55. *A Note on Norms of Compression Operators on Function Spaces*

By Tetsuya SHIMOGAKI
Department of Mathematics, Tokyo Institute of Technology

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1970)

1. In what follows, let $(X, \|\cdot\|)$ be a *rearrangement invariant Banach function space*, i.e. a Banach space of Lebesgue integrable functions over a (finite or infinite) interval $(0, l)$ which satisfies the following conditions:

(1.1) $|g|\leq|f|$,[1] $f\in X$ *implies* $g\in X$ *and* $\|g\|\leq\|f\|$;

(1.2) $0\leq f_n\uparrow$, $\|f_n\|\leq M$, $n\geq 1$ *implies* $f=\bigcup_{n\geq 1} f_n\in X$ *and* $\|f\|=\sup_{n\geq 1}\|f_n\|$;

(1.3) *If* $0\leq f\in X$ *and* g *is equimeasurable with* f, *then* $g\in X$ *and* $\|f\|=\|g\|$.

From (1.2) it follows that the norm $\|\cdot\|$ on X is *semicontinuous*, i.e. $0\leq f_n\uparrow f, f_n, f\in X$ implies $\|f\|=\sup_{n\geq 1}\|f_n\|$. We denote by σ_a $(a>0)$ *the compression operator* on X:

(1.4) $$\sigma_a f=f_a, \quad f\in X,$$

where f_a is given by $f_a(x)=f(ax)$, if $ax\leq l$, and $f_a(x)=0$ otherwise. Since X is rearrangement invariant, the linear operators $\sigma_a, a>0$ are bounded, and $\|\sigma_a\|\leq 1$, if $a\geq 1$, and $1\leq\|\sigma_a\|\leq a^{-1}$, if $0<a<1$ [8]. The values of $\|\sigma_a\|, a>0$ play an important role to describe some interesting properties of the function space X concerning some interpolation properties for classes of linear operators [4, 8, 9], the Hardy Littlewood maximal functions [7], or the conjugate functions [1, 5].

Now we put for $a>0$ and $n\geq 1$

(1.5) $$\gamma_a^n=\sup\{\|\sigma_a f\|;\ f\in S_n, \|f\|=1\},$$

where S_n denotes the set of all positive simple functions with at most n-distinct nonzero values. Then we have for every $a>0$

$$\gamma_a^1\leq\gamma_a^2\leq\cdots\leq\|\sigma_a\|.$$

When X is an $\Lambda(\varphi)$-space or an $M(\varphi)$-space [2], $\gamma_a^1=\|\sigma_a\|$ holds; When X is an Orlicz space L_Φ we have $\gamma_a^2=\|\sigma_a\|$ [4]. Since $\|\cdot\|$ on X is semicontinuous, $\|\sigma_a\|=\sup_{n\geq 1}\gamma_a^n$ holds for every $a>0$. Now the following questions are naturally raised:

i) *For every* $a>0$, *is* $\|\sigma_a\|=\gamma_a^2$ *true?*; *For an arbitrary* X, *does there exist an* $n\geq 1$ *such that* $\|\sigma_a\|=\gamma_a^n$ *holds for each* $a>0$?

---

1) $|f|$ denotes the function $|f(x)|, x\in(0, l)$. $f\leq g$ means that $f(x)\leq g(x)$ a. e. on $(0, l)$.

ii) *For an arbitrary* X, *do there exist an* $M>0$ *and an* $n\geq 2$ *such that* $\|\sigma_a\|\leq M\gamma_a^n$ *holds for every* $a>0$?

