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A Note on Nontrivial Periodic Solutions of Dynamical Systems with Subquadratic Potential

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ABSTRACT. We obtain non-constant periodic solutions for a class of second-order autonomous dynamic systems whose potential is subquadratic at infinity. We give a theorem on conjugate points for convex potentials.

1. INTRODUCTION

This paper is concerned with the existence of periodic solutions $u(t)$ of a conservative system of the form

$$u'' + \nabla F(u) = h(t) \tag{1}$$

where $F \in C^i(\mathbf{R}^N, \mathbf{R})$ $(i = 1, 2, 3)$ is subquadratic at infinity and $h(t)$ is a continuous periodic vector-valued function. Our aim is to show how the saddle-point theorem of Rabinowitz [11], together with results of

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Lazer and Solimini [9] that give supplementary information based on the Morse index of critical points, can be used to obtain in a quite simple way, not only the existence of a periodic solution of (1), but also some basic facts about nontrivial solutions of the autonomous counterpart of (1),

$$u'' + \nabla F(u) = 0. \tag{2}$$

Since our hypotheses (see §2) imply that F has at least one critical point in $\mathbf{R}^N$, the interesting question about (2) is whether there exist nontrivial (i.e. non constant) solutions. We give a condition for this to happen in theorem 2, which may be viewed as a generalization of the well-known fact that, for the pendulum scalar equation

$$u'' + a \sin u = 0,$$

nontrivial oscillations appear only with periods $T > 2\pi \ / \ \sqrt{a}$.

The text is organized as follows. In section 2 we present an existence result for (1) which will be used in the remaining sections and which lies upon assumptions closely related to those introduced by Ahmad, Lazer and Paul [1]. In section 3, we give a sufficient condition for the existence of non-constant T-periodic solutions of (2). Finally in section 4, we give a theorem on "conjugate points" for (2), in the convex case. We point out that combining this result with the forementioned background of critical point theory a simple proof of the existence of a solution, with a given minimal period T, of (2), can be given. This last result has been obtained for general subquadratic Hamiltonian systems by Clarke and Ekeland [5] (see also Ekeland and Hofer [6]). A simple approach in the case where the potential is even was given by Willem [15]. We use a device similar to that of Salvatore [13,14]. Since for convex potentials the Morse-Ekeland index is well-defined, these results may be worked out by adapting the method described in [6], [7] or [10, chapt.7]. Our approach is an alternative to this method; basically it differs from it in the sense that we study the Morse index of a "direct" rather than of a "dual" action functional.

Since many authors have studied the above mentioned problems it would become lengthy to quote a complete bibliography. We therefore confine ourselves to refer in addition to the work of Ambrosetti [2], Ambrosetti and Mancini [3], Benci, Cappozi and Fortunato [4], Rabinowitz [12], Girardi and Matzeu [8] and also to the book by Mawhin and Willem

[10] for a survey. Their references provide a complement of information in the existing research in this area.

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2. EXISTENCE OF A PERIODIC SOLUTION

Suppose that $F \in C^1(\mathbf{R}^N, \mathbf{R})$ and $h \in C([0,T], \mathbf{R}^N)$ satisfy the following assumptions. Here $|\cdot|$ denotes the Euclidean norm of $\mathbf{R}^N$.

(A) $F(u) = o\,(|u|^2)$ as $|u| \to \infty$;

(B) There exist $\eta, \epsilon, R > 0$ such that whenever $v \in S^{N-1}$, $w \in \mathbf{R}^N$ is such that $|w - v| \leq \epsilon$, and $\rho \geq R$ we have

$$\nabla F(\rho w) \cdot v \geq \eta.$$

(C) $\int_0^T h(t)dt = 0$.

Examples: (i) Let $\varphi \in C^1(\mathbf{R}, \mathbf{R})$ be even and set $F(u) = \varphi(|u|)$. Then F satisfies (A)-(B) if and only if $lim_{|x|\to\infty}\varphi(x)/x^2 = 0$ and $\liminf_{x\to+\infty} \varphi'(x) > 0$.

