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**A NOTE ON NONUNITAL ABSORBING EXTENSIONS**

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# A NOTE ON NONUNITAL ABSORBING EXTENSIONS

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**Elliott and Kucerovsky stated that a nonunital extension of separable  $C^*$ -algebras with a stable ideal is nuclearly absorbing if and only if the extension is purely large. However, their proof was flawed. We give a counterexample to their theorem as stated, but establish an equivalent formulation of nuclear absorption under a very mild additional assumption to being purely large. In particular, if the quotient algebra is nonunital, then we show that the original theorem applies. We also examine how this affects results in classification theory.**

## 1. Introduction and a counterexample

A (unital) extension of  $C^*$ -algebras  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  is called (unital) *weakly nuclear* if there is a (unital) completely positive splitting  $\sigma : \mathfrak{A} \rightarrow \mathfrak{E}$  which is weakly nuclear, i.e., for every  $b \in \mathfrak{B}$  the map  $b\sigma(-)b^* : \mathfrak{A} \rightarrow \mathfrak{B}$  is nuclear. Such an extension is called trivial if we may take the weakly nuclear splitting to be a  $*$ -homomorphism. An extension is called (unital) nuclearly absorbing if it absorbs every trivial, (unital) weakly nuclear extension, i.e., the Cuntz sum of our given extension  $\epsilon$  with any trivial, (unital) weakly nuclear extension is strongly unitarily equivalent to  $\epsilon$ . A remarkable result of Elliott and Kucerovsky [2001] shows that a unital, separable extension with a stable ideal is unital nuclearly absorbing if and only if the extension is *purely large*. Recall that an extension  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  of  $C^*$ -algebras with  $\mathfrak{B}$  stable is called purely large if for any  $x \in \mathfrak{E} \setminus \mathfrak{B}$ , the hereditary  $C^*$ -subalgebra  $\overline{x\mathfrak{B}x^*}$  of  $\mathfrak{B}$  contains a stable,  $\sigma$ -unital  $C^*$ -subalgebra  $\mathfrak{D}$  which is full in  $\mathfrak{B}$ . Note that we have added the requirement that  $\mathfrak{D}$  be  $\sigma$ -unital, since this was implicitly used in [op. cit., Lemma 7] and since this is automatic in the separable case, which is our main concern.

In their paper, Elliott and Kucerovsky use the unital version above to obtain a nonunital version of this result, i.e., that a nonunital extension is nuclearly absorbing

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if and only if it is purely large. Unfortunately this is not true. We provide a counterexample below.

A stable  $C^*$ -algebra is said to have the *corona factorisation property* if all full multiplier projections are Murray–von Neumann equivalent, or equivalently, all norm-full multiplier projections are properly infinite. As is shown in [Kucerovsky and Ng 2006a], any full extension by a  $\sigma$ -unital, stable  $C^*$ -algebra with the corona factorisation property is purely large in the sense of [Elliott and Kucerovsky 2001]. Here *full* means that the Busby map is full, i.e., that it maps nonzero elements to full elements in the corona algebra.

It is known that  $C^*$ -algebras which do not have the corona factorisation property have rather exotic properties; see, e.g., [Kucerovsky and Ng 2006b]. It follows by [Robert 2011, Corollary 1] that any  $\sigma$ -unital, stable  $C^*$ -algebra with finite nuclear dimension, or, more generally, nuclear dimension less than  $\omega$ , has the corona factorisation property. Thus for classification purposes, the corona factorisation property is not really any restriction.

After receiving an early version of this note, Efren Ruiz constructed a counterexample to [Eilers et al. 2014, Theorem 4.9]. In fact, by using results from this note, Ruiz has constructed two graphs such that the induced  $C^*$ -algebras have exactly one nontrivial ideal, have isomorphic six-term exact sequences in  $K$ -theory with order and scale, but for which the  $C^*$ -algebras are nonisomorphic. This implies that we do not have a complete classification of graph  $C^*$ -algebras with exactly one nontrivial ideal using the above  $K$ -theoretic invariant, as opposed to what was previously believed. The counterexample is provided in Section 4. Fortunately, all recent classification results of *stable* graph  $C^*$ -algebras are unaffected by the issues addressed in this note, and hence stand as given.

As for general notation in this note we let  $\pi$  denote the quotient map from the multiplier algebra of some  $C^*$ -algebra to its corona algebra, and we consider an essential extension algebra as a  $C^*$ -subalgebra of the multiplier algebra of the ideal. When referring to full elements in a multiplier algebra, we always mean with respect to the norm topology, and *not* the strict topology.

A counterexample to [Elliott and Kucerovsky 2001, Corollary 16] could be as follows.

