A NOTE ON ODD ZETA VALUES OVER ANY NUMBER FIELD AND EXTENDED EISENSTEIN SERIES

SOUMYARUP BANERJEE, RAJAT GUPTA AND RAHUL KUMAR

ABSTRACT. In this article, we have studied transformation formulas of zeta function at odd integers over an arbitrary number field which in turn generalizes Ramanujan's identity for the Riemann zeta function. The above transformation leads to a new number field extension of Eisenstein series, which satisfies the transformation $z \mapsto -1/z$ like an integral weight modular form over $SL_2(\mathbb{Z})$. The results provide number of important applications, which are important in studying the behaviour of odd zeta values as well as Lambert series in an arbitrary number field.

1. INTRODUCTION

One of the central objects in analytic number theory is the Riemann zeta function $\zeta(s)$, which plays a crucial role in studying the distribution of primes and has applications in various branches of physics, probability theory, and applied statistics. Over the years, the zeta values and its arithmetic behaviour have been studied extensively by many mathematicians. Euler discovered the zeta values at positive even integral arguments in 1734, which precisely states that for any natural number m,

$$\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m} B_{2m}}{2(2m)!},\tag{1.1}$$

where B_m denotes the *m*-th Bernoulli number. Thus, it follows from the above formula that the zeta values at any positive even integer are transcendental due to the well-known fact that π is transcendental and B_m is rational.

The arithmetic nature of $\zeta(s)$ at odd integers is still mysterious. Apéry [1], [2] established the irrationality of $\zeta(3)$. Recently, Zudilin [29] has shown that at least one of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. In this direction, a celebrated result was obtained by Lerch [21], which precisely states that, for $m \geq 0$,

$$\zeta(4m+3) = 2^{4m+2} \pi^{4m+3} \sum_{j=0}^{2m+2} (-1)^{j+1} \frac{B_{2j} B_{4m+4-2j}}{(2j)!(4m+4-2j)!} - 2\sum_{n=1}^{\infty} \frac{n^{-4m-3}}{e^{2\pi n}-1}.$$
 (1.2)

Since the infinite series is rapidly convergent in the above formula, Berndt and Straub [4] state that " $\zeta(4m+3)$ is almost a rational multiple of π^{4m+3} ".

Ramanujan [24, p. 173, Ch. 14, Entry 21(i)] obtained an elegant generalization of Lerch identity (1.2), namely, for any non-zero integer m and $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$,

$$\alpha^{-m} \left\{ \frac{1}{2} \zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2n\alpha} - 1} \right\} = (-\beta)^{-m} \left\{ \frac{1}{2} \zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2n\beta} - 1} \right\} + \frac{1}{\pi^{2m+2}} \sum_{j=0}^{m+1} (-1)^{m+j} \zeta(2j) \zeta(2m-2j+2) \alpha^{m+1-j} \beta^j.$$
(1.3)

The above identity encodes even integral weight Eisenstein series transformation $z \mapsto -1/z$ over the full modular group as well as the Eichler integrals and has some important applications in theoretical

²⁰¹⁰ Mathematics Subject Classification. Primary 11M06, 11R42, 33E20; Secondary 33C10.

Keywords and phrases. Odd zeta values, Dedekind zeta function, Ramanujan's formula, Extended Eisenstein series, Number field.

computer science [cf. [17]]. Ramanujan's identity (1.3) has been generalized in different directions (cf. [6], [8], [9], [10], [11], [12], [15] and [19]).

The zeta values at even integers over any number field have been studied extensively. Klingen [18] and Siegel [25] generalized the result (1.1) of Euler for any totally real number field. The result precisely states as for any totally real number field \mathbb{K} of degree d with discriminant $D_{\mathbb{K}}$,

$$\zeta_{\mathbb{K}}(2m) = \frac{q_m \pi^{2md}}{\sqrt{D_K}} \qquad (m \in \mathbb{N}), \tag{1.4}$$

where q_m is some fixed non-zero rational number. Zagier [28] obtained a closed form expression for $\zeta_{\mathbb{K}}(2)$ over any number field \mathbb{K} . Here, we investigate transformation formula for odd zeta values over an arbitrary number field, which in turn, generalizes (1.3). Before stating the new transformation formula, we define the Dedekind zeta function.

Let \mathbb{K} be any number field with extension degree $[\mathbb{K} : \mathbb{Q}] = d$ and signature (r_1, r_2) (i. e., $d = r_1 + 2r_2$) and $D_{\mathbb{K}}$ denotes the absolute value of the discriminant of \mathbb{K} . Let $\mathcal{O}_{\mathbb{K}}$ be its ring of integers and \mathfrak{N} be the norm map of \mathbb{K} over \mathbb{Q} . Then the *Dedekind zeta function* attached to number field \mathbb{K} is defined by

$$\zeta_{\mathbb{K}}(s) := \sum_{\mathsf{a} \subset \mathcal{O}_{\mathbb{K}}} \frac{1}{\mathfrak{N}(\mathsf{a})^s},$$

for $\operatorname{Re}(s) > 1$, where **a** runs over the non-zero integral ideals of $\mathcal{O}_{\mathbb{K}}$. If $V_{\mathbb{K}}(m)$ counts the number of non-zero integral ideals in $\mathcal{O}_{\mathbb{K}}$ with norm m, then $\zeta_{\mathbb{K}}(s)$ can also be written as

$$\zeta_{\mathbb{K}}(s) = \sum_{m=1}^{\infty} \frac{V_{\mathbb{K}}(m)}{m^s}, \qquad (\operatorname{Re}(s) > 1).$$
(1.5)

The function $\zeta_{\mathbb{K}}(s)$ extends analytically to the entire complex plane except for a simple pole at s = 1. We denote the residue at s = 1 by H, which can be expressed by the analytic class number formula,

$$H = \frac{2^{r_1} (2\pi)^{r_2} hR}{w\sqrt{D_{\mathbb{K}}}},$$

where h denotes the class number, R is the regulator and w is the number of roots of unity in \mathbb{K} . Throughout, we will stick with the above notations for an arbitrary number field \mathbb{K} .

Ramanujan's identity (1.3) mainly involves the Lambert series

$$\sum_{n=1}^{\infty} \frac{n^a}{e^{ny} - 1} = \sum_{n=1}^{\infty} \sigma_a(n) e^{-ny},$$
(1.6)

where $\sigma_a(n) := \sum_{d|n} d^a$. The above series plays a crucial role in the theory of modular forms as for a = 2m - 1 with $m \in \mathbb{N}$ and $y = -2\pi i z$ either of the above series is essentially an Eisenstein series of weight 2m over the full modular group. Here, the case m = 1 is Eisenstein series of weight 2, which is not exactly the modular form but the quasi-modular form. The series for a = -2m - 1 with $m \in \mathbb{N}$ and $y = -2\pi i z$ represents the Eichler integral corresponding to the weight 2m Eisenstein series.