The questions above are closely related to a problem concerning the Hardy Littlewood maximal functions. X is called to have the Hardy Littlewood property and is denoted by $X\in HLP$ [3], if X satisfies that $f\in X$ implies $\theta(f)\in X$, where $\theta(f)$ is the Hardy Littlewood maximal function of f. For any $x>0$ we put $x'=\min(x, l)$ and

(1.6) $$\tau_X(x)=\tau(x)=\|\chi_{(0,x')}\|,$$

and call it the *fundamental function* of X. Since X is rearrangement invariant, $\tau(x)=\|\chi_e\|$ holds for any measurable set $e\subset(0, l)$ with $mes(e)=x$. Recently R. O'Neil presented the following problem:[2]

iii) *Is it possible to characterize the property* $X\in HLP$ *in terms of the fundamental function* τ *of* X?

This problem can be stated in terms of compression operators, since it is known [7, 9] that $X\in HLP$ if and only if

(1.7) $$\lim_{a\to 0} a\|\sigma_a\|=0.$$

In this paper we shall show that there exists a *rearrangement invariant Banach function space* X *failing to satisfy* (1.7), *which has, however, the same fundamental function as the space* L^2. Since $L^2\in HLP$, this space gives the negative answer to the problem iii). At the same time, in view of $\gamma_a^n\leq n\gamma_a^1$ and $\gamma_a^1=\sup\{\tau(a^{-1}x)/\tau(x)\,;\,0<x\leq l\}$, $a>0$, it appears as a counter example to the question ii) (hence to i) also).

2. Let $l=1$ and define the functions κ_α, $0<\alpha\leq 1$ by

(2.1) $$\kappa_\alpha=\alpha^{-\frac{1}{2}}\chi_{(0,\alpha)}.$$

Let $n\geq 2$ be fixed, and put $\alpha_0=0$, $\alpha_1=2^{-2n(n-1)}\cdot n^{-1}$, and $\alpha_i=2^{2n(i-1)}\cdot\alpha_1=2^{2n(i-n)}\cdot n^{-1}$. Also define the functions ω_n by

(2.2) $$\omega_n=n^{-\frac{1}{2}}\Big(\bigcup_{i=1}^{n}\kappa_{\alpha_i}\Big)=n^{-\frac{1}{2}}\sum_{i=1}^{n}\kappa'_{\alpha_i},$$

where $\kappa'_{\alpha_i}=\kappa_{\alpha_i}\chi_{(\alpha_{i-1},\alpha_i)}$, $1\leq i\leq n$. By (2.1) and (2.2) we have

(2.3) $$\int_0^1\sum_{i<\nu}\kappa'_{\alpha_i}dx\leq 2^{n(\nu-n)}\cdot n^{-\frac{1}{2}}\cdot(2^n-1)^{-1},\quad 1<\nu\leq n.$$

We denote by $\langle f, g\rangle$ the integral $\int_0^1 fg\,dx$. Then, we have

(2.4) $$\langle\omega_n,\kappa_{\alpha_\nu}\rangle\leq n^{-\frac{1}{2}}+n^{-\frac{1}{2}}(2^n-1)^{-1},\quad 1\leq\nu\leq n.$$

In fact, $\langle\omega_n,\kappa_{\alpha_\nu}\rangle\leq n^{-\frac{1}{2}}\langle\kappa_{\alpha_\nu},\kappa_{\alpha_\nu}\rangle+\sum_{i<\nu}n^{-\frac{1}{2}}\langle\kappa'_{\alpha_i},\kappa_{\alpha_\nu}\rangle\leq n^{-\frac{1}{2}}+n^{-\frac{1}{2}}\alpha_\nu^{-\frac{1}{2}}\int_0^1\sum_{i<\nu}\kappa'_{\alpha_i}dx\leq n^{-\frac{1}{2}}+n^{-\frac{1}{2}}(2^n-1)^{-1}$. From this we can derive further by an elementary calulation

(2.5) $$\langle\omega_n,\kappa_\alpha\rangle\leq n^{-\frac{1}{2}}+n^{-\frac{1}{2}}(2^n-1)^{-1}\leq 1,\quad 0<\alpha\leq 1.$$

Also we have obviously

2) The author expresses his thanks to Professor J. Ishii for informing him of this problem raised by Professor R. O'Neil.