(ii) If $A(u)$ is a positive definite quadratic form in $\mathbf{R}^N$, the function $F(u) = (1 + A(u))^{1/2}$ satisfies (A)-(B).

(iii) Let $\varphi, \psi \in C^1(\mathbf{R}, \mathbf{R})$ be such that φ', ψ' are bounded,

$$\lim_{|x|\to\infty} \frac{\varphi(x)}{x^2} = 0, \ \lim_{|x|\to\infty} \frac{\psi(x)}{x^2} = 0,$$

$$\liminf_{x\to+\infty} \varphi'(x) > 0, \ \liminf_{x\to+\infty} \psi'(x) > 0,$$

$$\limsup_{x\to-\infty} \varphi'(x) < 0, \ \limsup_{x\to-\infty} \psi'(x) < 0.$$

Then $F(x,y) = \varphi(x) + \psi(y)$ satisfies (A)-(B) in $\mathbf{R}^2$, and the same holds for any perturbation of the form $F(x,y) + G(x,y)$ where $G \in C^1(\mathbf{R}^2, \mathbf{R})$ satisfies $\nabla G(u) \to 0$ as $|u| \to \infty$.

Assumptions (A), (B) are close to the following, which have been introduced by Ahmad, Lazer and Paul [1] in studying Dirichlet problems:

(A*) $\nabla F(u)$ is bounded in $\mathbf{R}^N$;

(B*) $\lim_{|u|\to\infty} F(u) = +\infty$.

In this sense our first theorem is a variation on the results of [1] which, as shown in [10,chap.4] still holds for the periodic boundary condition.

Theorem 1. *Let $F \in C^1(\mathbf{R}^N, \mathbf{R})$ and $h \in C([0,T];\mathbf{R}^N)$ satisfy (A)-(B)-(C), or (A*)-(B*)-(C). Then the system (1) has at least one solution $u(t)$ such that $u(0) = u(T)$ and $u'(0) = u'(T)$.*

Proof. We need only consider the first set of assumptions, since the proof in the other case is well known (cf. [10]). Throughout the paper we shall make use of the functional

$$J(u) = \int_0^T \left(\frac{|u'|^2}{2} - F(u) + h(t)\cdot u\right)dt$$

(where $\cdot$ denotes the scalar product of $\mathbf{R}^N$) which is well-defined in the Hilbert space $H^1_T \equiv \{u \in H^1(0,T;\mathbf{R}^N) : u(0) = u(T)\}$. Moreover $J \in C^1(H^1_T, \mathbf{R})$. We shall obtain a critical point of J by means of the saddle-point theorem of Rabinowitz [11]. By well-known arguments, such a critical point is a solution of class C^2 of (1) as in the statement of the theorem. We now verify the hypotheses of the saddle-point theorem with respect to the direct sum decomposition $H^1_T = \mathbf{R}^N \oplus \tilde{H}$ where $\mathbf{R}^N$ is identified with the subspace of constant functions and $\tilde{H}$ consists of those $u \in H^1_T$ such that $\int_0^T u\, dt = 0$. Namely, we must show that:

(i) J is bounded from below in $\tilde{H}$;

(ii) $lim_{|c|\to\infty} J(c) = -\infty$ if $c \in \mathbf{R}^N$;

(iii) J satisfies the Palais-Smale condition.

Proof of (i). This is a straightforward consequence of the fact that for any $\epsilon > 0$ we can find $C > 0$ such that $F(u) \le \epsilon|u|^2 + C$ for all $u \in \mathbf{R}^N$, together with the Wirtinger inequality

$$\frac{4\pi^2}{T^2}\int_0^T |u|^2 dt \le \int_0^T |u'|^2 dt,\ u \in \tilde{H},$$

and the hypothesis (C).

