Bulletin of the Seismological Society of America. Vol. 68, No. 1, pp. 125-132. February 1978

A NOTE ON NONEQUIVALENT QUADRUPOLE SOURCE CYLINDRICAL SHEAR POTENTIALS WHICH GIVE EQUAL DISPLACEMENTS

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ABSTRACT

Two standard integral forms of the cylindrical shear potentials for point quadrupole seismic sources, a frequency domain *k*-integral and a time domain Cagniard-de Hoop path *p*-integral, are shown not to be the frequency-time domain inverse of each other. Their relationship is derived and they are shown to be seismically equivalent in that they yield the same displacement field.

INTRODUCTION

During the last few years, two integral representation forms for the cylindrical shear potentials due to a point quadrupole source, e.g., point shear dislocation or double couple, have been used in seismic wave propagation formulations. One is a wave number or k-integral representation in the frequency domain (Sato, 1969, 1972; Harkrider, 1976), and the other is a ray parameter or p-integral over a Cagniard-de Hoop path representation in the time domain (Helmberger, 1974). Each form has its own advantages in investigating seismic waves in multilayered media.

Analytic and numerical evaluations have shown that the two forms are not the Fourier time-frequency transforms of each other. It is the purpose of this paper to relate the two forms and to show that they are seismically equivalent in that they both yield the same displacement field. It is shown that the discrepancy results from the omission of a residue contribution in the evaluation of the time domain potential. The omission affects only the potentials; the displacement derived from the potentials is not affected.

INTEGRAL RELATIONS

The cylindrical shear potentials for a quadrupole source in the frequency domain involve integrals of the form

$$\bar{A}_1 = \bar{\xi}_1 e^{\pm i\phi} = \int_0^\infty F_v \frac{J_1(kr)}{k}\, dk e^{\pm i\phi}$$

$$\bar{A}_2 = \xi_2 e^{\pm i2\phi} = \int_0^\infty F_v J_2(kr)\, dk e^{\pm i2\phi} \tag{1}$$

where

$$k_v = \frac{\omega}{v}$$

$$F_v = \frac{k \exp\left(-\nu_v |z - z_0|\right)}{\nu_v}$$

$$\nu_v = \begin{cases} (k^2 - k_v^2)^{1/2}; & k > k_v \\ i(k_v^2 - k^2)^{1/2}; & k < k_r \end{cases}$$

ω is the angular frequency, v is the body velocity and (r, ϕ, z) are the cylindrical coordinates (Sato, 1969, 1972, and Harkrider, 1976). Harkrider (1976) expressed the integrals in closed form as

$$\bar{\xi}_1 = \frac{1}{i\omega}\frac{v}{r}\,(e^{-ik_v|z-z_0|} - e^{ik_vR})$$

$$\bar{\xi}_2 = \frac{e^{-ik_vR}}{R} + \frac{2v}{i\omega r^2}\,(e^{-ik_v|z-z|_0} - e^{ik_vR}) \tag{2}$$

where

$$R^2 = (z - z_0)^2 + r^2.$$

Taking the inverse Fourier transform defined by

$$\xi_i = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\xi}_i e^{-i\omega t}\, d\omega$$

we have

$$\xi_1(t) = \frac{v}{r}\,[H(t - |z - z_0|/v) - H(t - R/v)]$$

$$\xi_2(t) = -\frac{\delta(t - R/v)}{R} - \frac{2v}{r^2} H(t - R/v) + \frac{2v}{r^2} H(t - |z - z_0|/v)$$

$$= -\frac{\partial}{\partial t}\left\{\left[\frac{1}{R} + \frac{2v}{r^2}\,(t - R/v)\right] H(t - R/v)\right\} + \frac{2v}{r^2} H(t - |z - z_0|/v). \tag{3}$$