**Example 1.1.** Let  $\mathfrak{A} = \mathbb{C}$ ,  $\mathfrak{B} = \mathbb{K} \oplus \mathbb{K}$ , and consider the trivial extension  $\mathfrak{E}$  with splitting  $\sigma(1) = P \oplus 1 \in \mathcal{M}(\mathbb{K}) \oplus \mathcal{M}(\mathbb{K}) \cong \mathcal{M}(\mathfrak{B})$ , where  $P$  is a full projection in  $\mathcal{M}(\mathbb{K})$  such that  $1 - P$  is also full. The extension  $\mathfrak{E}$  is clearly full, and since  $\mathfrak{B}$  has the corona factorisation property, this implies that  $\mathfrak{E}$  is a nonunital, purely large extension. However, it does not absorb the zero extension, i.e., the extension with the zero Busby map. This is easily seen by projecting to the second coordinate in the corona algebra  $\pi_2 : \mathcal{Q}(\mathfrak{B}) \cong \mathcal{Q}(\mathbb{K}) \oplus \mathcal{Q}(\mathbb{K}) \rightarrow \mathcal{Q}(\mathbb{K})$ , since  $\pi_2(\tau(1)) = 1$  and  $\pi_2((\tau \oplus 0)(1))$  is a nontrivial projection, where  $\tau$  denotes the Busby map.

The flaw in the original proof is the claim that a nonunital extension  $\mathfrak{E}$  is purely large if and only if its unitisation  $\mathfrak{E}^\dagger$  is purely large. The sufficiency is trivial but the necessity is incorrect.

**Lemma 1.2.** *There exists a nonunital purely large extension such that the unitisation is not purely large.*

*Proof.* Let  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  denote the extension of [Example 1.1](#). The unitisation  $\mathfrak{E}^\dagger$  has Busby map  $\tau^\dagger : \mathbb{C} \oplus \mathbb{C} \rightarrow \mathcal{Q}(\mathbb{K}) \oplus \mathcal{Q}(\mathbb{K})$  given by

$$\tau^\dagger(1 \oplus 0) = \pi(P) \oplus 1 \quad \text{and} \quad \tau^\dagger(0 \oplus 1) = \pi(1 - P) \oplus 0.$$

Since  $\pi(1 - P) \oplus 0$  is not full in  $\mathcal{Q}(\mathbb{K}) \oplus \mathcal{Q}(\mathbb{K})$ ,  $\tau^\dagger$  is not a full homomorphism and thus the extension can not be purely large. □

### 2. Fixing the theorem

We will start by showing that the original theorem still holds, if we assume that the quotient is nonunital.

**Theorem 2.1.** *Let  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  be an extension of separable  $C^*$ -algebras with  $\mathfrak{B}$  stable. Suppose that  $\mathfrak{A}$  is nonunital. Then the extension is nuclearly absorbing if and only if it is purely large.*

*Proof.* As in [\[Elliott and Kucerovsky 2001, Section 16\]](#) the extension is nuclearly absorbing if and only if the unitised extension is unitaly nuclearly absorbing, which in turn is equivalent to the unitised extension being purely large. Thus it suffices to show that this is equivalent to the nonunitised extension being purely large. We use the same proof as in the original paper. Clearly the extension is purely large if the unitisation is purely large. Assume that the nonunital extension is purely large. Note, in particular, that the Busby map  $\tau$  is injective. It suffices to show that  $\overline{(1 - x)\mathfrak{B}(1 - x)^*}$  contains a stable  $C^*$ -subalgebra which is full in  $\mathfrak{B}$  for any  $x \in \mathfrak{E}$ . Suppose that  $(1 - x)\mathfrak{E} \subset \mathfrak{B}$ . Then  $\pi(x)$  is a unit for  $\pi(\mathfrak{E}) = \tau(\mathfrak{A}) \subset \mathcal{Q}(\mathfrak{B})$ . However, this contradicts the fact that  $\mathfrak{A}$  is nonunital, since the Busby map  $\tau$  is injective. Hence we may find  $x' \in \mathfrak{E}$  such that  $(1 - x)x' \notin \mathfrak{B}$ . Since

$$\overline{(1 - x)x'\mathfrak{B}((1 - x)x')^*} \subset \overline{(1 - x)\mathfrak{B}(1 - x)^*}$$

and since the nonunital extension is purely large, the former of these contains a stable  $C^*$ -subalgebra which is full in  $\mathfrak{B}$ . □

To prove a stronger result, where the assumption that the quotient being unital is removed, we will use the following lemma.

**Lemma 2.2.** *Let  $\mathfrak{B}$  be a stable, separable  $C^*$ -algebra, and let  $P \in \mathcal{M}(\mathfrak{B})$  be a full, properly infinite projection. Then the trivial extension of  $\mathbb{C}$  by  $\mathfrak{B}$  with splitting  $\sigma$  given by  $\sigma(1) = P$  is purely large.*

*Proof.* If  $P = 1$  then the extension is the canonical unitisation extension

$$0 \rightarrow B \rightarrow B^\dagger \rightarrow \mathbb{C} \rightarrow 0,$$

which is clearly self-absorbing. It follows from [Elliott and Kucerovsky 2001] that it is purely large.