Here, we consider an infinite series involving Meijer G-function [cf. §2], which generalizes the above Lambert series (1.6) for an arbitrary number field \mathbb{K} as

$$\sum_{\mathsf{a}\subset\mathcal{O}_{\mathbb{K}}}\mathfrak{N}(\mathsf{a})^{a}\Omega_{\mathbb{K}}\left(\frac{\mathfrak{N}(\mathsf{a})y}{D_{\mathbb{K}}}\right) = \sum_{n=1}^{\infty} V_{\mathbb{K}}(n)n^{a}\Omega_{\mathbb{K}}\left(\frac{ny}{D_{\mathbb{K}}}\right)$$
(1.7)

where,

$$\Omega_{\mathbb{K}}(x) := \frac{2^{1-r_1-r_2}}{\pi^{1-\frac{r_1}{2}}} \sum_{j=1}^{\infty} V_{\mathbb{K}}(j) G_{0,2d}^{d+1,0} \left(\left(0 \right)_{r_1+r_2}, \left(\frac{1}{2} \right)_{r_2+1}; \left(\frac{1}{2} \right)_{r_1+r_2-1}, \left(0 \right)_{r_2} \right| \frac{x^2 j^2}{4^d} \right).$$
(1.8)

In particular, for $\mathbb{K} = \mathbb{Q}$, $\Omega_{\mathbb{K}}(x)$ reduces to $\frac{1}{e^x - 1}$, which has been derived in (3.15).

In the following theorem we obtain a transformation formula of Dedekind zeta function at odd integers over an arbitrary number field \mathbb{K} , which provides (1.3) as a special case when $\mathbb{K} = \mathbb{Q}$.

Theorem 1.1. Let \mathbb{K} be an arbitrary number field. For any non-zero integer m and $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$ with $\alpha\beta = \pi^{2d}$, the following identity holds

$$\alpha^{-m} \left\{ \frac{H}{2} \zeta_{\mathbb{K}}(2m+1) + \mathcal{C}_{\mathbb{K}} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \Omega_{\mathbb{K}} \left(\frac{2^{d} n \alpha}{D_{\mathbb{K}}} \right) \right\}$$

= $(-\beta)^{-m} \left\{ \frac{H}{2} \zeta_{\mathbb{K}}(2m+1) + \mathcal{C}_{\mathbb{K}} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \Omega_{\mathbb{K}} \left(\frac{2^{d} n \beta}{D_{\mathbb{K}}} \right) \right\}$
+ $\frac{1}{\pi^{(2m+1)d+1}} \sum_{j=0}^{m+1} (-1)^{m+j} \zeta_{\mathbb{K}}(2j) \zeta_{\mathbb{K}}(2m-2j+2) \alpha^{m+1-j} \beta^{j}.$ (1.9)

where $C_{\mathbb{K}} = \frac{2^{r_1-1}(2\pi)^{r_2}}{\sqrt{D_{\mathbb{K}}}}.$

The following corollary provides Ramanujan's identity as a special case of the above theorem.

Corollary 1.2. For $\mathbb{K} = \mathbb{Q}$, Ramanujan's identity (1.3) holds.

When \mathbb{K} is any real quadratic field, Franke [14] has studied an analogue of Ramanujan's identity but could not get the result in a symmetric form. The following corollary follows from Theorem 1.1, which provides the transformation formula in symmetric form.

Corollary 1.3. Let \mathbb{K} be any real quadratic field. Let

$$\Re(x) := \sum_{j=1}^{\infty} V_{\mathbb{K}}(j) \left(K_0(2\epsilon\sqrt{jx}) + K_0(2\overline{\epsilon}\sqrt{jx}) \right),$$

where $\epsilon = e^{i\pi/4}$ and $K_{\nu}(x)$ denotes the Bessel function of the second kind (cf. §2). If $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$ with $\alpha\beta = \pi^4$ and m be any non-zero integer. Then

$$\alpha^{-m} \left\{ \frac{H}{2} \zeta_{\mathbb{K}}(2m+1) + \frac{2}{\sqrt{D}_{\mathbb{K}}} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \Re\left(\frac{4n\alpha}{D_{\mathbb{K}}}\right) \right\}$$
$$= (-\beta)^{-m} \left\{ \frac{H}{2} \zeta_{\mathbb{K}}(2m+1) + \frac{2}{\sqrt{D}_{\mathbb{K}}} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \Re\left(\frac{4n\beta}{D_{\mathbb{K}}}\right) \right\}$$
$$+ \frac{1}{\pi^{4m+3}} \sum_{j=0}^{m+1} (-1)^{m+j} \zeta_{\mathbb{K}}(2j) \zeta_{\mathbb{K}}(2m-2j+2) \alpha^{m+1-j} \beta^{j}.$$
(1.10)

The following corollary provides transformation formula for $\zeta_{\mathbb{K}}(2m+1)$, when \mathbb{K} is an imaginary quadratic field.

Corollary 1.4. Let \mathbb{K} be any imaginary quadratic field. Let

$$\mathcal{I}(x) := \frac{2i}{\pi} \sum_{j=1}^{\infty} V_{\mathbb{K}}(j) \left(K_0(2\epsilon\sqrt{jx}) - K_0(2\overline{\epsilon}\sqrt{jx}) \right).$$
(1.11)

For $m \in \mathbb{Z} \setminus \{0\}$ and $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$ with $\alpha\beta = \pi^4$, we have

$$\alpha^{-m} \left\{ \frac{H}{2} \zeta_{\mathbb{K}}(2m+1) + \frac{\pi}{\sqrt{D}_{\mathbb{K}}} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \mathcal{I}\left(\frac{4n\alpha}{D_{\mathbb{K}}}\right) \right\}$$

$$= (-\beta)^{-m} \left\{ \frac{H}{2} \zeta_{\mathbb{K}}(2m+1) + \frac{\pi}{\sqrt{D}_{\mathbb{K}}} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \mathcal{I}\left(\frac{4n\beta}{D_{\mathbb{K}}}\right) \right\} + \frac{1}{\pi^{4m+3}} \sum_{j=0}^{m+1} (-1)^{m+j} \zeta_{\mathbb{K}}(2j) \zeta_{\mathbb{K}}(2m-2j+2) \alpha^{m+1-j} \beta^{j}.$$

We next generalize Lerch identity (1.2) for an arbitrary number field K.