Since $\bar{\xi}_1$ was obtained from an integral table, after a change of variable and some manipulation, the integral for $\bar{\xi}_1$ was integrated numerically for sets of (r, z, v, ω) and was found to be in good agreement with the given closed form expression. A further check was obtained by taking the limit of $\omega \to 0$ for the integral to obtain the area under ξ_1 and the value of ξ_1 as $t \to \infty$. The area under ξ_1 is

$$\lim_{\omega\to 0} \bar{\xi}_1(\omega) = \int_0^\infty e^{-k|z-z_0|}\,\frac{J_1(kr)}{k}\, dk$$

$$= \frac{(R - |z - z_0|)}{r}.$$

The long time value of ξ_1 is

$$\lim_{\omega\to 0} \{i\omega\xi_1(\omega)\} = 0.$$

Similarly

$$\lim_{\omega\to 0} \bar{\xi}_2(\omega) = \int_0^\infty e^{-k|z-z_0|} J_2(kr)\, dk$$

$$= \frac{1}{R}\,\frac{(R - |z - z_0|)^2}{r^2}$$

$$\lim_{\omega\to 0} \{i\omega\bar{\xi}_2(\omega)\} = 0$$

which agree with the closed form expressions in equations (3). The area relation for ξ_2 is found in Erdelyi *et al.* (1954) equation 4.14(1), i.e.

$$\int_0^\infty e^{-k|z-z_0|} J_n(kr)\, dk = \frac{1}{R}\left\{\frac{R-|z-z_0|}{r}\right\}^n$$

and $\bar{\xi}_1(0)$ follows from the relation

$$\bar{\xi}_2(\omega) = -\frac{e^{-ik_v R}}{R} + \frac{2}{r}\bar{\xi}_1(\omega)$$

(Harkrider, 1976).

Although the closed form relation for the integral form of $\bar{\xi}_2(0)$ was also obtained from tables, it is related to integral expressions which have been frequently used in static problems. For instance Sato and Matsu'ura (1973) present a table for various (m, n) of the integral form

$$I_{m,n} = \int_0^\infty k^m e^{-k|z-z_0|} J_n(kr)\, dk.$$

The tables were obtained by taking multiple z and r derivatives of the integral expression for $(1/R)$.

$\bar{\xi}_2(0)$ is related to $I_{1,2}$

by

$$\frac{\partial \bar{\xi}_2}{\partial z} = -\operatorname{sgn}(z-z_0) I_{1,2}.$$

From our previous relations

$$\frac{\partial \bar{\xi}_2(0)}{\partial z} = -\operatorname{sgn}\,(z-z_0)\,\frac{r^2(2R+|z-z_0|)}{R^3(R+|z-z_0|)^2}$$

which agrees with $I_{1,2}$ in the tables.

Generalized Ray Theory Integral Relations

Following Helmberger (1974), we express the inverse of the integrals in equation (1) as

$$\xi_1(t) = -\frac{2}{\pi}\operatorname{Im}\int_{i0}^{i\infty} \frac{H(t-\tau)(t-\tau+pr)}{(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}}\frac{1}{\eta_v pr}\, dp$$

$$\xi_2(t) = -\frac{2}{\pi}\frac{\partial}{\partial t}\operatorname{Im}\int_{i0}^{i\infty} \frac{H(t-\tau)c(t,\tau)}{(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}}\frac{p}{\eta_v}\, dp \tag{4}$$

where

$$p = \frac{k}{\omega}$$

$$\eta_v = \left(\frac{1}{v^2} - p^2\right)^{1/2}$$

$$\tau = pr + \eta_v|z-z_0|$$

and

$$c(t, \tau) = \cosh\left[2 \cosh^{-1}\left(\frac{t - \tau + pr}{pr}\right)\right].$$