Proof of (ii). It obviously suffices to show that $F(c) \to +\infty$ as $|c| \to \infty$ in $\mathbf{R}^N$. Now if $c \in \mathbf{R}^N$, $|c| > R$, let $d = Mc/|c|$ and write

$$F(c) - F(d) = |c-d| \int_0^1 \nabla F(d + t(c-d)) \cdot \frac{c-d}{|c-d|} dt.$$

Assumption (B) implies $F(c) \geq \eta|c-d| + K$, where $K = min\{F(z) : |z| = M\}$, and (ii) follows.

Proof of (iii). Let (u_n) be a sequence in H_T^1 such that $J(u_n)$ is bounded and $J'(u_n) \to 0$. Consider the decomposition $u_n = a_n + w_n$ where $a_n \in \mathbf{R}^N$, $w_n \in \tilde{H}$. Given $\epsilon > 0$ there exists $C_1 > 0$ such that

$$\int_0^T \left(\frac{|w_n'|^2}{2} + hw_n\right) dt \leq \epsilon \int_0^T |u_n|^2 dt + C_1$$
$$\leq \epsilon T|a_n|^2 + \epsilon \int_0^T |w_n|^2 dt + C_1.$$

Then if $\| \ \|$ denotes a norm in H_T^1 and C_i are constants independent on n we obtain

$$\|w_n\|^2 \leq \epsilon C_2 |a_n|^2 + C_3. \tag{3}$$

We claim that $|a_n|$ is bounded. If this is not the case, then along some subsequence (still denoted a_n) the preceeding inequality implies $\|w_n\|/|a_n| \to 0$. Now let $z_n(t) = w_n(t)/|a_n|$. We obtain

$$u_n(t) = |a_n|(v_n + z_n(t)) \tag{4}$$

where $v_n = a_n/|a_n|$ and $z_n(t) \to 0$ uniformly in $[0,T]$. We may suppose that $v_n \to v$ in S^{N-1}. Since

$$\langle -J'(u_n), v\rangle = \int_0^T \nabla F(u_n(t)) \cdot v \ dt,$$

(B), (4) and Fatou's lemma allow us to conclude that

$$\liminf_{n\to\infty} \langle -J'(u_n), v\rangle \geq T\eta > 0,$$

a contradiction which proves the claim. By virtue of (3), $u_n(t)$ is bounded in H^1_T. By standard results, (u_n) contains a convergent subsequence. The proof is complete. ■

3. AUTONOMOUS SYSTEMS

We now turn to the autonomous system (2). We suppose that F is C^2 and still satisfies (A)-(B). Theorem 1 is now a triviality since F has a minimum in $\mathbf{R}^N$, and each critical point of F is a T-periodic solution of (2) for any $T > 0$.

Let us introduce the following definition and notations. Given $T > 0$ and a critical point $\bar{u}$ of F, we say that $\bar{u}$ is T*−admissible* if the spectrum $\sigma(F''(\bar{u}))$ does not contain numbers of the form $4n^2\pi^2/T^2 (n \in \mathbf{Z})$. If $\bar{u}$ is a T-admissible critical point of F, we accordingly number the eigenvalues of $F''(\bar{u})$ as

$$\lambda_1(\bar{u}) \leq \cdots \leq \lambda_k(\bar{u}) < 0 < \lambda_{k+1}(\bar{u}) \leq \cdots \leq \lambda_N(\bar{u})$$

(so that k is the index of $\bar{u}$ as a critical point of F) and denote by $n = n(\bar{u})$ the greatest integer with the property $4n^2\pi^2/T^2 < \lambda_{k+1}(\bar{u})$, provided that $k < N$.

For the statement of the next theorem we also include the condition:

(C*) There exists $R > 0$ such that $\nabla F(u) \neq 0$ if $|u| \geq R$.

Theorem 2. *Let $F \in C^2(\mathbf{R}^N, \mathbf{R})$ satisfy (A)-(B) or (A*)-(B*)-(C*) and suppose that each critical point of F is T-admissible. If for each such critical point $\bar{u}$ one of the following conditions is satisfied*

(i) $(N-k) + 2n(N-k) > N$

(ii) $(N-k) + 2n(N-k) < N$ *and* $\lambda_N(\bar{u}) < 4(n+1)^2\pi^2/T^2$
then (2) has a non-constant T-periodic solution.

