conjunction.)

Now the proof follows relatively easily. It is also an induction but on the complexity of formulas.

- ϕ is $b \in c$ for $b, c \in D \cup \{a\}$. $v((b \in c)^-) \subseteq v^+(b \in c)$ by the definition of I^+.
- $1 \in v(\neg\phi^-)$ gives $0 \in v(\phi^-)$, which ensures that $0 \in v^+(\phi)$ by hypothesis, and so $1 \in v^+(\neg\phi)$. Similarly $0 \in v(\neg\phi^-)$ gives $0 \in v^+(\neg\phi)$.
- $1 \in v((\phi \vee \psi)^-)$ gives $1 \in v(\phi^-)$ or $1 \in v(\psi^-)$, and so $1 \in v^+(\phi)$ or $1 \in v^+(\psi)$, giving $1 \in v^+(\phi \vee \psi)$. Similarly if $0 \in v((\phi \vee \psi)^-)$ then $0 \in v(\phi^-)$ and $0 \in v(\psi^-)$, and so $0 \in v^+(\phi)$ and $0 \in v^+(\psi)$, giving $0 \in v^+(\phi \vee \psi)$.
- $1 \in v((\exists x)(\phi(x)^-)$ ensures that for some $d \in D$, $1 \in v(\phi(d)^-)$ and so, $1 \in v^+(\phi(d))$, so $1 \in v^+((\exists x)(\phi(x))$. On the other hand, if $0 \in v((\exists x)(\phi(x)^-)$, we have that for all $d \in D$, $0 \in v(\phi(d)^-)$, so $0 \in v^+(\phi(d))$ for each $d \in D$ by hypothesis, and so by our lemma, $0 \in v^+(\phi(a))$ as well. It follows that for each $d \in D \cup \{a\}$, $0 \in v^+(\phi(d))$ and thus, $0 \in v^+((\exists x)(\phi(x)))$ as desired. This completes the induction.

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