Before deforming the integral path to the Cagniard-de Hoop path (see Gilbert and Helmberger, 1972), it should be noted that there is a first order pole at $p = 0$ in both integrals. As $p \rightarrow 0$, we have the limits

$$\eta_v \rightarrow \frac{1}{v}$$

$$\tau \rightarrow \frac{|z - z_0|}{v} \equiv t_z$$

and since

$$c(t, \tau) = \frac{1}{2}\left[\frac{(A + \sqrt{A^2 - 1})^4 + 1}{(A + \sqrt{A^2 - 1})^2}\right]$$

where

$$A \equiv \frac{t - \tau + pr}{pr}$$

$$A \rightarrow \frac{t - t_z}{pr}$$

$$c(t, \tau) \rightarrow 2\,\frac{(t - t_z)^2}{p^2r^2}$$

the integrand of ξ_1

$$\rightarrow \frac{vH(t - t_z)}{r}\,\frac{1}{p}$$

and the integrand of ξ_2

$$\rightarrow 2\,\frac{v(t - t_z)H(t - t_z)}{r^2}\,\frac{1}{p}.$$

Thus the lower limit of the integrals is actually an infinitesimal distance above the origin on the positive imaginary axis. For the inverse of the Sommerfeldt integral, i.e., zero order Bessel function in the integrand

$$\xi_0(t) = \frac{2}{\pi}\,\mathrm{Im}\int_0^{i\infty} \frac{H(t - \tau)}{(t - \tau)^{1/2}(t - \tau + 2pr)^{1/2}}\,\frac{p}{\eta_v}\,dp$$

there is no pole at the origin and the lower limit is at the origin.

Deforming the contour from the positive imaginary axis to the Cagniard-de Hoop contour, Γ (Figure 1) defined by $\tau(p)$ positive real, we have

$$\int_{i0}^{i\infty} [\quad]\,dp = \int_C [\quad]\,dp + \int_\Gamma [\quad]\,dp$$

where the C contour is the infinitesimal quarter circle about the origin from $+i0$ to $+0$. Now

$$\int_C \frac{H(t-\tau)(t-\tau+pr)}{(t-\tau)^{1/2}(t-\tau+2pr)}\frac{1}{\eta_v pr}\,dp = -\frac{\pi i}{2}\,\text{Residue}\,(p=0) = -\frac{\pi i}{2}\frac{v}{r}H(t-t_z)$$

and similarly

$$\int_C \frac{H(t-\tau)c(t,\tau)}{(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}}\frac{p}{\eta_v}\,dp = -\frac{\pi i}{2}2\frac{v}{r^2}(t-t_z)H(t-t_z)$$

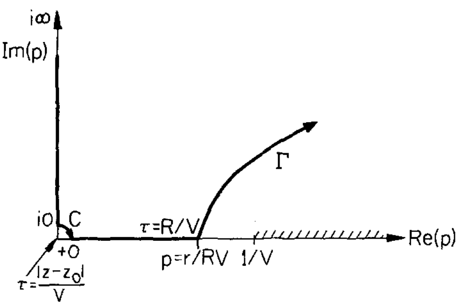

FIG. 1. First quadrant of the complex p plane showing contours C and Γ.

or

$$\xi_1(t) = \frac{v}{r}H(t-t_z) - \frac{2}{\pi}\,\text{Im}\int_\Gamma \frac{H(t-\tau)(t-\tau+pr)}{(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}}\frac{1}{\eta_v pr}\,dp$$

$$\xi_2(t) = \frac{\partial}{\partial t}\left[2\frac{v}{r^2}(t-t_z)H(t-t_z)\right] - \frac{2}{\pi}\frac{\partial}{\partial t}\,\text{Im}\int_\Gamma \frac{H(t-\tau)c(t,\tau)}{(t-\tau)^{1/2}(t-\tau+2pr)}\frac{p}{\eta_v}\,dp$$

$$= \frac{2v}{r^2}H(t-t_z) - \frac{2}{\pi}\frac{\partial}{\partial t}\,\text{Im}\int_\Gamma \frac{H(t-\tau)c(t,\tau)}{(t-\tau)^{1/2}(t-\tau+2pr)}\frac{p}{\eta_v}\,dp.$$