Remark. (a) (i) always holds at a local minimum provided $n = n(\bar{u}) \geq 1$. (ii) always holds at a local maximum. In $\mathbf{R}^2$ each critical point of index 1 satisfies (i) or (ii). If $k = N$, n is undefined but we take $2n(N-k) = 0$.

(b) If F has only one critical point $\bar{u}$ (a minimum) the theorem asserts that there exists a T-periodic non-constant solution provided that $T > 2\pi\sqrt{\lambda_1(\bar{u})}$.

Proof. Since each critical point of F is T-admissible, it is nondegenerate; assumption (B) or assumption (C*) thus implies that the set of critical points of F is finite. On the other hand the second derivative of J at $\bar{u}$ is the quadratic form

$$J''(\bar{u})(v) = \int_0^T (|v'|^2 - F''(\bar{u})v \cdot v)dt, \ v \in H^1_T.$$

Since the linear system

$$v'' + F''(\bar{u})v = 0$$

has no nontrivial T-periodic solution, $J''(\bar{u})$ is nondegenerate. On the other hand the Morse index of $\bar{u}$ as a critical point of J (that is the index of $J''(\bar{u})$) is (see [10],chap.9) no smaller than $(N-k) + 2n(N-k)$, with equality if $n = n(\bar{u})$ satisfies $\lambda_N(\bar{u}) < 4(n+1)^2\pi^2/T^2$. It follows that if either (i) or (ii) holds, the Morse index of $\bar{u}$ is different from N.

Assume that (2) has no solution distinct from the critical points of F. Then lemma 1.1 in [9] is applicable and it implies the existence of a critical point of J with Morse index N, a contradiction. This ends the proof. ■

Remark. Theorem 2 may be proved by using Morse inequalities instead of explicit resource to lemma 1.1. in [9]. In fact, assume that the set of critical points of J coincides with that of F, let $c_1 < \ldots < c_p$ be the distinct critical values of J and choose $a < c_1$ such that $a < \inf\{J(u) : u \in \tilde{H}\}$. Let D be a closed disk centered at the origin in $\mathbf{R}^N$ such that its boundary S is contained in J^a, and choose $b > max\{c_p, max_D J\}$. Then we have a commutative diagram of homomorphisms

$$\begin{array}{ccc} H_N(D,S) & \overset{\partial_N}{\longrightarrow} & \tilde{H}_{N-1}(S) \\ i_N\downarrow & & \downarrow j_N \\ H_N(J^b,J^a) & \underset{\partial'_N}{\longrightarrow} & \tilde{H}_{N-1}(J^a) \end{array}$$

where the homology groups are taken over the real numbers and the vertical arrows are induced by inclusions. ($\tilde{H}$ denotes reduced homology.) The arguments in the proof of lemma 1.1. of [9] show that $j_N \neq 0$; since ∂_N is an isomorphism it follows that $H_N(J^b,J^a) \neq 0$. If $m_N = m_N(J^b,J^a)$ is the number of critical points of J with index N it follows that $m_N \geq \dim H_N(J^b,J^a) \geq 1$, a contradiction with (i)-(ii). Thus we see that the saddle-point theorem geometrical setting might be replaced by the more general condition that for some regular value a of J we have $S \subset J^a$ and that this inclusion induces a nontrivial homomorphism j_N in homology.

4. CONJUGATE POINTS. SOLUTIONS WITH GIVEN MINIMAL PERIOD.

In this section we assume that F is convex. More precisely we introduce the following assumption

(D) $F \in C^3(\mathbf{R}^N,\mathbf{R})$ and $F''(\bar{u})$ is positive definite for each $\bar{u} \in \mathbf{R}^N$.

In the sequel we shall use the following form of a theorem of Benci and Fortunato. See Salvatore [13] for a more general statement and proof.