It is well known, since  $\mathfrak{B}$  is stable, that  $P$  is full and properly infinite exactly when it is Murray–von Neumann equivalent to 1. Let  $v$  be an isometry such that  $vv^* = P$  and let  $t_1, t_2 \in \mathcal{M}(\mathfrak{B})$  be such that  $t_1 t_1^* + t_2 t_2^* = P = t_1^* t_1 = t_2^* t_2$ . Then  $s_1 := t_1 v$  and  $s_2 := t_2 + (1 - P)$  are the canonical generators of a unital copy of  $\mathcal{O}_2$  in  $\mathcal{M}(\mathfrak{B})$ , for which  $P = s_1 s_1^* + s_2 P s_2^*$ . Hence,

$$\pi(\sigma(1)) = \pi(s_1)1\pi(s_1)^* + \pi(s_2)\pi(P)\pi(s_2)^*,$$

which implies that our extension is the Cuntz sum of the unitisation extension and itself. It follows from [op. cit., Lemma 13] that our extension is purely large.  $\square$

Now for the stronger case where we allow the quotient to be unital.

**Theorem 2.3.** *Let  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  be an extension of separable  $C^*$ -algebras with  $\mathfrak{B}$  stable. The extension is nuclearly absorbing if and only if it is purely large and there is a full, properly infinite projection  $P \in \mathcal{M}(\mathfrak{B})$  such that  $P\mathfrak{E} \subset \mathfrak{B}$ .*

*Proof.* Assume that the extension is nuclearly absorbing. Then it absorbs the zero extension so we may assume that the Busby map is of the form  $\tau \oplus 0$ , where the symbol  $\oplus$  denotes a Cuntz sum. Let  $P = 0 \oplus 1$ . Then  $P\mathfrak{E} \subset \mathfrak{B}$  since  $\pi(P)$  annihilates the image of the Busby map. Moreover, the extension absorbs some purely large extension and is thus itself purely large by [loc. cit.].

Now suppose that the extension is purely large and that  $P$  is a full, properly infinite projection such that  $P\mathfrak{E} \subset \mathfrak{B}$ . As in the proof of Theorem 2.1 it suffices to show that the unitised extension is purely large. It is enough to show that  $\overline{(1-x)\mathfrak{B}(1-x)^*}$  contains a stable  $C^*$ -subalgebra which is full in  $\mathfrak{B}$  for any  $x \in \mathfrak{E}$ . Observe that

$$\overline{(1-x)P\mathfrak{B}P(1-x)^*} \subset \overline{(1-x)\mathfrak{B}(1-x)^*}.$$

Since  $(1-x)P = P - xP$  and  $xP \in \mathfrak{B}$ , it suffices to show that the extension

$$0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{B} + \mathbb{C}P \rightarrow \mathbb{C} \rightarrow 0$$

is purely large. This follows from Lemma 2.2.  $\square$

Note that an extension must be nonunital in order to satisfy the equivalent conditions in the above theorem. We immediately get the following corollary.

**Corollary 2.4.** *Let  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  be an extension of separable  $C^*$ -algebras with  $\mathfrak{B}$  stable. Then the extension is nuclearly absorbing if and only if it is purely large and absorbs the zero extension.*

When we assume that the ideal has the corona factorisation property, then we get a perhaps more hands-on way of checking if a full extension is nuclearly absorbing. To exhibit this we introduce the following definition.

**Definition 2.5.** Let  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  be an extension of  $C^*$ -algebras. We say that the extension is *unitisably full* if the unitised extension  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E}^\dagger \rightarrow \mathfrak{A}^\dagger \rightarrow 0$  is full.

It is clear that if an extension is unitisably full, then it is full and nonunital. If the quotient algebra  $\mathfrak{A}$  is unital, then the extension is unitisably full if and only if the extension is full and  $1_{\mathcal{Q}(\mathfrak{B})} - \tau(1_{\mathfrak{A}})$  is full, where  $\tau$  denotes the Busby map. Note that this case is our main concern due to [Theorem 2.1](#).

**Theorem 2.6.** *Let  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  be an extension of separable  $C^*$ -algebras, such that  $\mathfrak{B}$  is stable and has the corona factorisation property. Then the extension is nuclearly absorbing if and only if the extension is unitisably full.*

*Proof.* As in the proof of [Theorem 2.1](#) the extension is nuclearly absorbing if and only if the unitised extension is purely large. Since  $\mathfrak{B}$  has the corona factorisation property, this is the case if and only if the extension is unitisably full. □

We will end this section by showing that, in the absence of the corona factorisation property, there are purely large, unitisably full extensions which are not nuclearly absorbing. We will need a converse of [Lemma 2.2](#).

