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A note on noetherian Hilbert rings

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Introduction. All rings considered here are commutative with identity. In this note, we give two examples of noetherian Hilbert rings. The most famous example of a noetherian Hilbert domain is an affine domain over a field. Such an integral domain is equicodimensional, i.e. its all maximal ideals have the same height. Noetherian Hilbert domains with maximal ideals of different height are given in [1], [5], [6], [10] and [11]. Krull's example in [6] is obtained by a localization of $K[X, Y]$, where K is a countable, algebraically closed field. Heinzer in [5] constructs a noetherian Hilbert domain with maximal ideals of preassigned height, and subsequently in [1] and [10] the same examples as Heinzer's are constructed by making use of the following proposition in [4, (10. 5. 8)]: Let A be a noetherian ring and let s be a non-nilpotent element contained in rad(A). Then A_s is a Hilbert ring.

By the way, in [6] and [11], two dimensional noetherian Hilbert domains with only a finite number of height one maximal ideals are constructed. However almost all noetherian Hilbert domains already known have the following property: Let $\mathfrak{M}$ be a maximal ideal of a noetherian Hilbert domain A. Then, if $n = ht(\mathfrak{M}) \geq 2$, A has infinitely many height n maximal ideals.

In Section 1, we show that if A is a noetherian ring containing an uncountable field and if S is a multiplicative subset of A generated by countably many elements of rad(A), then $S^{-1}A$ is a Hilbert ring. In Section 2, we construct a noetherian Hilbert domain with a preassigned number of maximal ideals of preassinged height by making use of a modification of Krull's method in [6, p. 371].

Notation. Let A be a ring. Then

$\mathrm{Max}(A) = \{\mathfrak{P} \in \mathrm{Spec}(A);\ \mathfrak{P}$ is a maximal ideal in $A\}$,

$Ht_n(A) = \{\mathfrak{P} \in \mathrm{Spec}(A);\ ht(\mathfrak{P}) = n\}$,

$\mathrm{rad}(A) = \bigcap_{\mathfrak{P} \in \mathrm{Max}(A)} \mathfrak{P}$.

Let $\mathfrak{p}$ be a prime ideal in a ring A. Then

$U(\mathfrak{p}) = \{\mathfrak{P} \in \mathrm{Spec}(A);\ \mathfrak{P} \supset \mathfrak{p}$ and $ht(\mathfrak{P}/\mathfrak{p}) = 1\}$.

$\mathbf{C}$ = the field of complex numbers.

$\mathbf{N}$ = the set of natural numbers.

1. We need some preliminary results.

LEMMA 1. *Let A be a noetherian ring. Then A is a Hilbert ring if and* *only if $U(\mathfrak{p})$ is an infinite set for any non-maximal prime ideal $\mathfrak{p}$ in A.*

PROOF. The assertion follows from Theorem 4 in [3].

LEMMA 2. *Let A be a ring containing an uncountable set E such that $a-b$ is a unit of A for all $a \neq b$ in E. Then the following statements hold.*

(a) Let I, I_n $(n=1, 2,...)$ be ideals in A. If I is finitely generated and if $I \subseteq \sum_{n=1}^{\infty} I_n$, then $I \subseteq I_n$ holds for some n.

(b) If A is noetherian, and if $\mathfrak{P} \supset \mathfrak{p}$ are prime ideals in A such that $ht(\mathfrak{P}/\mathfrak{p}) \geq 2$, then $U(\mathfrak{p}A_{\mathfrak{P}})$ is an uncountable set.

PROOF. (a) Let $x_1,..., x_r$ *be generators for I. Set* $H=\{x_1+ax_2+\cdots+a^{j-1}x_j+\cdots+a^{r-1}x_r;\ a \in E\}$. Since E is an uncountable set, there is an integer n such that I_n contains r-elements $x_1+\cdots+(a_i)^{j-1}x_j+\cdots+(a_i)^{r-1}x_r$ $(i=1, 2,..., r)$ of H, where $a_i \neq a_j$ if $i \neq j$. As is well-known, the determinant of $r \times r$-matrix $((a_i)^{j-1})$ is $\prod_{i>j}(a_i-a_j)$. This is a unit in A by our assumption. Therefore I_n contains $x_1,..., x_r$. Thus $I \subseteq I_n$.