Theorem 3. *Let X be a Hilbert space, δ_0 a given positive number and $\{a_\delta : |\delta| \leq \delta_0\}$ a family of continuous, quadratic forms such that*

(i) for each δ there exist $m_\delta > 0$ and a weakly sequentially continuous quadratic form b_δ such that $a_\delta + m_\delta d_\delta$ is a inner product equivalent to the one given in X.

(ii) there exists $\nu > 0$ such that

$$\left.\frac{d}{d\delta}\right|_{\delta=0} a_\delta(x,x) \leq -\nu\|x\|^2 \qquad \forall x \in X_0$$

where $X_0 = \{u \in X : a_0(u,v) = 0 \ \forall v \in X\}$.

(iii) there exists $M > 0$ *such that*

$$|a_\delta(x,y) - a_0(x,y)| \leq M\delta \|x\| \, \|y\|, \ \forall x,y \in X.$$

Then there exists $\delta_1 > 0$ *such that whenever* $-\delta_1 < \xi < 0 < \eta < \delta_1$ *we have*

$$\text{index of } a_\eta = \text{index of } a_0 + \dim X_0,$$
$$\text{index of } a_\xi = \text{index of } a_0. \quad \blacksquare$$

Now let $\bar{u}(t)$ be a *non constant* solution of (3) with period $T > 0$. The linear system

$$z'' + F''(\bar{u}(t))z = 0 \tag{5}$$

has the nontrivial solution $z = \bar{u}'(t)$.

We say that a number $S \in (0,T]$ is *conjugate* to 0 with respect to $\bar{u}$ if and only if (5) admits a nontrivial S-periodic solution. The *multiplicity* of S as a point conjugate to 0 is, by definition, the dimension of the subspace of S-periodic solutions to (5).

By performing the change of independent variable $t = (S/T)\tau$ we may reformulate the above definition as follows: S is conjugate to 0 with respect to $\bar{u}$ if and only if the system

$$(T^2/S^2)v'' + F''(\bar{u}_S(\tau))v = 0, \ \bar{u}_S(\tau) = \bar{u}\left(\frac{S}{T}\tau\right), \tag{6}$$

admits a nontrivial T-periodic solution $v(\tau)$, the multiplicity of S as a conjugate of 0 being the dimension of the subspace formed by such solutions. Let us define a function $m : (0,T] \to \mathbf{N}_0$ by setting $m(S) =$ index in H^1_T of the quadratic form

$$Q_S(v) = \int_0^T \left[(T^2/S^2)\,|v'|^2 - F''(\bar{u}_S)v \cdot v)\right] d\tau.$$

Let us remark that $Q_T = J''(\bar{u})$, so that $m(T)$ is the Morse index of $\bar{u}$ as a critical point of J. The study of the function m will be based on theorem 3. We proceed to show that the family $\{Q_{S+\delta}\}$, where $|\delta|$ is small, satisfies the hypotheses of that theorem.

Lemma 1. *Let F satisfy (D). Then given $S \in (0,T]$ there exists $k > 0$ such that, for any T-periodic solution $v(\tau)$ of (5):*

$$\frac{d}{d\delta}\bigg|_{\delta=0} Q_{S+\delta}(v) \leq -K\|v\|^2.$$

Remark. Since the space of solutions of (6) is finite-dimensional the choice of the norm for v is irrelevant.

Proof. We have

$$\frac{d}{d\delta}Q_{S+\delta}(v) = \int_0^T \Big[-\frac{2T^2}{(S+\delta)^3}|v'|^2 - \frac{\tau}{T}\sum_{i,j,k}\frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}$$

$$(\bar{u}(\frac{S+\delta}{T}\tau))\bar{u}'_k(\frac{S+\delta}{T}\tau)v_i(\tau)v_j(\tau)\Big]d\tau =$$

$$= \int_0^T \Big[-\frac{2T^2}{(S+\delta)^3}|v'|^2 - \frac{T}{S+\delta}\frac{\tau}{T}\sum_{i,j,k}\frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}$$