(2.6) $$\langle \omega_n, \omega_n \rangle = 1 - 2^{-2n}.$$

Next, we estimate the value $\langle \sigma_{n^{-1}}\omega_n, \omega_n \rangle$ from above. Decomposing $\sigma_n{}^{-1}\omega_n$ into $\omega'_n + \omega''_n$, where

$$\omega'_n = n^{-\frac{1}{2}} \sum_{\nu=1}^{n} \alpha_\nu^{-\frac{1}{2}} \chi_{(n\alpha_{\nu-1}, \beta_{\nu-1})}, \quad \omega''_n = n^{-\frac{1}{2}} \sum_{\nu=1}^{n} \alpha_\nu^{-\frac{1}{2}} \chi_{(\beta_{\nu-1}, n\alpha_\nu)} \text{ and}$$

$\beta_{\nu-1} = n\alpha_{\nu-1} + \alpha_\nu - \alpha_{\nu-1}$, we get

$$\begin{aligned} \langle \sigma_{n^{-1}}\omega_n, \omega_n \rangle &= \langle \omega'_n, \omega_n \rangle + \langle \omega''_n, \omega_n \rangle \\ &\leq \langle \omega_n, \omega_n \rangle + n^{-1} \sum_{\nu=1}^{n-1} \alpha_\nu^{-\frac{1}{2}} \alpha_{\nu+1}^{-\frac{1}{2}} (n-1)(\alpha_\nu - \alpha_{\nu-1}) \\ &\leq 1 + (n-1)n^{-1} \sum_{\nu=1}^{n-1} \alpha_\nu^{-1} 2^{-n} (\alpha_\nu - \alpha_{\nu-1}), \end{aligned}$$

which implies

(2.7) $$\langle \sigma_{n^{-1}}\omega_n, \omega_n \rangle \leq 1 + 2^{-n}(n-1).$$

Since $\langle \sigma_{n^{-1}}\omega_n, \kappa_\alpha \rangle = n^{\frac{1}{2}} \langle \omega_n, \kappa_{\alpha n^{-1}} \rangle$, we obtain by (2.5)

(2.8) $$\langle \sigma_{n^{-1}}\omega_n, \kappa_\alpha \rangle \leq 1 + (2^n - 1)^{-1}.$$

Thus, for every $n \geq 2$, we can define ω_n by (2.2) satisfying all the conditions (2.4)$\sim$(2.8). Now we pick up a subsequence $\{\omega_{n_\nu}\}$ of $\{\omega_n\}$ in such a way that $n_{\nu+1} > 2^{(2n_\nu^2)} \cdot n_\nu$, $\nu \geq 1$, and put $\bar{\omega}_\nu = \omega_{n_\nu}$. Then we have

(2.9) $$\begin{cases} \langle \bar{\omega}_\nu, \kappa_\alpha \rangle \leq n_\nu^{-\frac{1}{2}} + n_\nu^{-\frac{1}{2}} (2^{n_\nu} - 1)^{-1}, & 0 < \alpha \leq 1; \\ \langle \bar{\omega}_\nu, \bar{\omega}_\nu \rangle = 1 - 2^{-2n_\nu}, & \nu \geq 1; \\ \langle \bar{\omega}_\nu, \bar{\omega}_\mu \rangle \leq n_\nu^{-\frac{1}{2}}, & \text{if } \mu > \nu. \end{cases}$$

The last inequality of (2.9) is derived from (2.5) and the fact that $\bar{\omega}_\mu \chi_{(0,\beta)} = \bar{\omega}_\mu$ and $\bar{\omega}_\nu \chi_{(0,\beta)} \leq n_\nu^{-\frac{1}{2}} \kappa_\beta$, where $\beta = \alpha_1$ defined above for $n = n_\nu$. Putting $g_\nu = \sigma_{n_\nu^{-1}} \bar{\omega}_\nu$, we get from (2,7), (2.8) and (2.9)