**Proposition 2.7.** *Let  $\mathfrak{B}$  be a stable, separable  $C^*$ -algebra, and let  $P \in \mathcal{M}(\mathfrak{B})$  be a full projection. Then the trivial extension of  $\mathbb{C}$  by  $\mathfrak{B}$  with splitting  $\sigma$  given by  $\sigma(1) = P$  is purely large if and only if  $P$  is properly infinite.*

*Proof.* One direction is [Lemma 2.2](#). Suppose that the extension is purely large. It suffices to show that the Cuntz sum  $P \oplus 0$  is properly infinite. The extension with splitting  $\sigma'(1) = P \oplus 0$  is purely large and absorbs the zero extension, and thus it is absorbing by [Corollary 2.4](#). Since the extension with splitting  $\sigma_0(1) = 1 \oplus 0$  is also absorbing, there is a unitary  $U \in \mathcal{M}(\mathfrak{B})$  such that  $U^*(P \oplus 0)U - 1 \oplus 0 \in \mathfrak{B}$ . Pick an isometry  $V \in \mathcal{M}(\mathfrak{B})$  such that  $V^*(1 \oplus 0)V = 1$ . Then

$$V^*(U^*(P \oplus 0)U - 1 \oplus 0)V = (UV)^*(P \oplus 0)UV - 1 \in \mathfrak{B}.$$

Since  $\mathfrak{B}$  is stable, we may find an isometry  $W$  such that

$$\|(UVW)^*(P \oplus 0)UVW - 1\| = \|W^*((UV)^*(P \oplus 0)UV - 1)W\| < 1.$$

This implies that  $P \oplus 0$ , and thus also  $P$ , is properly infinite. □

We can now extend our class of counterexamples to include purely large, unitisably full extensions  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  which are not nuclearly absorbing. In fact, such an extension can be made for any  $\mathfrak{B}$  without the corona factorisation property.

**Proposition 2.8.** *Let  $\mathfrak{B}$  be a stable, separable  $C^*$ -algebra which does not have the corona factorisation property. Then there is a purely large, unitisably full extension of  $\mathbb{C}$  by  $\mathfrak{B}$  which is not nuclearly absorbing.*

*Proof.* Let  $Q$  be a full multiplier projection which is not properly infinite, but where  $P := 1 - Q$  is properly infinite and full. Such a projection can be obtained by taking any full multiplier projection  $Q'$  which is not properly infinite, and letting  $Q = Q' \oplus 0$  be a Cuntz sum. In fact,  $P = 1 - Q$  will be properly infinite since it majorises the properly infinite, full projection  $0 \oplus 1$ . Consider the trivial extension  $\mathfrak{E}$  of  $\mathbb{C}$  by  $\mathfrak{B}$  with splitting  $\sigma(1) = P$ . The unitised extension has a splitting  $\sigma_1 : \mathbb{C} \oplus \mathbb{C} \rightarrow \mathcal{M}(\mathfrak{B})$  given by  $\sigma_1(1 \oplus 0) = P$  and  $\sigma_1(0 \oplus 1) = Q$ . Since both  $P$  and  $Q$  are full and orthogonal, the unitised extension is full.

By Proposition 2.7 the extension is purely large. Such an extension is nuclearly absorbing exactly when its unitisation is purely large [Elliott and Kucerovsky 2001]. If the unitisation was purely large, then  $(Q - b)\mathfrak{B}(Q - b)^*$  would contain a stable  $C^*$ -subalgebra full in  $\mathfrak{B}$ , for every  $b \in \mathfrak{B}$ . However, this would imply that the extension of  $\mathbb{C}$  by  $\mathfrak{B}$  with splitting  $\sigma_0(1) = Q$  is purely large, which it is not by Proposition 2.7. Hence the extension is not nuclearly absorbing.  $\square$

### 3. How this affects classification results

In the classification of nonsimple  $C^*$ -algebras, a popular result has been a result of Kucerovsky and Ng, which says that under the mild condition of the corona factorisation property on a stable, separable  $C^*$ -algebra  $\mathfrak{B}$ ,  $KK^1(\mathfrak{A}, \mathfrak{B})$  is the group of unitary equivalence classes of full extensions  $\mathfrak{E}$  of  $\mathfrak{A}$  by  $\mathfrak{B}$  for any nuclear separable  $C^*$ -algebra  $\mathfrak{A}$ . This is unfortunately not the case. The theorem only remains true if one adds the condition that the extensions are unitisably full as in Definition 2.5. See Theorem 3.2 below.

A counterexample of the original result could be as follows.

**Example 3.1.** Let  $0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{E} \rightarrow \mathfrak{A} \rightarrow 0$  be the extension from Example 1.1 with Busby map  $\tau$ . Then  $\mathfrak{B}$  has the corona factorisation property and the extension is full. As seen in Example 1.1,  $\tau$  and  $\tau \oplus 0$  are both nonunital and are *not* unitarily equivalent. However, they define the same element in  $KK^1(\mathfrak{A}, \mathfrak{B})$ .

The closest we get to fixing the theorem would be the following.

**Theorem 3.2.** *Let  $\mathfrak{B}$  be a separable, stable  $C^*$ -algebra. Then the following are equivalent:*

- (i)  $\mathfrak{B}$  has the corona factorisation property.
- (ii) For any separable  $C^*$ -algebra  $\mathfrak{A}$ ,  $KK_{\text{nuc}}^1(\mathfrak{A}, \mathfrak{B})$  is the group of strong unitary equivalence classes of all full, weakly nuclear extensions of  $\mathfrak{A}$  by  $\mathfrak{B}$  which absorb the zero extension.