(b) Since A is noetherian, $\mathfrak{P} = \bigcup_{\mathfrak{Q} \in W} \mathfrak{Q}$ hold, where $W=\{\mathfrak{Q} \in U(\mathfrak{p});\ \mathfrak{Q} \subset \mathfrak{P}\}$. Suppose that $U(\mathfrak{p}A_{\mathfrak{P}})$ is countable. Then by (a) $\mathfrak{P}=\mathfrak{Q}$ for some $\mathfrak{Q} \in W$. This contradicts the assumption that $ht(\mathfrak{P}/\mathfrak{p}) \geq 2$.

LEMMA 3. *Let A be a ring satisfying the following conditions either (a) or (b):*

(a) A contains an uncountable field.

(b) A is a semi local ring such that $A/\mathfrak{M}$ is uncountable for each maximal ideal $\mathfrak{M}$ in A.

Then A contains an uncountable set E such that $a-b$ is a unit in A for all $a \neq b$ in E.

PROOF. If A satisfies the condition (a), there is nothing to prove. Suppose that A satisfies the condition (b). Let $\mathfrak{M}_1,..., \mathfrak{M}_r$ be the maximal ideals in A. Let J be a set such that card $(J)=$ min $\{$card$(A/\mathfrak{M}_i);\ i=1,..., r\}$, where card $(*)$ stands for the cardinality of $*$. Let S_i be a complete set of representatives for the non-zero elements of $A/\mathfrak{M}_i$, and let $\{a_{ij};\ j \in J,\ a_{ij} \neq a_{ih}$ if $j \neq h\}$ be a subset of S_i. For each j, there exists a_j of A such that $a_j \equiv a_{ij} \pmod{\mathfrak{M}_i}$ for $i=1,..., r$. Then $E=\{a_j;\ j \in J\}$ is a desired set (cf. [8, p. 94]).

LEMMA 4. *Let A be a noetherian ring and let S be a multiplicative subset of A generated by countably many elements. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \cap S=\phi$. If $U(\mathfrak{p})$ is an uncountable set, then $U(S^{-1}\mathfrak{p})$ is also an uncountable set.*

PROOF. Considering $A/\mathfrak{p}$, we may assume that A is an integral domain and $\mathfrak{p}=0$. Let S be the multiplicative subset in A generated by $s_1, s_2,..., s_n,...$.

For each s_n, there is only a finite number of height one prime ideals in A which contain s_n. Therefore $Ht_1(S^{-1}A)$ is uncountable.

We can now prove the following:

THEOREM 5. *Let A be a noetherian ring satisfying the following conditions either (a) or (b):*

(a) A contains an uncountable field.

(b) A is a semi local ring such that $A/\mathfrak{M}$ is uncountable for each maximal ideal $\mathfrak{M}$ in A.

Then, if S is a multiplicative subset of A generated by countably many elements of rad (A), $S^{-1}A$ is a Hilbert ring.

PROOF. Let $S^{-1}\mathfrak{p}$ be an arbitrary non-maximal prime ideal in $S^{-1}A$, and let $S^{-1}\mathfrak{P}$ be a maximal ideal containing $S^{-1}\mathfrak{p}$, where $\mathfrak{p}, \mathfrak{P} \in \mathrm{Spec}(A)$. Since S is a subset of rad(A), $\mathfrak{P}$ is a non-maximal prime ideal in A. Hence $\dim(A/\mathfrak{p}) \geq 2$. Therefore by Lemma 3 and (b) of Lemma 2, $U(\mathfrak{p})$ is uncountable, so $U(S^{-1}\mathfrak{p})$ is infinite by Lemma 4. Thus $S^{-1}A$ is a Hilbert ring by Lemma 1.