Changing the variable of integration to τ we have

$$\xi_1(t) = \frac{v}{r}H(t-t_z) - \frac{1}{r}\frac{2}{\pi}\,\text{Im}\int_{t_R}^{t} \frac{(t-\tau+pr)}{(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}}\left(\frac{dp}{d\tau}\right)\frac{1}{\eta_v p}\,d\tau$$

$$\xi_2(t) = \frac{2v}{r^2}H(t-t_z) - \frac{2}{\pi}\frac{\partial}{\partial t}\,\text{Im}\int_{t_R}^{t} \frac{c(t,\tau)}{(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}}\left(\frac{dp}{d\tau}\right)\frac{p}{\eta_v}\,d\tau \qquad (5)$$

where

$$\frac{dp}{d\tau} = \frac{i\eta_v}{(\tau^2-t_R^2)^{1/2}}$$

$$t_R \equiv \frac{R}{v}$$

and the lower limit is t_R since the integrands are real for the interval $t_z < \tau < t_R$.

Comparing equations (5) with equations (3), we see that

$$\frac{2}{\pi}\,\mathrm{Re}\int_{t_R}^{t}\frac{(t-\tau+pr)}{p(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}(\tau^2-t_R{}^2)^{1/2}}\,d\tau = vH(t-t_R)$$

$$\frac{2}{\pi}\,\mathrm{Re}\int_{t_R}^{t}\frac{c(t,\tau)p}{(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}(\tau^2-t_R{}^2)^{1/2}}\,d\tau = \left[\frac{1}{R}+\frac{2v}{r^2}(t-t_R)\right]H(t-t_R). \quad (6)$$

Equations (6) were integrated numerically for various sets of (r, z, v, t) and were found to be in good agreement with their closed form expressions.

Helmberger (1974) ignored the residue contribution in equations (5) since, as we shall see in the next section, these terms do not contribute to the displacement field. They are present with the Cagniard-de Hoop contour integral in order to yield a finite spectral amplitude at zero frequency for the shear potentials.

Displacement Contribution

For purposes of discussion we will use the following superscript notations

$$\bar{A}_1 = \bar{A}_1{}^{(1)} + \bar{A}_1{}^{(2)}$$

$$\bar{A}_2 = \bar{A}_2{}^{(1)} + \bar{A}_2{}^{(2)}$$

where from equation (2)

$$\bar{A}_1{}^{(1)} = -\frac{1}{i\omega}\frac{v}{r}e^{-ik_vR}e^{\pm i\phi}$$

$$\bar{A}_1{}^{(2)} = \frac{1}{i\omega}\frac{v}{r}e^{-ik_v|z-z_0|}e^{\pm i\phi}$$

$$\bar{A}_2{}^{(1)} = -\left[\frac{1}{R}+\frac{2v}{i\omega r^2}\right]e^{ik_vR}e^{\pm i2\phi} \quad (7)$$

and

$$\bar{A}_2{}^{(2)} = \frac{2v}{i\omega r^2}e^{-ik_v|z-z_0|}e^{\pm i2\phi}.$$

By substitution it is easy to verify that not only $\bar{A}_1$ and $\bar{A}_2$ satisfy Helmholtz equations, i.e.

$$\nabla^2\bar{A} = -k_v{}^2\bar{A}$$

as stated in Harkrider (1976), but that each superscripted term also satisfies the Helmholtz equation.

The integral forms of equation (1) are associated with the SV cylindrical potential, $\bar{\Psi}$, and the SH cylindrical potential, $\bar{X}$, when the spectral displacement field, $\bar{u}$ is expressed as

$$\bar{u} = \mathrm{grad}\,\bar{\Phi} + \mathrm{curl}\,\mathrm{curl}\,(0, 0, \bar{\Psi}) + \mathrm{curl}\,(0, 0, \bar{X}). \quad (8)$$

Forming the shear potentials from equations (7), it is easy to verify by direct substitution in equation (8) that the superscript (2) terms do not contribute to the displacement field. The elimination results from the special form of $A_i^{(2)}$, i.e.