- (iii) For any separable  $C^*$ -algebra  $\mathfrak{A}$ ,  $KK_{\text{nuc}}^1(\mathfrak{A}, \mathfrak{B})$  is the group of strong unitary equivalence classes of all full, weakly nuclear extensions  $\mathfrak{E}$  of  $\mathfrak{A}$  by  $\mathfrak{B}$ , for which there is a full projection  $P \in \mathcal{M}(\mathfrak{B})$  such that  $P\mathfrak{E} \subset \mathfrak{B}$ .
- (iv) For any separable  $C^*$ -algebra  $\mathfrak{A}$ ,  $KK_{\text{nuc}}^1(\mathfrak{A}, \mathfrak{B})$  is the group of strong unitary equivalence classes of all unitisably full, weakly nuclear extensions  $\mathfrak{E}$  of  $\mathfrak{A}$  by  $\mathfrak{B}$ .

*Proof.* It is well-known that  $KK_{\text{nuc}}^1(\mathfrak{A}, \mathfrak{B})$  is (isomorphic to) the group of strong unitary equivalence classes of weakly nuclear extensions of  $\mathfrak{A}$  by  $\mathfrak{B}$  which are nuclearly absorbing. This is proved in [Kirchberg 2000, Sections 3 and 4], though in a much more general setting. Alternatively, one can prove this exactly as one proves that  $KK^1(A, B) \cong \text{Ext}^{-1}(A, B)$ , and then note that the latter can be viewed as strong unitary equivalence classes of semisplit extensions of  $A$  by  $B$  which are absorbing. Note that weakly nuclear extensions are automatically semisplit. Thus (i)  $\Rightarrow$  (iv) by Theorem 2.6, and (iv)  $\Rightarrow$  (i) follows from Proposition 2.8.

If  $\mathfrak{B}$  has the corona factorisation property, then any full extension by  $\mathfrak{B}$  is purely large. Thus (i)  $\Rightarrow$  (iii) follows from Theorem 2.3.

Clearly (iii) is equivalent to the condition that for any  $C^*$ -algebra  $\mathfrak{A}$ , any full, weakly nuclear extension  $\mathfrak{E}$  of  $\mathfrak{A}$  by  $\mathfrak{B}$ , for which there is a full projection  $P \in \mathcal{M}(\mathfrak{B})$  such that  $P\mathfrak{E} \subset \mathfrak{B}$ , is nuclearly absorbing. If the extension  $\mathfrak{E}$  has Busby map  $\tau \oplus 0$ , then  $(0 \oplus 1)\mathfrak{E} \subset \mathfrak{B}$ , and thus (iii)  $\Rightarrow$  (ii).

It remains to show (ii)  $\Rightarrow$  (i). Let  $P \in \mathcal{M}(\mathfrak{B})$  be a full projection, and let  $P \oplus 0$  be the Cuntz sum. Note that  $Q \sim Q \oplus 0$  for any projection  $Q$ . By (ii), the extension with the Busby map  $\tau : \mathbb{C} \rightarrow \mathcal{Q}(\mathfrak{B})$  given by  $\tau(1) = \pi(P \oplus 0)$  is nuclearly absorbing. In particular, it absorbs the unitisation extension of  $\mathfrak{B}$ . Consider the lift  $\rho(1) = P \oplus 0$  of  $\tau$  and the canonical lift of the unitisation extension of  $\mathfrak{B}$ . We may find a unitary  $u \in \mathcal{M}(\mathfrak{B})$  such that  $u^*(P \oplus 0 \oplus 0)u - 0 \oplus 0 \oplus 1 \in \mathfrak{B}$ . If  $v$  is an isometry such that  $vv^* = 0 \oplus 0 \oplus 1$ , then  $(uv)^*(P \oplus 0 \oplus 0)uv - 1 \in \mathfrak{B}$ . Thus we may pick an isometry  $w$  such that

$$\|w^*((uv)^*(P \oplus 0 \oplus 0)uv - 1)w\| = \|s^*(P \oplus 0 \oplus 0)s - 1\| < 1,$$

where  $s$  is the isometry  $uvw$ . Hence  $P$  is Murray–von Neumann equivalent to  $s^*(P \oplus 0 \oplus 0)s$ , which is equivalent to 1.  $\square$

**Remark 3.3.** It clearly follows from the proof above that we could restrict our attention only to nuclear  $C^*$ -algebras  $\mathfrak{A}$  if desired. In this case we can remove the weakly nuclear condition, since all extensions of a separable, nuclear  $C^*$ -algebra are weakly nuclear by the lifting theorem of Choi and Effros [1976], and also we would have  $KK_{\text{nuc}}^1(\mathfrak{A}, \mathfrak{B}) = KK^1(\mathfrak{A}, \mathfrak{B})$ .

We still get some nice results for classification. This follows from the above theorem and Theorem 2.1.