THEOREM 6. *Let A be a noetherian Hilbert ring such that $U(\mathfrak{p})$ is uncountable for any non-maximal prime ideal $\mathfrak{p}$ in A. Let S be a multiplicative subset of A generated by countably many elements. Then $S^{-1}A$ is a Hilbert ring.*

PROOF. Let $S^{-1}\mathfrak{p}$ be a non-maximal prime ideal in $S^{-1}A$, where $\mathfrak{p} \in \mathrm{Spec}(A)$. Since $U(\mathfrak{p})$ is uncountable, $U(S^{-1}\mathfrak{p})$ is infinite by Lemma 4. Therefore $S^{-1}A$ is a Hilbert ring.

REMARK. Let T be the multiplicative subset of $\mathbf{C}[X]$ generated by $\{X-a;\ a \in \mathbf{C}-\mathbf{N}\}$, and let $A=(T^{-1}\mathbf{C}[X])[Y]$, where X, Y are indeterminates. Let S be the multiplicative subset in A generated by $\{X-n;\ n=2, 3,\dots\}$. Then A contains an uncountable field and Max(A) is uncountable, but $S^{-1}A$ is not a Hilbert ring.

2. We shall consider the following question: Let $\mathfrak{M}$ be a maximal ideal of a noetherian Hilbert domain A, and let $n=ht(M)$. Then, do there exist infinitely many height n maximal ideals in A?

Krull [6, p. 371] and Roberts [11] constructed two dimensional noetherian Hilbert domains with only a finite number of height one maximal ideals. Hence, if $n=1$, the above question is not true. We begin with some affirmative cases.

PROPOSITION 7. *(a) If A is a noetherian Hilbert domain, then the above question is true for $A[X]$, where X is an indeterminate.*

(b) Let $(A, \mathfrak{M})$ be a noetherian local domain, and let s be a non-zero element of $\mathfrak{M}$. Then the above question is true for A_s (A_s is a Hilbert domain by

[4, (10. 5. 8)]).

For the proof of this proposition, we need the following lemmas.

LEMMA 8. (Theorem 1 in [7]). *Let $\mathfrak{P}$ be a height n prime ideal in a noetherian ring. Then almost all the prime ideals directly above $\mathfrak{P}$ have height $n+1$.*

LEMMA 9. *Let $(A, \mathfrak{M})$ be a noetherian local domain. If there exists a height one prime ideal $\mathfrak{q}$ in A such that $ht(\mathfrak{M}/\mathfrak{q})=1$, then $\{\mathfrak{p} \in \mathrm{Spec}(A); \mathfrak{M} \supset \mathfrak{p} \supset 0$ is saturated$\}$ is an infinite set.*

PROOF. This is immediate from Proposition 1 in [7].

PROOF of Proposition 7. (a) Let $\mathfrak{N}$ be a maximal ideal in $A[X]$. By Theorem 5 in [3], $\mathfrak{M}=\mathfrak{N} \cap A$ is a maximal ideal in A. Then it is easy to see that there exist infinitely many maximal ideals in $A[X]$, containing $\mathfrak{M}A[X]$, of height $ht(\mathfrak{N})=ht(\mathfrak{M})+1$.

(b) Let $\mathfrak{p}A_s$ be a maximal ideal in A_s, where $\mathfrak{p} \in \mathrm{Spec}(A)$. We see then immediately that $\dim(A/\mathfrak{p})=1$. Let $\mathfrak{M} \supset \mathfrak{p} \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_{n-1} \supset 0$ be a saturated chain of prime ideals in A, where $n=ht(\mathfrak{p})$. By applying Lemma 9 for $A/\mathfrak{p}_1$, we see that $W_1=\{\mathfrak{q} \in \mathrm{Spec}(A); \mathfrak{M} \supset \mathfrak{q} \supset \mathfrak{p}_1, ht(\mathfrak{M}/\mathfrak{q})=ht(\mathfrak{q}/\mathfrak{p}_1)=1\}$ is an infinite set. Hence $W_2=\{\mathfrak{q} \in W_1; \mathfrak{q} \not\ni s, ht(\mathfrak{q})=n\}$ is infinite by Lemma 8. Therefore $\mathfrak{q}A_s$ is a height n maximal ideal in A_s for each $\mathfrak{q} \in W_2$.