Theorem 1.5. Let \mathbb{K} be an arbitrary number field. For any integer m, the following identity holds :

$$\zeta_{\mathbb{K}}(4m+3) = \frac{1}{\pi H} \sum_{j=0}^{2m+2} (-1)^{j+1} \zeta_{\mathbb{K}}(2j) \zeta_{\mathbb{K}}(4m-2j+4) - \frac{2^{r_1}(2\pi)^{r_2}}{H\sqrt{D_{\mathbb{K}}}} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{4m+3}} \Omega_{\mathbb{K}}\left(\frac{(2\pi)^d n}{D_{\mathbb{K}}}\right).$$
(1.12)

The arithmetic nature of $\zeta_{\mathbb{K}}(2m+1)$ for m odd, follows from the above theorem.

Corollary 1.6. For \mathbb{K} totally real, atleast one of $\zeta_{\mathbb{K}}(4m+3)$ and $\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{4m+3}} \Omega_{\mathbb{K}}\left(\frac{(2\pi)^d n}{D_{\mathbb{K}}}\right)$ is transcendental.

Remark. When \mathbb{K} is any real quadratic field, the asymptotics of $K_0(x)$ as $x \to \infty$ readily implies that the infinite series in the above corollary is rapidly convergent. Thus, we can conclude that $\zeta_{\mathbb{K}}(4m+3)$ is "almost" an algebraic multiple of π^n for some fixed positive integer n.

We next investigate the applications of Theorem 1.1 in the theory of modular form. Let f be a holomorphic function on an Poincare upper half-plane \mathfrak{H} . For any positive integer k, f is said to be a modular form of weight k over $SL_2(\mathbb{Z})$, if it satisfies

$$(i) f(z+1) = f(z)$$

(ii) $f(-1/z) = (-z)^k f(z).$ (1.13)

In the theory of modular forms, the crucial example of modular forms is the Eisenstein series. For $z \in \mathfrak{H}$, the 'normalized Eisenstein series' of even integral weight k over $\mathrm{SL}_2(\mathbb{Z})$ can be defined by the following Fourier series expansion :

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1}}{e^{-2\pi i n z} - 1}.$$

We here extend the above series definition of Eisenstein series over an arbitrary number field K. For an even positive integer k and $z \in \mathfrak{H}$, we define

$$\mathcal{G}_{k,\mathbb{K}}(z) := \frac{H}{2\mathcal{C}_{\mathbb{K}}} \zeta_{\mathbb{K}}(1-k) + \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-k}} \Omega_{\mathbb{K}} \left(-\frac{(2\pi)^d inz}{D_{\mathbb{K}}}\right).$$
(1.14)

If \mathbb{K} is totally real, the above definition can be normalized as

$$\mathcal{E}_{k,\mathbb{K}}(z) := \frac{2\mathcal{C}_{\mathbb{K}}}{H\zeta_{\mathbb{K}}(1-k)}\mathcal{G}_{k,\mathbb{K}}(z) = 1 + \frac{2\mathcal{C}_{\mathbb{K}}}{H\zeta_{\mathbb{K}}(1-k)}\sum_{n=1}^{\infty}\frac{V_{\mathbb{K}}(n)}{n^{1-k}}\Omega_{\mathbb{K}}\left(-\frac{(2\pi)^d inz}{D_{\mathbb{K}}}\right).$$

In particular, for $\mathbb{K} = \mathbb{Q}$, $\mathcal{E}_{k,\mathbb{Q}}(z) = E_k(z)$ (using (3.15) below). For \mathbb{K} not totally real, the first term in (1.14) vanishes. Thus we have

$$\mathcal{G}_{k,\mathbb{K}}(z) = \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-k}} \Omega_{\mathbb{K}} \left(-\frac{(2\pi)^d inz}{D_{\mathbb{K}}} \right).$$

In the following theorem, we have shown that the extended Eisenstein series satisfies the transformation formula (1.13) of an integral weight modular form over $SL_2(\mathbb{Z})$.

Theorem 1.7. Let \mathbb{K} be an arbitrary number field. For k be an even integer greater than 2 and $z \in \mathfrak{H}$, we have

$$\mathcal{G}_{k,\mathbb{K}}(-1/z) = z^k \, \mathcal{G}_{k,\mathbb{K}}(z).$$

Remark. In general, $\mathcal{G}_{k,\mathbb{K}}(z)$ is not a periodic function. Thus, it is not exactly a modular form over $\mathrm{SL}_2(\mathbb{Z})$.

The above theorem can be re-stated in the following symmetric form.

Theorem 1.8. Let K be an arbitrary number field. For any non-zero positive integer m > 1 and $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$ with $\alpha\beta = \pi^{2d}$, we have

$$\alpha^m \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}}\left(\frac{2^d n\alpha}{D_{\mathbb{K}}}\right) - (-\beta)^m \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}}\left(\frac{2^d n\beta}{D_{\mathbb{K}}}\right) = -\frac{H}{2\mathcal{C}_{\mathbb{K}}} \{\alpha^m - (-\beta)^m\} \zeta_{\mathbb{K}}(1-2m).$$
(1.15)

In particular, for \mathbb{K} not totally real, we have

$$\alpha^m \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}}\left(\frac{2^d n\alpha}{D_{\mathbb{K}}}\right) = (-\beta)^m \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}}\left(\frac{2^d n\beta}{D_{\mathbb{K}}}\right).$$

As an easy consequence of the above theorem, the following corollary provides an explicit evaluation of generalized Lambert series.

Corollary 1.9. For an arbitrary number field \mathbb{K} and for any odd natural number m > 1,

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}}\left(\frac{(2\pi)^d n}{D_{\mathbb{K}}}\right) = -\frac{H}{2\mathcal{C}_{\mathbb{K}}} \zeta_{\mathbb{K}}(1-2m).$$
(1.16)

In particular, If \mathbb{K} is not totally real,

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}}\left(\frac{(2\pi)^d n}{D_{\mathbb{K}}}\right) = 0$$

The weight-2 Eisenstein series $E_2(z)$ over $SL_2(\mathbb{Z})$ is not exactly a modular form since instead of (1.13), it satisfies

$$E_2(-1/z) = z^2 E_2(z) + \frac{6z}{\pi i}$$

Therefore, it sometimes known as quasi-modular form. In the following theorem, we investigate the transformation of $\mathcal{G}_{2,\mathbb{K}}(z)$.