**Corollary 3.4.** *Let  $\mathfrak{B}$  be a separable, stable  $C^*$ -algebra with the corona factorisation property and let  $\mathfrak{A}$  be a nonunital, separable  $C^*$ -algebra. Then  $KK_{\text{nuc}}^1(\mathfrak{A}, \mathfrak{B})$  is the group of strong unitary equivalence classes of all full, weakly nuclear extensions of  $\mathfrak{A}$  by  $\mathfrak{B}$ .*

**Corollary 3.5.** *Let  $\mathfrak{B}$  be a separable, stable  $C^*$ -algebra with the corona factorisation property and let  $\mathfrak{A}$  be a separable  $C^*$ -algebra. Let  $\mathfrak{E}_i$  be full, weakly nuclear extensions of  $\mathfrak{A}$  by  $\mathfrak{B}$ , with Busby maps  $\tau_i$ , for  $i = 1, 2$ . If  $[\tau_1] = [\tau_2] \in KK_{\text{nuc}}^1(\mathfrak{A}, \mathfrak{B})$ , then  $\mathfrak{E}_1 \otimes \mathbb{K} \cong \mathfrak{E}_2 \otimes \mathbb{K}$ .*

*Proof.* Given a Busby map  $\tau : \mathfrak{A} \rightarrow \mathcal{Q}(\mathfrak{B})$ , let  $\tau^s$  be the composition

$$\mathfrak{A} \otimes \mathbb{K} \xrightarrow{\tau \otimes \text{id}} \mathcal{Q}(\mathfrak{B}) \otimes \mathbb{K} \hookrightarrow \mathcal{Q}(\mathfrak{B} \otimes \mathbb{K}).$$

It is well known that the map  $KK_{\text{nuc}}^1(\mathfrak{A}, \mathfrak{B}) \rightarrow KK_{\text{nuc}}^1(\mathfrak{A} \otimes \mathbb{K}, \mathfrak{B} \otimes \mathbb{K})$  given by  $[\tau] \mapsto [\tau^s]$ , is an isomorphism (the proof is identical to the similar result in classical  $KK$ -theory). Thus  $\tau_1^s$  and  $\tau_2^s$  are strongly unitarily equivalent by [Corollary 3.4](#), and since their corresponding extension algebras are  $\mathfrak{E}_1 \otimes \mathbb{K}$  and  $\mathfrak{E}_2 \otimes \mathbb{K}$  respectively, it follows that  $\mathfrak{E}_1 \otimes \mathbb{K} \cong \mathfrak{E}_2 \otimes \mathbb{K}$ . □

**Remark 3.6.** Every result in this note holds with the ideal  $\mathfrak{B}$  being  $\sigma$ -unital instead of separable. The quotient  $\mathfrak{A}$  should still be separable. This is a special case of a much more general result in [\[Gabe and Ruiz 2015\]](#).

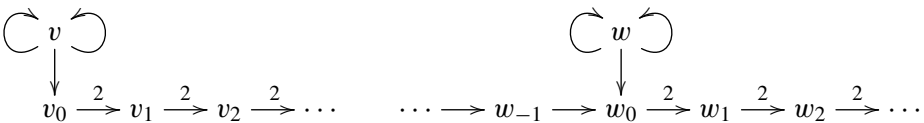
### 4. The counterexample of Ruiz

**Definition 4.1.** Let  $E = (E^0, E^1, r, s)$  be a (countable, directed) graph. The graph  $C^*$ -algebra  $C^*(E)$  is the universal  $C^*$ -algebra generated by mutually orthogonal projections  $p_v$  for  $v \in E^0$ , and isometries  $s_e$  for  $e \in E^1$ , which satisfy the relations

- $s_e^* s_f = \delta_{ef} p_{r(e)}$  for all  $e, f \in E^1$ ,
- $s_e s_e^* \leq p_{s(e)}$  for all  $e \in E^1$ ,
- $p_v = \sum_{e \in s^{-1}(\{v\})} s_e s_e^*$  for all  $v \in E^0$  satisfying  $0 < |s^{-1}(\{v\})| < \infty$ .

**Example 4.2** (the counterexample). Theorem 4.9 of [\[Eilers et al. 2014\]](#) states that if  $C^*(E)$  and  $C^*(F)$  are nonunital and both have exactly one nontrivial, two-sided, closed ideal, and the induced six-term exact sequences in  $K$ -theory are isomorphic, such that the isomorphisms on all  $K_0$ -groups preserve order and scale, then  $C^*(E) \cong C^*(F)$ . We will provide a counterexample to this result.