Our main aim in this note is to construct a noetherian Hilbert domain with a preassinged number of maximal ideals of preassigned height. For the construction of this example we require several lemmas.

For the rest of this section we assume that k is an algebraically closed field contained in the field of complex numbers $\mathbf{C}$. We denote by $k^n (n \geq 2)$ the affine n-space over k, and also donote by $A=k[X_1, \ldots, X_n]$ the affine coordinate ring of k^n. For each pair of integers (r, m), where $1 \leq r \leq n-1$ and $m \geq 0$, $U_{r,m}$ is the linear subvariety of k^n defined as follows: $U_{r,0}=\{(z_1, \ldots, z_n) \in k^n; z_{r+1}=\cdots=z_n=0\}$ and $U_{r,m}=\{(z_1, \ldots, z_n) \in k^n; z_{r+1}=m^{-1}, z_{r+2}=\cdots=z_n=0\}$ if $m \neq 0$. It is clear that $U_{r,0} \supset U_{r-1,m}$.

Let V be a linear subvariety of k^n. We say that an irreducible closed subset L (in Zariski topology) of V is a hypersurface in V if $\dim(L)=\dim(V)-1$.

LEMMA 9. *Let V and V_0 be linear subvarieties of k^n given by $X_n=u$ and $X_n=0$ respectively, where $u \in k-\{0\}$. If L is a hypersurface in V and $Q_1, \ldots, Q_s$ are points of k^n-V, then there is a hypersurface H in k^n such that (i) $H \cap V=L$, (ii) $Q_1, \ldots, Q_s \notin H$ and (iii) $H \cap V_0$ does not meet the set $E=\{(z_1, \ldots, z_n) \in k^n; z_n=0$ and $|z_i| \leq 1$ for $i=1, \ldots, n-1\}$.*

PROOF. Since L is a hypersurface in V, L is defined by an irreducible polynomial f in $k[X_1,\dots, X_{n-1}]$. Let $(a_1,\dots, a_{n-1}, u)$ be a point of $V-L$. We then put $F_t(X_1,\dots, X_n)=(X_n-ut)^{\deg(f)}f((X_n-ut)^{-1}u(1-t)(X_1-a_1t)+a_1,\dots, (X_n-ut)^{-1}u(1-t)(X_{n-1}-a_{n-1}t)+a_{n-1}t)$, where $t\in k$. We first show that $S=\{t\in k;\ F_t(Q_i)=0$ for some $i\}$ is a finite set. Suppose that $F_t(b_1,\dots, b_n)$ is zero as a polynomial in t for some $(b_1,\dots, b_n)\in k^n$. Since $F_1(b_1,\dots, b_n)=(b_n-u)^{\deg(f)}\cdot f(a_1,\dots, a_{n-1})$, we must have $b_n=u$. Therefore $F_t(b_1,\dots, b_n)=(u-ut)^{\deg(f)}\cdot f(b_1,\dots, b_{n-1})$, and hence $(b_1,\dots, b_n)\in L$. This shows that $F_t(Q_i)$ is not zero as a polynomial in $t(i=1,\dots, s)$. Thus S is a finite set. On the other hand, F_t is an irreducible polynomial in A if $t\neq 1$. Therefore it defines a hypersurface H_t in k^n. Since $F_t(X_1,\dots, X_{n-1}, u)=(u-ut)^{\deg(f)}f(X_1,\dots, X_{n-1})$, we have $L=H_t\cap V$. Moreover $H_t\cap V_0$ is defined by $F_t(X_1,\dots, X_{n-1}, 0)=(-ut)^{\deg(f)}f(t^{-1}(t-1)X_1+a_1,\dots, t^{-1}(t-1)X_{n-1}+a_{n-1})$. Now choose a positive rational number ε so that $\sum_{i=1}^{n-1}|z_i-a_i|^2<\varepsilon$ implies $f(z_1,\dots, z_{n-1})\neq 0$, and also choose $t\in k-S\cup\{1\}$ such that $1<\varepsilon|t(t-1)^{-1}|^2n^{-1}$. Then $\mathrm{Max}\{|z_1|,\dots, |z_{n-1}|\}\leq 1$ implies $\sum_{i=1}^{n-1}|(t^{-1}(t-1)z_i+a_i)-a_i|^2=\sum_{i=1}^{n-1}|t^{-1}(t-1)z_i|^2<|t^{-1}(t-1)|^2n<\varepsilon$, hence $F_t(z_1,\dots, z_{n-1}, 0)\neq 0$. This shows that $H_t\cap E=\phi$. Therefore the proof is complete.