Theorem 1.10. Let \mathbb{K} be an arbitrary number field. For $z \in \mathfrak{H}$,

$$\mathcal{G}_{2,\mathbb{K}}(-1/z) = z^2 \mathcal{G}_{2,\mathbb{K}}(z) - \frac{z}{\pi i} \zeta_{\mathbb{K}}^2(0).$$

$$(1.17)$$

Equivalently, for $\operatorname{Re}(\alpha)$, $\operatorname{Re}(\beta) > 0$ with $\alpha\beta = \pi^{2d}$,

$$\alpha \sum_{n=1}^{\infty} n V_{\mathbb{K}}(n) \Omega_{\mathbb{K}}\left(\frac{2^d n \alpha}{D_{\mathbb{K}}}\right) + \beta \sum_{n=1}^{\infty} n V_{\mathbb{K}}(n) \Omega_{\mathbb{K}}\left(\frac{2^d n \beta}{D_{\mathbb{K}}}\right) = -\frac{H\zeta_{\mathbb{K}}(-1)}{2\mathcal{C}_{\mathbb{K}}}(\alpha + \beta) - \frac{\zeta_{\mathbb{K}}^2(0)}{\pi^{1-d}}.$$
 (1.18)

Remark. If \mathbb{K} is not \mathbb{Q} or an imaginary quadratic field, it is well-known that $\zeta_{\mathbb{K}}(0)$ vanishes, thus $\mathcal{G}_{2,\mathbb{K}}(z)$ satisfies (1.13).

The final result generalizes the transformation formula satisfied by the logarithm of the Dedekind eta function [3, Equation (3.10)] for an arbitrary number field \mathbb{K} .

Theorem 1.11. Let \mathbb{K} be an arbitrary number field. For $\operatorname{Re}(\alpha)$, $\operatorname{Re}(\beta) > 0$ with $\alpha\beta = \pi^{2d}$, we have

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n} \Omega_{\mathbb{K}}\left(\frac{2^d n\alpha}{D_{\mathbb{K}}}\right) - \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n} \Omega_{\mathbb{K}}\left(\frac{2^d n\beta}{D_{\mathbb{K}}}\right) = \frac{H^2}{4\mathcal{C}_{\mathbb{K}}} \log\left(\frac{\alpha}{\beta}\right) + \frac{\zeta_{\mathbb{K}}(0)\zeta_{\mathbb{K}}(2)}{\pi^{d+1}\mathcal{C}_{\mathbb{K}}}(\alpha-\beta).$$

2. Preliminaries

In this section, we will gather known results from the literature that will be used in the subsequent proofs. One such result is the functional equation satisfied by the Dedekind zeta function $\zeta_{\mathbb{K}}(s)$, which is given by: [20, pp. 254-255].

$$\zeta_{\mathbb{K}}(s) = D_{\mathbb{K}}^{\frac{1}{2}-s} 2^{ds-r_2} \pi^{ds-r_1-r_2} \frac{\Gamma(1-s)^{r_1+r_2}}{\Gamma(s)^{r_2}} \sin\left(\frac{\pi s}{2}\right)^{r_1} \zeta_{\mathbb{K}}(1-s).$$
(2.1)

The following lemma gives the convexity bound of the Dedekind zeta function in the critical region.

Lemma 2.1. Let \mathbb{K} be any number field. Then

$$\zeta_{\mathbb{K}}(\sigma+it) \ll \begin{cases} |t|^{\frac{d}{2}-d\sigma+\epsilon} D_{\mathbb{K}}^{\frac{1}{2}-\sigma+\epsilon}, & \text{if } \sigma \leq 0, \\ |t|^{\frac{d(1-\sigma)}{2}+\epsilon} D_{\mathbb{K}}^{\frac{1-\sigma}{2}+\epsilon}, & \text{if } 0 \leq \sigma \leq 1, \\ |t|^{\epsilon} D_{\mathbb{K}}^{\epsilon}, & \text{if } \sigma \geq 1 \end{cases}$$

holds true for any $\epsilon > 0$.

Proof. This follows from a standard argument by applying the Phragmen-Lindelöf principle and the functional equation (2.1) of the Dedekind zeta function. The details may be found in [16, Chapter 5], for example.

The well-known duplication and reflection formulas of the gamma function [26, p. 46, Equations (3.4) and (3.5)] are as following

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s)$$
(2.2)

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$
(2.3)

We now wrote down the definitions of the Bessel functions. The Bessel functions of the first kind and the second kind of order ν are defined by [27, p. 40, 64]

$$J_{\nu}(z) := \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu}}{m! \Gamma(m+1+\nu)} \qquad (z,\nu \in \mathbb{C}),$$
$$Y_{\nu}(z) := \frac{J_{\nu}(z) \cos(\pi\nu) - J_{-\nu}(z)}{\sin \pi\nu} \qquad (z \in \mathbb{C}, \nu \notin \mathbb{Z}),$$

along with $Y_n(z) = \lim_{\nu \to n} Y_{\nu}(z)$ for $n \in \mathbb{Z}$. The modified Bessel functions of the first and second kinds are defined by [27, p. 77, 78]

$$I_{\nu}(z) := \begin{cases} e^{-\frac{1}{2}\pi\nu i} J_{\nu}(e^{\frac{1}{2}\pi i}z), & \text{if } -\pi < \arg(z) \le \frac{\pi}{2}, \\ e^{\frac{3}{2}\pi\nu i} J_{\nu}(e^{-\frac{3}{2}\pi i}z), & \text{if } \frac{\pi}{2} < \arg(z) \le \pi, \end{cases}$$
$$K_{\nu}(z) := \frac{\pi}{2} \frac{I_{-\nu}(z) - I_{\nu}(z)}{\sin\nu\pi}$$
(2.4)

respectively. When $\nu \in \mathbb{Z}$, $K_{\nu}(z)$ is interpreted as a limit of the right-hand side of (2.4).

We next define the Meijer G-function. The G-function was introduced initially by Meijer as a very general function using a series. Later, it was defined more generally via a line integral in the complex plane (cf. [13]) given by

$$G_{p,q}^{m,n} \begin{pmatrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{pmatrix} | z = \frac{1}{2\pi i} \int_{(C)} \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s \mathrm{d}s,$$
(2.5)

where $z \neq 0$ and m, n, p, q are integers which satisfy $0 \leq m \leq q$ and $0 \leq n \leq p$. The poles of the integrand must be all simple. Here (C) in the integral denotes the vertical line from $C - i\infty$ to $C + i\infty$ such that all poles of $\Gamma(b_j - s)$ for $j = 1, \ldots, m$, must lie on one side of the vertical line while all poles of $\Gamma(1 - a_j + s)$ for $j = 1, \ldots, n$ must lie on the other side. The integral then converges for $|\arg z| < \delta \pi$ where

$$\delta = m + n - \frac{1}{2}(p+q).$$

The integral additionally converges for $|\arg z| = \delta \pi$ if $(q-p)(\operatorname{Re}(s) + 1/2) > \operatorname{Re}(v) + 1$, where

$$v = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j$$

Special cases of the G-function include many other special functions.