Let  $E$  and  $F$  be the respective graphs



Both  $C^*(E)$  and  $C^*(F)$  are nonunital, full extensions of the Cuntz algebra  $\mathcal{O}_2$  by the stabilisation of the CAR algebra  $M_{2^\infty} \otimes \mathbb{K}$ . The six-term exact sequences of the induced extensions, where we write the  $K_0$ -groups with order and scale as  $(K_0(\mathfrak{A}), K_0(\mathfrak{A})^+, \Sigma K_0(\mathfrak{A}))$ , are both isomorphic to

$$\begin{array}{ccc}
 (\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]_+, \mathbb{Z}[\frac{1}{2}]_+) & \xrightarrow{(\text{id}, \iota, \iota)} & (\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]) \longrightarrow (0, 0, 0) \\
 \uparrow & & \downarrow \\
 0 & \longleftarrow & 0 \longleftarrow 0
 \end{array}$$

where  $\mathbb{Z}[\frac{1}{2}]_+ = \mathbb{Z}[\frac{1}{2}] \cap [0, \infty)$  and  $\iota : \mathbb{Z}[\frac{1}{2}]_+ \hookrightarrow \mathbb{Z}[\frac{1}{2}]$  is the canonical inclusion. To compute the order and scale of  $K_0(C^*(E))$  and  $K_0(C^*(F))$  we simply use that both  $C^*(E)$  and  $C^*(F)$  contain full, properly infinite projections,  $p_v$  and  $p_w$ , respectively, and apply [Rørddam 2002, Proposition 4.1.4]. That  $p_v$  and  $p_w$  are properly infinite follows since  $v$  and  $w$  both support two loops, so it follows easily from the defining relations that they are properly infinite. Thus if [Eilers et al. 2014, Theorem 4.9] were true, it should follow that  $C^*(E) \cong C^*(F)$ . We will show that this is not the case, by showing that one extension with  $C^*(F)$  is nuclearly absorbing, but that the extension with  $C^*(E)$  is not.

**The extension with  $C^*(F)$  is nuclearly absorbing.** Recall that  $F^*$  denotes the set of paths in  $F$ , and that if  $\alpha = e_1 \cdots e_n \in F^*$  then  $s_\alpha := s_{e_1} \cdots s_{e_n}$ , and that  $r(\alpha) = r(e_n)$  and  $s(\alpha) = s(e_1)$ . Let  $\mathfrak{I}_F$  denote the unique nontrivial ideal in  $C^*(F)$ , which is isomorphic to  $M_{2^\infty} \otimes \mathbb{K}$ . By [Ruiz and Tomforde 2014], we may describe  $\mathfrak{I}_F$  as

$$\mathfrak{I}_F = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^*, r(\alpha) = r(\beta) = w_n \text{ for some } n \in \mathbb{Z}\}.$$

Let  $P = \sum_{n=1}^\infty w_{-n}$  which is easily seen to converge strictly in the multiplier algebra of  $\mathfrak{I}_F$ . We clearly have that  $PC^*(F) \subset \mathfrak{I}_F$ . Thus, if  $P$  is a full, properly infinite projection in  $\mathcal{M}(\mathfrak{I}_F)$ , then it follows from Theorem 2.3 that the extension with  $C^*(F)$  is nuclearly absorbing. Since  $\mathfrak{I}_F$  has the corona factorisation property, it suffices to show that  $P$  is full.

Note that  $M_{2^\infty} \cong p_{w_0} \mathfrak{I}_F p_{w_0}$ . Let  $\rho$  denote the unique tracial state on  $p_{w_0} \mathfrak{I}_F p_{w_0}$ , and  $\rho_\infty$  denote the induced trace function on  $\mathcal{M}(\mathfrak{I}_F)_+$ . It follows from [Rørddam 1991, Theorem 4.4] that  $P$  is full if and only if  $\rho_\infty(P) = \infty$ . Since  $p_{w_{-n}}$  for  $n > 0$  is Murray–von Neumann equivalent to  $p_{w_0}$ , it follows that  $\rho(p_{w_{-n}}) = \rho(p_{w_0}) = 1$  and thus  $\rho_\infty(P) = \sum_{n=1}^\infty \rho(p_{w_{-n}}) = \infty$ . Thus the extension

$$0 \rightarrow \mathfrak{I}_F \rightarrow C^*(F) \rightarrow C^*(F)/\mathfrak{I}_F \rightarrow 0$$

is nuclearly absorbing.

**The extension with  $C^*(E)$  is not nuclearly absorbing.** Let  $\mathfrak{J}_E$  denote the unique nontrivial ideal in  $C^*(E)$ , which is isomorphic to  $M_{2^\infty} \otimes \mathbb{K}$ . As above, we may describe  $\mathfrak{J}_E$  as

$$\mathfrak{J}_E = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) = v_n \text{ for some } n \geq 0\}.$$