LEMMA 10. *Let V be a linear subvariety of k^n, L a hypersurface in V. Assume that $\dim(V)\geq 1$. If $Q_1,\dots, Q_s\in k^n-V$, then there is a hypersurface H in k^n such that (i) $H\cap V=L$ and (ii) $Q_1,\dots, Q_s\notin H$. Moreover assume that V is defined by $X_{r+1}=\cdots=X_n=0$ $(r\geq 1)$. Then H can be chosen so that it does not contain any linear subvarieties given by $X_q=v$, $X_{q+1}=\cdots=X_n=0$ where $n\geq q\geq r+1$ and $v\in k$.*

PROOF. We may assume that V is defined by $X_{r+1}=\cdots=X_n=0$ $(r\geq 1)$. Then L is defined by an irreducible polynomial f in $k[X_1,\dots, X_r]$. For each point $\alpha=(a_{ij})$ of $k^{r(n-r)}(1\leq i\leq r$ and $r+1\leq j\leq n)$, we put $G(\alpha; X_1,\dots, X_n)=f(X_1+\sum_{j=r+1}^{n}a_{1j}X_j,\dots, X_r+\sum_{j=r+1}^{n}a_{rj}X_j)$. Since f is also an irreducible polynomial in $k[X_1,\dots, X_n]$, so is $G(\alpha; X_1,\dots, X_n)$, and therefore it defines a hypersurface H_α in k^n. It is obvious that $H_\alpha\cap V=L$ and $Q_1,\dots, Q_s\notin H_\alpha$ for a suitable choice of $\alpha\in k^{r(n-r)}$. Finally assume that H_α contains a linear subvariety given by $X_q=v$, $X_{q+1}=\cdots=X_n=0$ $(q\geq r+1, v\in k)$. Then $G(\alpha; X_1,\dots, X_r, 0,\dots, v, 0,\dots, 0)=f(X_1+a_{1q}v,\dots, X_r+a_{rq}v)=0$, which is impossible. This completes the proof.

LEMMA 11. *Assume that L is a hypersurface in $U_{r,m}(m\neq 0)$. Let $Q_1,\dots, Q_s\in k^n-U_{r,m}$. Then there is a hypersurface H in k^n such that (i) $H\cap U_{r,m}=L$, (ii) $Q_1,\dots, Q_s\notin H$ and (iii) H does not contain any $U_{r',m'}$ (where $r'=1,\dots, n-1$ and $m'=0, 1,\dots$).*

PROOF. It is clear that $U_{r,0}$, $U_{r,m}\subset U_{r+1,0}$. We may assume that $Q_1,\dots, Q_s$

$\in U_{r+1,0}$ and $Q_{s'+1},\dots, Q_s \notin U_{r+1,0}$. By Lemma 9, there is a hypersurface H_1 in $U_{r+1,0}$ such that $H_1 \cap U_{r,m}=L$, $Q_1,\dots, Q_{s'} \notin H_1$ and $U_{r',m'} \not\subset H_1$ for $r' \leq r$ and $m' \geq 0$. By Lemma 10, there is a hypersurface H in k^n such that $H \cap U_{r+1,0}=H_1$, $Q_s,\dots, Q_s \notin H$ and $U_{r',m'} \not\subset H$ for $r'>r$ and $m' \geqq 0$. It is now obvious that H satisfies the above properties (i), (ii) and (iii).

Let Z be a subset of $\mathbf{C}^n$. We denote by Z' and Z^* the closures of Z with respect to the Zariski topology and the usual topology on $\mathbf{C}^n$ respectively.