(2.10) $$\begin{cases} \langle g_\nu, \kappa_\alpha \rangle \leq 1 + (2^{n_\nu} - 1)^{-1}; \\ \langle g_\nu, \bar{\omega}_\nu \rangle \leq 1 + 2^{-n_\nu}(n_\nu - 1); \\ \langle g_\nu, \bar{\omega}_\mu \rangle \leq 1, \quad \text{if } \mu \neq \nu. \end{cases}$$

Now let C be the set: $\{\kappa_\alpha : 0 < \alpha \leq 1\} \cup \{\bar{\omega}_\nu : \nu \geq 2\}$, and define a space X of integrable functions by

(2.11) $$X = \left\{ f : \sup_{c \in C} \int c f^* dx < \infty \right\} = \bigcap_{c \in C} \Lambda(c),$$

where f^* is the decreasing rearrangement of the function $|f|$. The space X, equipped with the norm: $\|f\| = \sup_{c \in C} \int c f^* dx, f \in X$, is a rearrangement invariant Banach function space including the space L^2. Since, in virtue of (2.9), $\kappa_\alpha \in X$ and $\|\kappa_\alpha\| = 1$ for all $0 < \alpha \leq 1$, $\tau_X(\alpha) = \alpha^{\frac{1}{2}} = \tau_{L^2}(\alpha)$. On account of (2.10), $g_\nu \in X$ and $\lim_{\nu \to \infty} \|g_\nu\| \leq 1$. On the other hand, $\lim_{\nu \to \infty} \|\sigma_{n_\nu} g_\nu\| = \lim_{\nu \to \infty} \|\bar{\omega}_\nu\| \geq 1$ by (2.9). Hence, $\lim_{\nu \to \infty} \|\sigma_{n_\nu}\| = 1$. Consequently, the fundamental function τ_X of X coincides with τ_{L^2} of the space L^2, but the following condition (2.12) fails to be true in X:

(2.12) $$\lim_{a \to \infty} \|\sigma_a\| = 0.$$

The conjugate space $Y=\bar{X}$ of X is a rearrangement invariant Banach function space in which the condition (1.7) is violated, since the conditions (1.7) and (2.12) are mutually conjugate. Since Y is also rearrangement invariant, the fundamental function $\tau_Y(x)$ of Y is $\tau_X(x)^{-1}\cdot x$, hence $\tau_Y(x)=\tau_X(x)=x^{\frac{1}{2}}$ for all $x\in(0,1)$. The fundamental function of Y coincides with that of L^2, but the condition (1.7) is not satisfied. Therefore, *the construction of the space Y gives the negative answer to the problem* iii), *and hence both X and Y provide counter examples to the question* ii) *at the same time.*

References

[1] D. W. Boyd: The Hilbert transform on rearrangement-invariant spaces. Canad. J., **19**, No. 3, 599–616 (1967).
[2] G. G. Lorentz: Some new functional spaces. Ann. Math., **51**, 27–55 (1950).
[3] ——: Majorants in spaces of integrable functions. Amer. J. Math., **77**, 484–492 (1955).
[4] G. G. Lorentz and T. Shimogaki: Majorants for interpolation theorems. Publ. of Ramanujan Inst., No. 1, 115–122 (1969).
[5] R. O'Neil and G. Weiss: The Hilbert transform and rearrangement of functions. Studia Math., **23**, 189–198 (1963).
[6] W. A. J. Luxemburg: Rearrangement invariant Banach function spaces. Proc. Symposium in Analysis, Queen's University, **10**, 83–144 (1967).
[7] T. Shimogaki: Hardy-Littlewood majorants in function spaces. J. Math. Soc. Japan, **17**, 365–373 (1965).
[8] ——: On the complete continuity of operators in an interpolation theorem. J. Fac. Sci. Hokkaido Univ., Ser. 1, **20**, No. 3, 109–114 (1968).
[9] ——: An interpolation theorem on Banach function spaces. Studia Math., **31**, 233–240 (1968).