To show that the extension  $\epsilon : 0 \rightarrow \mathfrak{J}_E \rightarrow C^*(E) \rightarrow \mathcal{O}_2 \rightarrow 0$  is not nuclearly absorbing, it suffices to show that the unitised extension  $\epsilon^\dagger : 0 \rightarrow \mathfrak{J}_E \rightarrow C^*(E)^\dagger \rightarrow \mathcal{O}_2 \oplus \mathbb{C} \rightarrow 0$  is *not* full. Let  $\sigma : C^*(E) \rightarrow \mathcal{M}(\mathfrak{J}_E)$  be the canonical  $*$ -homomorphism. Then  $C^*(E)^\dagger \cong \sigma(C^*(E)) + \mathbb{C}1_{\mathcal{M}(\mathfrak{J}_E)}$ . Note that  $1 - \sigma(p_v)$  is a lift of  $(0, 1) \in \mathcal{O}_2 \oplus \mathbb{C}$  (under the obvious identifications), so if  $\epsilon^\dagger$  is full, we should have that  $1 - \sigma(p_v) + \mathfrak{J}_E$  is full in  $\mathcal{Q}(\mathfrak{J}_E)$  by [Kucerovsky and Ng 2006a, Proposition 3.3]. Since  $\mathfrak{J}_E$  is stable, fullness of  $1 - \sigma(p_v) + \mathfrak{J}_E$  is equivalent to fullness of  $1 - \sigma(p_v)$  in  $\mathcal{M}(\mathfrak{J}_E)$ .

The corner in  $\mathfrak{J}_E$  generated by  $1 - \sigma(p_v)$  is easily seen to be

$$\overline{\text{span}}\{s_\alpha s_\beta^* : s(\alpha) \neq v \neq s(\beta)\},$$

which has an approximate unit  $(\sum_{n=0}^k p_{v_n})_{k=1}^\infty$ . Thus  $1 - \sigma(p_v) = \sum_{n=0}^\infty p_{v_n}$ . As above,  $M_{2^\infty} \cong p_{v_0} \mathfrak{J}_E p_{v_0}$ , so let  $\rho$  be the unique tracial state and  $\rho_\infty$  be the induced trace function on  $\mathcal{M}(\mathfrak{J}_E)_+$ . We have that  $\rho(p_{v_n}) = 2^{-n}$  so

$$\rho_\infty(1 - \sigma(p_v)) = \sum_{n=0}^\infty 2^{-n} < \infty.$$

It follows that  $1 - \sigma(p_v)$  is not full, and thus  $\epsilon$  is not nuclearly absorbing.

In particular,  $C^*(E) \not\cong C^*(F)$ .

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### References

- [Choi and Effros 1976] M. D. Choi and E. G. Effros, “The completely positive lifting problem for  $C^*$ -algebras”, *Ann. of Math.* (2) **104**:3 (1976), 585–609. [MR 0417795](#) [Zbl 0361.46067](#)
- [Eilers et al. 2014] S. Eilers, G. Restorff, and E. Ruiz, “The ordered  $K$ -theory of a full extension”, *Canad. J. Math.* **66**:3 (2014), 596–625. [MR 3194162](#) [Zbl 1309.46038](#)
- [Elliott and Kucerovsky 2001] G. A. Elliott and D. Kucerovsky, “An abstract Voiculescu–Brown–Douglas–Fillmore absorption theorem”, *Pacific J. Math.* **198**:2 (2001), 385–409. [MR 1835515](#) [Zbl 1058.46041](#)
- [Gabe and Ruiz 2015] J. Gabe and E. Ruiz, “Absorbing representations with respect to closed operator convex cones”, preprint, 2015. [arXiv 1503.07799](#)

- [Kirchberg 2000] E. Kirchberg, “Das nicht-kommutative Michael-Auswahlprinzip und die Klassifikation nicht-einfacher Algebren”, pp. 92–141 in *C\*-algebras* (Münster, 1999), edited by J. Cuntz and S. Echterhoff, Springer, Berlin, 2000. [MR 1796912](#) [Zbl 0976.46051](#)
- [Kucerovsky and Ng 2006a] D. Kucerovsky and P. W. Ng, “The corona factorization property and approximate unitary equivalence”, *Houston J. Math.* **32**:2 (2006), 531–550. [MR 2219330](#) [Zbl 1111.46050](#)
- [Kucerovsky and Ng 2006b] D. Kucerovsky and P. W. Ng, “S-regularity and the corona factorization property”, *Math. Scand.* **99**:2 (2006), 204–216. [MR 2289022](#) [Zbl 1139.46037](#)
- [Robert 2011] L. Robert, “Nuclear dimension and  $n$ -comparison”, *Münster J. Math.* **4** (2011), 65–72. [MR 2869254](#) [Zbl 1248.46040](#)
- [Rørdam 1991] M. Rørdam, “Ideals in the multiplier algebra of a stable  $C^*$ -algebra”, *J. Operator Theory* **25**:2 (1991), 283–298. [MR 1203034](#) [Zbl 0785.46049](#)
- [Rørdam 2002] M. Rørdam, “Classification of nuclear, simple  $C^*$ -algebras”, pp. 1–145 in *Classification of nuclear  $C^*$ -algebras. Entropy in operator algebras*, Encyclopaedia of Mathematical Sciences **126**, Springer, Berlin, 2002. [MR 1878882](#) [Zbl 1016.46037](#)
- [Ruiz and Tomforde 2014] E. Ruiz and M. Tomforde, “Ideals in graph algebras”, *Algebr. Represent. Theory* **17**:3 (2014), 849–861. [MR 3254772](#) [Zbl 1316.46043](#)

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