LEMMA 12. *Let Z be an irreducible closed subvariety of k^n. Then $Z'=Z^*$ in $\mathbf{C}^n$.*

PROOF. See [12].

LEMMA 13. *Let $H_1,\dots, H_s$ be hypersurfaces in k^n, and let Z be an irreducible closed subvariety of k^n with $\dim(Z) \geq 1$. Further let r be an integer $(1 \leq r \leq n-1)$. If $Z \not\subset U_{r,m}$ and $Z \not\subset H_i$ for every $m(=0, 1,\dots)$ and $i(=1,\dots, s)$, then $Z-\bigcup_{m=0}^{\infty} U_{r,m} \cup H_1 \cup \cdots \cup H_s$ is an infinite set.*

PROOF. We may assume that $k=\mathbf{C}$. In fact, by Lemma 12, $(\bigcup_{m=0}^{\infty} U_{r,m} \cup H_1 \cup \cdots \cup H_s)^* = \bigcup_{m=0}^{\infty} U_{r,m}^* \cup H_1^* \cup \cdots \cup H_s^* = \bigcup_{m=0}^{\infty} U'_{r,m} \cup H'_1 \cup \cdots \cup H'_s$. Therefore to prove that $Z-\bigcup_{m=0}^{\infty} U_{r,m} \cup H_1 \cup \cdots \cup H_s$ is an infinite set, it is enough to show that $Z'-\bigcup_{m=0}^{\infty} U'_{r,m} \cup H'_1 \cup \cdots \cup H'_s$ is an infinite set. Thus we may assume that $k=\mathbf{C}$. If $Z \not\subset U_{r+1,0}$, then $Z \not\subset U_{r+1,0} \cup H \cup \cdots \cup H_s$. Consequently $Z-U_{r+1,0} \cup H_1 \cup \cdots \cup H_s$ is an infinite set and therefore so is $Z-\bigcup_{m=0}^{\infty} U_{r,m} \cup H_1 \cup \cdots \cup H_s$. We now consider the case $Z \subset U_{r+1,0}$. Replacing k^n by $U_{r+1,0}$, we may assume that $r+1=n$. We then use induction on $d=\operatorname{sim}(Z)$. First suppose that $d=1$. Since Z is uncountable and $\bigcup_{m=0}^{\infty}(Z \cap U_{n-1,m}) \cup (Z \cap H_1) \cup \cdots \cup (Z \cap H_s)$ is countable, $Z-\bigcup_{m=0}^{\infty} U_{r,m} \cup H_1 \cup \cdots \cup H_s$ is uncountable. Therefore the assertion has been established for the case $d=1$. Suppose next that $d>1$. We put $\boldsymbol{W}=$the set of all irreducible closed subvarieties W of Z with $\dim(W)=d-1$. $\boldsymbol{W}$ is an uncountable set. Assume that $Z \subset \bigcup_{m=0}^{\infty} U_{n-1,m} \cup H_1 \cup \cdots \cup H_s$; then every element of $\boldsymbol{W}$ is contained in at least one of $U_{n-1,m}$ or H_i by the induction hypothesis. Therefore it is an irreducible component of some of $Z \cap U_{n-1,m}$ or $Z \cap H_i$. Therefore $\boldsymbol{W}$ is a countable set. This is a contradiction. This shows that $Z \not\subset \bigcup_{m=0}^{\infty} U_{n-1,m} \cup H_1 \cup \cdots \cup H_s$. We can now choose an irreducible closed subvariety Z_1 of Z so that $\dim(Z_1)=1$ and $Z_1 \not\subset \bigcup_{m=0}^{\infty} U_{n-1,m} \cup H_1 \cup \cdots \cup H_s$. Since $Z_1-\bigcup_{m=0}^{\infty} U_{n-1,m} \cup H_1 \cup \cdots \cup H_s$ is an infinite set, so is $Z-\bigcup_{m=0}^{\infty} U_{n-1,m} \cup H_1 \cup \cdots \cup H_s$. The lemma is thereby proved.

The following is a corollary to Lemma 13.