3. Transformation formulas for odd zeta values over an arbitrary number field

In this section, we will prove all the results presented in the introduction. We commence with a lemma that will be utilized as a crucial step in establishing our general Theorem 1.1.

Lemma 3.1. For c = Re(s) > 1, we have

$$\Omega_{\mathbb{K}}(x) = \frac{1}{2\pi i} \int_{(c)} \frac{\zeta_{\mathbb{K}}(s)\Gamma(s)^{r_1+r_2}}{\Gamma(1-s)^{r_2}} \cos\left(\frac{\pi s}{2}\right)^{r_1-1} x^{-s} ds$$

Here, and throughout, $\int_{(c)} ds$ denotes the line integral $\int_{c-i\infty}^{c+i\infty} ds$ with $c = \operatorname{Re}(s)$.

Proof. The definition of Meijer G-function, given in (2.5), implies that

$$G_{0,2d}^{d+1,0}\left(\begin{array}{c} & -\\ (0)_{r_1+r_2}, \left(\frac{1}{2}\right)_{r_2+1}; \left(\frac{1}{2}\right)_{r_1+r_2-1}, (0)_{r_2} \\ \end{array} \right| \frac{x^2 j^2}{4^d} = \frac{1}{2\pi i} \int_{(L)} \frac{\Gamma(-s)^{r_1+r_2} \Gamma\left(\frac{1}{2}-s\right)^{r_2+1}}{\Gamma\left(\frac{1}{2}+s\right)^{r_1+r_2-1} \Gamma(1+s)^{r_2}} \left(\frac{x^2 j^2}{4^d}\right)^s ds, \quad (3.1)$$

where $L = \operatorname{Re}(s) < -\frac{1}{2}$, We next replace s by -s/2 in (3.1) so as to obtain for $c = \operatorname{Re}(s) > 1$,

$$G_{0,2d}^{d+1,0} \left(\begin{array}{c} & - \\ (0)_{r_1+r_2}, (\frac{1}{2})_{r_2+1}; (\frac{1}{2})_{r_1+r_2-1}, (0)_{r_2} \end{array} \middle| \frac{x^2 j^2}{4^d} \right) = \frac{1}{4\pi i} \int_{(c)} \frac{\Gamma\left(\frac{s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{1}{2}+\frac{s}{2}\right)^{r_2+1}}{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)^{r_1+r_2-1} \Gamma\left(1-\frac{s}{2}\right)^{r_2}} \left(\frac{xj}{2^d} \right)^{-s} ds \\
= \frac{1}{4\pi i} \int_{(c)} \frac{\left(\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\right)^{r_1+r_2} \Gamma\left(\frac{1}{2}+\frac{s}{2}\right)^{1-r_1}}{\left(\Gamma\left(\frac{1}{2}-\frac{s}{2}\right) \Gamma\left(1-\frac{s}{2}\right)\right)^{r_2} \Gamma\left(\frac{1}{2}-\frac{s}{2}\right)^{r_1-1}} \left(\frac{xj}{2^d}\right)^{-s} ds \\$$
(3.2)

Invoking the duplication and reflection formulas (2.2) and (2.3) in (3.2) and simplifying, we deduce that

$$G_{0,2d}^{d+1,0}\left(\begin{array}{c} & -\\ (0)_{r_1+r_2}, \left(\frac{1}{2}\right)_{r_2+1}; \left(\frac{1}{2}\right)_{r_1+r_2-1}, (0)_{r_2} \\ \end{array} \right| \frac{x^2 j^2}{4^d} = \frac{2^{r_1+r_2-1}}{\pi^{\frac{r_1}{2}-1}} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s)^{r_1+r_2}}{\Gamma(1-s)^{r_2}} \cos\left(\frac{\pi s}{2}\right)^{r_1-1} (xj)^{-s} ds.$$

$$(3.3)$$

We substitute the above expression in (1.8) to see that

$$\Omega_{\mathbb{K}}(x) = \sum_{j=1}^{\infty} V_{\mathbb{K}}(j) \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s)^{r_1+r_2}}{\Gamma(1-s)^{r_2}} \cos\left(\frac{\pi s}{2}\right)^{r_1-1} (xj)^{-s} ds$$

Now, by interchanging the order of summation and integration in the above equation, which is justified due to [26, p. 30, Theorem 2.1], and subsequently using (1.5), we arrive at the lemma. \Box

We now provide a proof of our general transformation formula over arbitrary number field.

Proof of Theorem 1.1. To begin, we establish the convergence of the series on both sides of (1.9). Notably, it suffices to demonstrate the convergence of the series in (1.7). From (3.3), for c > 1 and large j, we have

$$\left| G_{0,2d}^{d+1,0} \left(\begin{array}{c} - \\ (0)_{r_1+r_2}, \left(\frac{1}{2}\right)_{r_2+1}; \left(\frac{1}{2}\right)_{r_1+r_2-1}, (0)_{r_2} \\ \end{array} \right| \frac{x^2 j^2}{4^d} \right) \right| < \frac{2^{r_1+r_2-1}}{\pi^{\frac{r_1}{2}-1}(xj)^c} \int_{-\infty}^{\infty} \left| \frac{\Gamma(c+it)^{r_1+r_2}}{\Gamma(1-c-it)^{r_2}} \cos\left(\frac{\pi(c+it)}{2}\right)^{r_1-1} dt \\ \ll j^{-c},$$

$$(3.4)$$

which follows upon observing that the integral on the right-hand side of the initial step converges, a fact that can be verified using the Stirling formula [7, p. 224], namely, for $p \leq \sigma \leq q$,

$$|\Gamma(s)| = \sqrt{2\pi} |t|^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}\pi|t|} \left(1 + O\left(\frac{1}{|t|}\right) \right)$$
(3.5)

as $|t| \to \infty$. Utilizing the bound given in (3.4) in conjunction with [5, p. 430, Corollary 7.119], we establish the well-definedness of $\Omega_{\mathbb{K}}(x)$ defined in (1.8). Furthermore, on account of the formula (3.5) and Lemma 3.1, for $c > \max\{1, 1 + \operatorname{Re}(a)\}$, we can deduce that,

$$\Omega_{\mathbb{K}}(ny) \ll_y n^{-c}$$

as $n \to \infty$. Thus, we obtain

$$\sum_{n=1}^{\infty} V_{\mathbb{K}}(n) n^{a} \Omega_{\mathbb{K}}(ny) \ll \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{c-a}}.$$

As the series on the right-hand side converges, we have established our claim.