$$(\bar{u}_{S+\delta}(\tau))\bar{u}'_{S+\delta,k}(\tau)v_i(\tau)v_j(\tau)\Big]d\tau$$

so that

$$\frac{d}{d\delta}\bigg|_{\delta=0} Q_{S+\delta}(v) = \int_0^T \Big[-\frac{2T^2}{S^3}|v'|^2 - \frac{\tau}{S}\sum_{i,j,k}\frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}(\bar{u}_S)\bar{u}'_{S,k}v_i v_j\Big]d\tau.$$

In order to compute the triple sum let us note that (6) implies

$$\frac{T^2}{S^2}\frac{d}{d\tau}(v\cdot v'') + \sum_{i,j,k}\frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}(\bar{u}_S)\bar{u}'_{S,k}v_i v_j + 2\sum_{i,j}\frac{\partial^2 F}{\partial x_i \partial x_j}(\bar{u}_S)v'_i v_j = 0$$

and on account of (6) again the last summand equals $-(2T^2/S^2)v' \cdot v''$, whence

$$\sum_{i,j,k} \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}(\bar{u}_S)\bar{u}'_{S,k} v_i v_j = \frac{T^2}{S^2}\frac{d}{d\tau}(|v'|^2 - v \cdot v''),$$

$$\begin{aligned}
\left.\frac{d}{d\delta}\right|_{\delta=0} Q_{S+\delta}(v) &= \int_0^T \left[-\frac{2T^2}{S^3}|v'|^2 - \frac{T^2}{S^3}\tau\frac{d}{d\tau}(|v'|^2 - v \cdot v'')\right] d\tau \\
&= \frac{T^2}{S^3}\Bigg(\int_0^T -2|v'|^2 d\tau - \Big[\tau(|v'|^2 - v \cdot v'')\Big]_0^T + \\
&\qquad + \int_0^T (|v'|^2 - v \cdot v'')d\tau \Bigg) \\
&= -\frac{T^3}{S^3}\Big(|v'(T)|^2 - v(T) \cdot v''(T) \Big) \\
&= -\frac{T^3}{S^3}\Big(|v'(T)|^2 + \frac{S^2}{T^2}F''(\bar{u}_S(T))v(T) \cdot v(T) \Big).
\end{aligned}$$

Therefore the lemma follows from the facts that F'' is positive definite and that the expression $(|v'(T)|^2 + a|v(T)|^2)^{1/2}$, where $a > 0$, is a norm in the space of solutions of (6). ■

Lemma 2. *Let $S \in (0,T]$ and $\delta_0 > 0$ be given, so that $|\delta_0| < S/2$. Then there exists $M > 0$ so that*

$$|Q_{S+\delta}(x,y) - Q_S(x,y)| \le M|\delta| \, ||x|| \, ||y||, \; x, y \in H^1_T, \; |\delta| \le |\delta_0|.$$

Proof. We have

$$Q_{S+\delta}(x,y) - Q_S(x,y) =$$

$$\int_0^T \left[\frac{T^2}{S+\delta^2}x' \cdot y' - \frac{T^2}{S^2}x' \cdot y' - (F''(\bar{u}_{S+\delta}) - F''(\bar{u}_S))x \cdot y \right] d\tau.$$

There exists $C > 0$ so that

$$\left|\frac{T^2}{(S+\delta)^2} - \frac{T^2}{S^2}\right| = \frac{T|\delta|}{S(S+\delta)} \cdot \left(\frac{T}{S+\delta} + \frac{T}{S}\right) \leq C|\delta|$$

if $|\delta| \leq \delta_0$. On the other hand, the (i,j)-entry of the matrix $F''(\bar{u}_{S+\delta}) - F''(\bar{u}_S)$ is

$$\frac{\partial^2 F}{\partial x_i \partial x_j}\left(\bar{u}\left(\frac{S+\delta}{T}\tau\right)\right) - \frac{\partial^2 F}{\partial x_i \partial x_j}\left(\bar{u}\left(\frac{S}{T}\tau\right)\right) =$$