LEMMA 14. *Let $H_1,\dots, H_s$ be hypersurfaces in k^n, and let Z an irreducible closed subvariety of k^n with $\dim(Z) \geq 1$. Further let $\boldsymbol{V}$ be a subset of $\{U_{r,m}$; r*

$=1,\ldots, n-1$ *and* $m=1, 2,\ldots\}$. *If* $Z\not\subseteq V$ *and* $Z\not\subseteq H_i$ *for every* $V\in \boldsymbol{V}$ *and* i, *then* $Z-\bigcup_{V\in\boldsymbol{V}} V\cup H_1\cup\cdots\cup H_s$ *is an infinite set.*

PROOF. Choose a positive integer r such that $Z\not\subseteq U_{r,0}$ and $Z\subseteq U_{r+1,0}$. Since $Z\cap V=\phi$ for every $V\in\boldsymbol{V}$ such that $\dim(V)\geq r+1$, and since $V\subset U_{r,0}$ for every $V\in\boldsymbol{V}$ such that $\dim(V)<r$, it is sufficient to prove that $Z-\bigcup_{m=0}^{\infty} U_{r,m}\cup H_1\cup\cdots\cup H_s$ is an infinite set, but this is obvious by Lemma 13.

We now proceed to the construction of a noetherian Hilbert domain with a preassigned number of maximal ideals of preassigned height.

Assume that k is a countable, algebraically closed field contained in $\mathbf{C}$. Let $1\leq r_1<\cdots<r_s<n$ be a sequence of positive integers, and let $m_1,\ldots, m_s$ be a sequence of positive integers or ∞. Choose a subset $\boldsymbol{V}$ of $\{U_{r,m};\ r=1,\ldots, n-1, m=1, 2, 3,\ldots\}$ so that the number of elements V of $\boldsymbol{V}$ with $\dim(V)=n-r_i$ is m_i for each $i=1,\ldots, s$. We now define $\boldsymbol{P}_1=$the set of all irreducible closed subvarieties of k^n which are hypersurfaces in some elements of $\boldsymbol{V}$, and $\boldsymbol{P}_2=$the product of $\mathbf{N}$ and the set of all irreducible closed subvarieties C of k^n such that $\dim(C)\geq 1$ and $C\not\subseteq\bigcup_{V\in\boldsymbol{V}} V$. Since k is countable, so are $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. Therefore we can put $\boldsymbol{P}_1=\{P_i;\ i\in\mathbf{N}\}$ and $\boldsymbol{P}_2=\{(\xi_i, C_i);\ i\in\mathbf{N}\}$. We shall construct, in succession, positive integers $e_1, e_2,\ldots$, points $Q_1, Q_2,\ldots$ of k^n and hypersurfaces $H_1, H_2,\ldots$ in k^n as follows. We put $e_1=1$. Let $Q_1\in C_1-\bigcup_{V\in\boldsymbol{V}} V$, and let V_1 be an element of $\boldsymbol{V}$ in which P_1 is a hypersurface. By Lemma 11, there is a hypersurface H_1 in k^n such that $V\not\subseteq H_1$ for every $V\in\boldsymbol{V}$, $H_1\cap V_1=P_1$ and $Q_1\notin H_1$. For each positive integer $i(>1)$, we choose an element V_i of $\boldsymbol{V}$ in which P_i is a hypersurface. Suppose that we can choose positive integers $e_1,\ldots, e_{t-1}$, points $Q_1,\ldots, Q_{t-1}$ of k^n and hypersurfaces $H_1,\ldots, H_{t-1}$ in k^n such that $Q_i\in C_{e_i}-\bigcup_{V\in\boldsymbol{V}} V\cup H_1\cup\cdots\cup H_{i-1}$, $Q_1,\ldots, Q_i\notin H_i$, $V\not\subseteq H_i$ for every $V\in\boldsymbol{V}$ and $H_i\cap V_i=P_i$ for $i=1,\ldots, t-1$. Since $\{i;\ C=C_i\}$ is an infinite set for every irreducible closed subvariety C fo k^n such that $\dim(C)\geq 1$ and $C\not\subseteq\bigcup_{V\in\boldsymbol{V}} V$, the set $\{i>e_{t-1};\ C_i\not\subseteq H_1\cup\cdots\cup H_{t-1}\}$ is not empty. Then we put $e_t=\mathrm{Min}\{i>e_{t-1};\ C_i\not\subseteq H_1\cup\cdots\cup H_{t-1}\}$. By Lemma 14, we can choose a point Q_t of $C_{e_t}-\bigcup_{V\in\boldsymbol{V}} V\cup H_1\cup\cdots\cup H_{t-1}$, $Q_t\neq Q_1,\ldots, Q_{t-1}$. Then by Lemma 11, there is a hypersurface H_t in k^n such that $V\not\subseteq H_t$ for every $V\in\boldsymbol{V}$, $H_t\cap V_t=P_t$ and $Q_1,\ldots, Q_t\notin H_t$. We shall now prove that the above sequence $H_1, H_2,\ldots$ of hypersurfaces in k^n has the following properties:

(a) if V is an element of $\boldsymbol{V}$, then every proper closed subvariety of V is contained in some H_t;

(b) $V\not\subseteq H_t$ for all $t\in\mathbf{N}$ and $V\in\boldsymbol{V}$;

(c) $L-\bigcup_{t=1}^{\infty} H_t$ is an infinite set for every positive dimensional irreducible closed subvariety L of k^n which is not contained in any $V\in\boldsymbol{V}$ and H_t.

In fact (a) and (b) are obvious by the construction of H_t. To prove (c), note that $\{i\in\mathbf{N};\ L=C_i\}$ is an infinite set. Suppose now that $L=C_i$ for some $i\in\mathbf{N}$.

Then there is an integer j such that $e_{j-1}<i\leq e_j$. If $i<e_j$, then $L=C_i\subset H_1\cup\cdots\cup H_{j-1}$ from the definition of e_j, which is a contradiction. Therefore $i=e_j$; hence $J=\{i\in\mathbf{N};\ L=C_{e_i}\}$ is an infinite set. Since $Q_j\in L-\bigcup_{t=1}^{\infty}H_t$ for $j\in J$, the assertion (c) is proved.

Let now f_t be a defining polynomial of H_t in $A=k[X_1,\dots,X_n]$ for each $t\in\mathbf{N}$, and let S be a multiplicative subset of A generated by $f_1, f_2,\dots$. We then put $R=S^{-1}A$.

THEOREM 15. *R is a notherian Hilbert domain such that (i) dim $(R)=n$, (ii) $\{r_1,\dots,r_s,n\}=\{ht(\mathfrak{m});\ \mathfrak{m}\in \mathrm{Max}(R)\}$ and (iii) m_i is the number of maximal ideals in R with height r_i for each $i=1,\dots,s$.*

PROOF. First, (a) and (b) imply that $\mathfrak{p}(V)R$ is a maximal ideal in R for every $V\in\boldsymbol{V}$, where $\mathfrak{p}(V)$ is the prime ideal in A corresponding to V. Next, (c) implies that if $\mathfrak{p}$ is a prime ideal in R such that $ht(\mathfrak{p})<n$ and $\mathfrak{p}\neq\mathfrak{p}(V)R$ for every $V\in\boldsymbol{V}$, then $\mathfrak{p}$ is contained in infinitely many height n maximal ideals in R. Therefore the proof is complete.

REMARK. By the above property (c), $ht(\mathfrak{p})+\dim(R/\mathfrak{p})=n$ for any non-maximal prime ideal $\mathfrak{p}$ in R, but R has a maximal ideal, of which height is less than n. Therefore R is another counterexample with relation to Remark 2.6 in [9]: If A is a noetherian ring such that $ht(\mathfrak{p})+\dim(A/\mathfrak{p})=\dim(A)$ for any non-maximal prime ideal $\mathfrak{p}$ in A, then does A satisfy that $ht(\mathfrak{M})=\dim(A)$ or 1 for any maximal ideal $\mathfrak{M}$ in A? (cf. [2], p. 478).

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