We now prove the transformation formula (1.9). Invoking Lemma 3.1 in the first step below, we have

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)\Omega_{\mathbb{K}}(ny)}{n^{2m+1}} = \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \frac{1}{2\pi i} \int_{(c)} \zeta_{\mathbb{K}}(s) \frac{\Gamma(s)^{r_1+r_2}}{\Gamma(1-s)^{r_2}} \cos\left(\frac{\pi s}{2}\right)^{r_1-1} (ny)^{-s} ds$$
$$= \frac{1}{2\pi i} \int_{(c)} \zeta_{\mathbb{K}}(2m+1+s) \zeta_{\mathbb{K}}(s) \frac{\Gamma(s)^{r_1+r_2}}{\Gamma(1-s)^{r_2}} \cos\left(\frac{\pi s}{2}\right)^{r_1-1} y^{-s} ds, \tag{3.6}$$

where in the last step we first interchanged the order of summation and integration which can be justified by using [26, p. 30, Theorem 2.1], and then used (1.5). The functional equation (2.1) can be written in the form

$$\frac{\Gamma(s)^{r_1+r_2}}{\Gamma(1-s)^{r_2}}\cos\left(\frac{\pi s}{2}\right)^{r_1-1}\zeta_{\mathbb{K}}(s) = \frac{D_{\mathbb{K}}^{\frac{1}{2}-s}2^{r_2-d(1-s)}\pi^{r_1+r_2-d(1-s)}}{\cos\left(\frac{\pi s}{2}\right)}\zeta_{\mathbb{K}}(1-s),\tag{3.7}$$

which transforms (3.6) into

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)\Omega_{\mathbb{K}}(ny)}{n^{2m+1}} = \sqrt{D_{\mathbb{K}}} 2^{r_2 - d} \pi^{r_1 + r_2 - d} \frac{1}{2\pi i} \int_{(c)} \frac{\zeta_{\mathbb{K}}(2m+1+s)\zeta_{\mathbb{K}}(1-s)}{\cos\left(\frac{\pi s}{2}\right)} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right)^{-s} ds.$$
(3.8)

We next shift the line of integration from c > 1 to $-2m - 3 < \mu = \operatorname{Re}(s) < -2m - 1$. To that end, let us consider the positively oriented contour formed by the line segments $[\mu - iT, c - iT]$, [c - iT, c + iT], $[c + iT, \mu + iT]$, $[\mu + iT, \mu - iT]$. Note that the integrand has simple poles at s = 0, -2m and s = 1 - 2k, for $0 \le k \le m + 1$. By Cauchy's residue theorem, we have

$$\frac{1}{2\pi i} \left(\int_{c-iT}^{c+iT} + \int_{c+iT}^{\mu+iT} + \int_{\mu+iT}^{\mu-iT} + \int_{\mu-iT}^{c-iT} \right) \frac{\zeta_{\mathbb{K}}(2m+1+s)\zeta_{\mathbb{K}}(1-s)}{\cos\left(\frac{\pi s}{2}\right)} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right)^{-s} ds$$

$$= R_0 + R_{-2m} + \sum_{k=0}^{m+1} R_{1-2k}.$$
(3.9)

It is easy to see that the integrals along the horizontal segments vanish as $T \to \infty$ with the help of Lemma 2.1 and the Stirling's formula (3.5) since the cosine function reduces to the gamma factors using (2.3).

Therefore, using the above fact, (3.8) and (3.9), we arrive at

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)\Omega_{\mathbb{K}}(ny)}{n^{2m+1}} = \sqrt{D_{\mathbb{K}}}2^{r_2-d}\pi^{r_1+r_2-d} \left(R_0 + R_{-2m} + \sum_{k=0}^{m+1} R_{1-2k} + I(y)\right),$$
(3.10)

where

$$I(y) := \frac{1}{2\pi i} \int_{(\mu)} \frac{\zeta_{\mathbb{K}}(2m+1+s)\zeta_{\mathbb{K}}(1-s)}{\cos\left(\frac{\pi s}{2}\right)} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right)^{-s} ds.$$
(3.11)

The residues involved in (3.10) can be easily evaluated to

$$R_{0} = -H\zeta_{\mathbb{K}}(2m+1)$$

$$R_{-2m} = (-1)^{m}H\zeta_{\mathbb{K}}(2m+1)\left(\frac{yD_{\mathbb{K}}}{(2\pi)^{d}}\right)^{2m}$$

$$R_{1-2k} = (-1)^{k+1}\frac{2}{\pi}\zeta_{\mathbb{K}}(2m-2k+2)\zeta_{\mathbb{K}}(2k)\left(\frac{yD_{\mathbb{K}}}{(2\pi)^{d}}\right)^{2k-1}.$$
(3.12)

Employing the change of variable s = -2m - w in (3.11), we have, for 1 < c = Re(s) < 3,

$$I(y) = (-1)^{m} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^{d}}\right)^{2m} \frac{1}{2\pi i} \int_{(c)} \frac{\zeta_{\mathbb{K}}(1-w)\zeta_{\mathbb{K}}(2m+1+w)}{\cos\left(\frac{\pi w}{2}\right)} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^{d}}\right)^{w} ds$$

$$= (-1)^{m} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^{d}}\right)^{2m} \left(\sqrt{D_{\mathbb{K}}}2^{r_{2}-d}\pi^{r_{1}+r_{2}-d}\right)^{-1} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \Omega_{\mathbb{K}} \left(\frac{n(2\pi)^{2d}}{yD_{\mathbb{K}}^{2}}\right),$$
(3.13)

where we used (3.8). Now equations (3.10) (3.12) and (3.13) together yield

$$\begin{split} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)\Omega_{\mathbb{K}}(ny)}{n^{2m+1}} &= \sqrt{D_{\mathbb{K}}} 2^{r_2 - d} \pi^{r_1 + r_2 - d} \left\{ -H\zeta_{\mathbb{K}}(2m+1) + (-1)^m H\zeta_{\mathbb{K}}(2m+1) \left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right)^{2m} \right. \\ &+ \frac{2}{\pi} \sum_{k=0}^{m+1} (-1)^{k+1} \zeta_{\mathbb{K}}(2m-2k+2) \zeta_{\mathbb{K}}(2k) \left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right)^{2k-1} \right\} \\ &+ (-1)^m \left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right)^{2m} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \Omega_{\mathbb{K}} \left(\frac{n(2\pi)^{2d}}{yD_{\mathbb{K}}^2}\right). \end{split}$$