$$A_i^{(2)} \sim \frac{1}{r^m} e^{\pm im\phi} e^{-ik_v|z-z_0|}$$

which in addition to satisfying the Helmholtz equation also satisfies

$$\frac{\partial^2 A_i^{(2)}}{\partial z^2} = -k_v^2 A_i^{(2)}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_i^{(2)}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_i^{(2)}}{\partial \phi^2} = 0.$$

It should be noted that the standard cartesian shear potentials $(\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3)$ implicitly defined by

$$\bar{u} = \text{grad}\,\bar{\Phi} + \text{curl}\,(\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3) = 0$$

with

$$\text{div}\,(\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3) = 0$$

do not contain terms involving $\exp\,[-ik_\beta\,|\,z - z_0\,|]$ since

$$\bar{\Psi}_i = \frac{1}{k_\beta^2}\,(\text{curl}\,\bar{u}_i).$$

Conclusions

We have shown that the forms of the cylindrical shear potentials used by Harkrider (1976) and Helmberger (1974) are seismically equivalent in that they both yield the same displacement field.

Even though

$$\int_{-\infty}^{\infty}\int_0^{\infty} F_v \frac{J_1(kr)}{k}\,dk e^{i\omega t}\,d\omega \neq -\frac{1}{r}\frac{2}{\pi}\,\text{Im}\int_{t_R}^{t} \frac{(t-\tau+pr)}{(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}}\left(\frac{dp}{d\tau}\right)\frac{1}{\eta_v p}\,d\tau$$

and

$$\int_{-\infty}^{\infty}\int_0^{\infty} F_v J_2(kr)\,dk e^{i\omega t}\,d\omega \neq -\frac{2}{\pi}\frac{\partial}{\partial t}\,\text{Im}\int_{t_R}^{t} \frac{c(t,\tau)}{(t-\tau)^{1/2}(t-\tau+2pr)^{1/2}}\left(\frac{dp}{d\tau}\right)\frac{p}{\eta_v}\,d\tau$$

cylindrical shear potentials expressed in terms of the right-hand side do not contain noncontributing terms to the displacement field as do the left-hand side relations. However, the displacement field expressed as an integral over k does not contain these plane-phase waves. Only the cylindrical shear potential fields expressed as k-integrals contain them. Thus analytic or numerical evaluations of the displacement field as a k-integral do not yield these nonpoint source terms. In generalized ray theory for quadrupole sources (Helmberger, 1974), the ability to isolate and eliminate these terms in the time domain for shear potentials is based on the recognition that they are given by the residue contribution of the pole at $p = 0$ in the Cagniard-de Hoop inversion technique. Since there are no poles or unusual singularities in the k-integrals of equation (1), it is not obvious which part of the integration path contributes them or even if a path can be chosen which will separate them from the displacement contributing part of these source shear potential representations.

Acknowledgments

This research was supported by the Advanced Research Projects Agency of the Department of Defense and was monitored by the Air Force Office of Scientific Research under Contract F49620-77-C-0022.

References

Erdelyi, A., W. Magnus, F. Oberhettinger, and F. G. Tricomi (1954). *Tables of Integral Transforms*, vols. 1–2, *Bateman Manuscript Project*, California Institute of Technology, McGraw-Hill, New York.

Gilbert, F. J. and D. V. Helmberger (1972). Generalized ray theory for a layered sphere, *Geophys. J.* **27**, 57–80.

Harkrider, D. G. (1976). Potentials and displacements for two theoretical sources, *Geophys. J.* **47**, 97–133.

Helmberger, D. V. (1974). Generalized ray theory for shear dislocations, *Bull. Seism. Soc. Am.* **64**, 45–64.

Sato, R. (1969). Formulations of solutions for earthquake source models and some related problems, *J. Phys. Earth* (*Tokyo*) **17**, 101–110.

Sato, R. (1972). Seismic waves in the near field, *J. Phys. Earth* (*Tokyo*) **20**, 357–375.

Sato, R. and M. Matsu'ura (1973). Static deformations due to the fault spreading over several layers in a multi-layered medium. Part I: Displacement, *J. Phys. Earth* (*Tokyo*) **21**, 227–249.

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Manuscript received February 8, 1977