$$\frac{\delta}{T}\tau \sum_k \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}(\bar{u}(\xi))\bar{u}_k'(\xi)$$

for some ξ in the interval with end points $(S/T)\tau$ and $((S+\delta)/T)\tau$. Then there exists $C_1 > 0$ such that (denoting by $|\;|$ a norm in the space of $N \times N$ matrices)

$$|F''(\bar{u}_{S+\delta}) - F''(\bar{u}_S)| \leq \frac{C_1\tau}{T}|\delta|.$$

Hence

$$|Q_{S+\delta}(x,y) - Q_S(x,y)| \leq |\delta| \int_0^T \left(C|x' \cdot y'| + \frac{C_1\tau}{T}|x \cdot y|\right) d\tau$$
$$\leq |\delta| \; M \; \|x\| \; \|y\| \quad \blacksquare$$

where $M = \max(C, C_1)$.

Lemma 3. *If $S > 0$ is sufficiently small we have $m(S) = N$.*

Proof. Since Q_S is negative definite in the subspace $\mathbf{R}^N$, it follows that $m(Q_S) \geq N$ for all $S \in (0,T]$. However, if $b > 0$ is such that $F''(\bar{u}_S(\tau)) \leq b$ for all $\tau \in [0,T]$ it turns out that

$$Q_S(v) \geq \int_0^T [(T^2/S^2)|v'|^2 - b|v|^2]\, dt.$$

Now the quadratic form in the right-hand side is positive definite in $\tilde{H}$ whenever $b < 4\pi^2/S^2$, as follows from the Wirtinger inequality. Thus clearly $m(S) = N$ for these values of S. ■

Following a pattern similar to that of Salvatore [13,14] in order to study minimality for some wave equations, we can now establish:

Theorem 4. *Let $F \in C^3(\mathbf{R}^N, \mathbf{R})$ be such that $F''(u)$ is positive definite for all $u \in \mathbf{R}^N$ and $\bar{u} = \bar{u}(t)$ a non constant T-periodic solution of* (2). *Then the Morse index of $\bar{u}$ as a critical point of J is given by*

$$m(Q_T) = \gamma + N$$

where γ is the count of conjugate points to 0 in $(0, T)$ with respect to $\bar{u}$, each one taken as many times as its multiplicity.

Proof. According to Theorem 3 and Lemmas 1 and 2, the integer valued function $S \mapsto m(Q_S)$ is increasing and left continuous; it has a discontinuity at a point S if and only if S is conjugate to 0 with respect to $\bar{u}$, its jump being given by the multiplicity of S as a conjugate point. The theorem follows from these remarks together with Lemma 3. ■

We illustrate the use of this theorem on conjugate points by giving a proof of a version for (2) of a theorem of Clarke and Ekeland [5] (see also [6,2]), still using the theory of Lazer and Solimini [9].

When F'' is positive definite, F can have only one critical point. We therefore suppose that the only critical point of F is the origin and we let $0 < \lambda_1 \leq \cdots \leq \lambda_N$ be the eigenvalues of $F''(0)$.

Theorem 5. *Assume that F satisfies (A),(B),(D). Then if $T > 2\pi/\sqrt{\lambda_1}$, (2) has at least one solution with minimal period T.*

Proof. The hypothesis $T > 2\pi/\sqrt{\lambda_1}$ clearly implies that the Morse index of 0 as a critical point of J is at least $3N$. By Theorem 2.4 in [9] we conclude that J has a (non constant) critical point $\bar{u}(t)$ with Morse index $\leq N$. Suppose that the minimal period of $\bar{u}(t)$ is T/m, where $m \in \mathbf{N}$. Then there exist at least $m - 1$ conjugate points to 0 in $(0, T)$. Theorem 4 implies $N \geq m - 1 + N$. Therefore $m = 1$ and the proof is complete. ■

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