First let $y = \frac{2^d \alpha}{D_{\mathbb{K}}}$ with $\alpha \beta = \pi^{2d}$ in the above equation and then multiply both sides by α^{-m} and simplify to see that

$$\alpha^{-m} \left\{ \sqrt{D_{\mathbb{K}}} 2^{r_2 - d} \pi^{r_1 + r_2 - d} H \zeta_{\mathbb{K}}(2m+1) + \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \Omega_{\mathbb{K}} \left(\frac{2^d n \alpha}{D_{\mathbb{K}}} \right) \right\}$$

= $(-\beta)^{-m} \left\{ \sqrt{D_{\mathbb{K}}} 2^{r_2 - d} \pi^{r_1 + r_2 - d} H \zeta_{\mathbb{K}}(2m+1) + \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{2m+1}} \Omega_{\mathbb{K}} \left(\frac{2^d n \beta}{D_{\mathbb{K}}} \right) \right\}$
+ $\sqrt{D_{\mathbb{K}}} 2^{r_2 - d+1} \pi^{r_1 + r_2 - d-1} \alpha^{-m} \sum_{k=0}^{m+1} (-1)^{k+1} \zeta_{\mathbb{K}}(2m-2k+2) \zeta_{\mathbb{K}}(2k) \left(\frac{\alpha}{\pi^d} \right)^{2k-1}.$ (3.14)

Finally (1.9) follows upon replacing k by m+1-k in the finite sum of (3.14) and multiplying both sides by $\mathcal{C}_{\mathbb{K}}$.

We next prove (1.3) from Theorem 1.1.

Proof of Corollary 1.2. Applying [22, p.231, Equation (8)], the following Meijer G-function reduces to

$$G_{0,2}^{2,0} \left(\left. \begin{array}{c} -\\ 0,\frac{1}{2} \end{array} \right| \frac{x^2 j^2}{4} \right) = \sqrt{2xj} K_{1/2}(xj)$$
$$= \sqrt{\pi} e^{-jx}.$$

For $\mathbb{K} = \mathbb{Q}$, the degree d is 1 with $r_1 = 1$ and $r_2 = 0$. Thus definition (1.8) yields

$$\Omega_{\mathbb{K}}(x) = \frac{1}{e^x - 1},\tag{3.15}$$

which concludes our corollary.

We next present the proof of the transformation formulas of $\zeta_{\mathbb{K}}(2m+1)$ for quadratic fields.

Proof of Corollary 1.3. To prove (1.10), it is enough to see that $\Omega_{\mathbb{K}}(jx) = \mathcal{R}(jx)$ when we take $r_1 = 2$ and $r_2 = 0$ in Theorem 1.1. To show that, note that, for c > 0,

$$G_{0,4}^{3,0} \left(\begin{array}{c} -\\ 0,0,\frac{1}{2};\frac{1}{2} \end{array} \right) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma\left(s\right)^2 \Gamma\left(\frac{1}{2}+s\right)}{\Gamma\left(\frac{1}{2}-s\right)} \left(\frac{x^2 j^2}{16}\right)^{-s} ds$$
$$= \frac{1}{2\pi i} \int_{(c)} \frac{(\Gamma\left(s\right) \Gamma\left(\frac{1}{2}+s\right))^2}{\Gamma\left(\frac{1}{2}-s\right) \Gamma\left(\frac{1}{2}+s\right)} \left(\frac{x^2 j^2}{16}\right)^{-s} ds$$
$$= \frac{4}{2\pi i} \int_{(c)} \Gamma^2\left(2s\right) \cos\left(\pi s\right) (xj)^{-2s} ds, \tag{3.16}$$

which follows by applying (2.2) and (2.3). Employing the change of variable $s \to s/2$ in (3.16), we obtain

$$G_{0,4}^{3,0} \begin{pmatrix} -\\ 0,0,\frac{1}{2};\frac{1}{2} \end{pmatrix} = \frac{2}{2\pi i} \int_{(2c)} \Gamma^2(s) \cos\left(\frac{\pi s}{2}\right) (xj)^{-s} ds$$
$$= \frac{1}{2} \left\{ \frac{1}{2\pi i} \int_{(2c)} \Gamma^2(s) \left(e^{\frac{\pi i}{2}} jx\right)^{-s} ds + \frac{1}{2\pi i} \int_{(c)} \Gamma^2(s) \left(e^{-\frac{\pi i}{2}} jx\right)^{-s} ds \right\}$$
$$= 2 \left\{ K_0 \left(2\epsilon\sqrt{jx}\right) + K_0 \left(2\overline{\epsilon}\sqrt{jx}\right) \right\},$$

where we used [23, p. 115, Formula (11.1)]

$$K_{z}(x) = \frac{1}{2\pi i} \int_{(c_{1})} \Gamma\left(\frac{s-z}{2}\right) \Gamma\left(\frac{s+z}{2}\right) 2^{s-2} x^{-s} ds, \quad c_{1} > |\operatorname{Re}(z)|.$$
(3.17)
aim $\Omega_{\mathbb{K}}(jx) = \mathcal{R}(jx)$, which concludes our result.

This proves our claim $\Omega_{\mathbb{K}}(jx) = \mathcal{R}(jx)$, which concludes our result.

Proof of Corollary 1.4. Let $r_1 = 0$ and $r_2 = 1$ in Theorem 1.1. First we show that $\Omega_{\mathbb{K}}(x)$ reduces to $\mathcal{I}(x)$. Note that

$$G_{0,4}^{3,0} \left(\begin{array}{c} -\\ 0,\frac{1}{2},\frac{1}{2};0 \end{array} \middle| \frac{x^2 j^2}{16} \right) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma\left(s\right) \Gamma^2\left(s+\frac{1}{2}\right)}{\Gamma\left(1-s\right)} \left(\frac{x^2 j^2}{16}\right)^{-s} ds$$
$$= \frac{1}{2\pi i} \int_{(c)} \frac{\left(\Gamma\left(s\right) \Gamma\left(s+\frac{1}{2}\right)\right)^2}{\Gamma(s) \Gamma\left(1-s\right)} \left(\frac{x^2 j^2}{16}\right)^{-s} ds$$
$$= 4 \frac{1}{2\pi i} \int_{(c)} \Gamma^2\left(2s\right) \sin(\pi s) \left(xj\right)^{-2s} ds,$$

where in the last step we used (2.2) and (2.3). The change of variable $s \to s/2$ in the above expression yields

$$G_{0,4}^{3,0}\left(\begin{array}{c}-\\0,\frac{1}{2},\frac{1}{2};0\end{array}\right|\frac{x^{2}j^{2}}{16}\right) = 2\frac{1}{2\pi i}\int_{(2c)}\Gamma^{2}\left(s\right)\sin(\pi s/2)\left(xj\right)^{-s}ds$$

$$= \frac{1}{i} \frac{1}{2\pi i} \int_{(2c)} \Gamma^2(s) \left(e^{\frac{\pi i s}{2}} - e^{-\frac{\pi i s}{2}} \right) (xj)^{-s} ds$$

$$= i \frac{1}{2\pi i} \left\{ \int_{(2c)} \Gamma^2(s) \left(e^{\frac{\pi i}{2}} xj \right)^{-s} ds - \int_{(2c)} \Gamma^2(s) \left(e^{-\frac{\pi i}{2}} xj \right)^{-s} ds \right\}$$

$$= 2i \left\{ K_0 \left(2\epsilon \sqrt{jx} \right) - K_0 \left(2\overline{\epsilon} \sqrt{jx} \right) \right\},$$

where we invoked (3.17). Combining this with (1.11) and (1.8) gives $\Omega_{\mathbb{K}}(jx) = \mathcal{I}(jx)$. This completes the proof of our result.

Next, we establish Lerch's result and its consequential Corollary 1.6.

Proof of Theorem 1.5. Upon setting $\alpha = \beta = \pi^d$, substituting *m* with 2m + 1, and simplifying the expression, we arrive at (1.12).

Proof of Corollary 1.6. For a totally real field, the result of Klingen [18] and Siegel [25] (given in (1.4) above) leads to the conclusion that the finite sum on the right-hand side of Equation (1.12) is transcendental. As a result, this implies that either $\zeta_k(4m+3)$ or $\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{4m+3}} \Omega_{\mathbb{K}}\left(\frac{(2\pi)^d n}{D_{\mathbb{K}}}\right)$ must also be transcendental.

4. EXTENDED EISENSTEIN SERIES OVER AN ARBITRARY NUMBER FIELD

We first establish the symmetric form of the transformation formula for Eisenstein series $\mathcal{G}_{k,\mathbb{K}}(z)$, as stated in Theorem 1.8. Subsequently, we present the proof of Theorem 1.7.

Proof of Theorem 1.8. By substituting m with -m in (1.9) and considering the case where $m \in \mathbb{N}$, we observe that the finite sum becomes empty and hence evaluates to zero. Further rearrangement of the terms leads to the proof of the desired result.

The second part of the proof follows as a straightforward consequence of (1.15), since for a non-totally real field, it is known that $\zeta_{\mathbb{K}}(1-2m) = 0$.

Proof of Theorem 1.7. Note that (1.15) can be re-written as

$$\alpha^m \left\{ \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}} \left(\frac{2^d n \alpha}{D_{\mathbb{K}}} \right) + \frac{H}{2\mathcal{C}_{\mathbb{K}}} \zeta_{\mathbb{K}} (1-2m) \right\} = (-\beta)^m \left\{ \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}} \left(\frac{2^d n \beta}{D_{\mathbb{K}}} \right) + \frac{H}{2\mathcal{C}_{\mathbb{K}}} \zeta_{\mathbb{K}} (1-2m) \right\}.$$

Letting $\alpha = -\pi^d i z$ and $\beta = \pi^d i / z$ in the above equation and then simplifying, we are led to

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}}\left(\frac{(2\pi)^d inz}{D_{\mathbb{K}}}\right) + \frac{H}{2\mathcal{C}_{\mathbb{K}}} \zeta_{\mathbb{K}}(1-2m) = z^{-2m} \left\{\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n^{1-2m}} \Omega_{\mathbb{K}}\left(\frac{(2\pi)^d in}{zD_{\mathbb{K}}}\right) + \frac{H}{2\mathcal{C}_{\mathbb{K}}} \zeta_{\mathbb{K}}(1-2m)\right\}.$$

By utilizing the definition of the Eisenstein series $\mathcal{G}_{k,\mathbb{K}}(z)$ as given in (1.14), we now conclude the proof of the theorem.

Proof of Theorem 1.9. After setting $\alpha = \pi^d = \beta$ and making the substitution $m \to 2m + 1$, we can simplify the expression, which leads to the closed-form evaluation as expressed in (1.16).

We now supply a proof of the result for quasi-modular form.

Proof of Theorem 1.10. The result in (1.18) can be readily derived from (1.9) by setting m = 1 and rearranging the terms accordingly.

For a proof of (1.17), we rewrite (1.18) as

$$\alpha \left\{ \sum_{n=1}^{\infty} n V_{\mathbb{K}}(n) \Omega_{\mathbb{K}} \left(\frac{2^d n \alpha}{D_{\mathbb{K}}} \right) + \frac{H \zeta_{\mathbb{K}}(-1)}{2\mathcal{C}_{\mathbb{K}}} \right\} = -\beta \left\{ \sum_{n=1}^{\infty} n V_{\mathbb{K}}(n) \Omega_{\mathbb{K}} \left(\frac{2^d n \beta}{D_{\mathbb{K}}} \right) + \frac{H \zeta_{\mathbb{K}}(-1)}{2\mathcal{C}_{\mathbb{K}}} \right\} - \frac{\zeta_{\mathbb{K}}^2(0)}{\pi^{1-d}}.$$

If we take $\alpha = -\pi^d i z$ and $\beta = \pi^d i / z$ in the above equation and simplify then it becomes

$$\sum_{n=1}^{\infty} n V_{\mathbb{K}}(n) \Omega_{\mathbb{K}}\left(\frac{(2\pi)^d n i z}{D_{\mathbb{K}}}\right) + \frac{H \zeta_{\mathbb{K}}(-1)}{2\mathcal{C}_{\mathbb{K}}} = -\frac{1}{z} \left\{ \sum_{n=1}^{\infty} n V_{\mathbb{K}}(n) \Omega_{\mathbb{K}}\left(\frac{(2\pi)^d n i}{z D_{\mathbb{K}}}\right) + \frac{H \zeta_{\mathbb{K}}(-1)}{2\mathcal{C}_{\mathbb{K}}} \right\} + \frac{\zeta_{\mathbb{K}}^2(0)}{\pi i}.$$

By employing the definition of $\mathcal{G}_{2,\mathbb{K}}(z)$ as given in (1.14) and performing simplifications, we obtain (1.17).

The transformation formula for logarithmic of Dedekind eta function over fields is proved next.

Proof of Theorem 1.11. The argument presented here is similar to that given for the proof of Theorem 1.1, and thus our explanation will be terse. We invoke Lemma 3.1 so that

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)\Omega_{\mathbb{K}}(ny)}{n} = \frac{1}{2\pi i} \int_{(c)} \zeta_{\mathbb{K}}(1+s)\zeta_{\mathbb{K}}(s) \frac{\Gamma(s)^{r_1+r_2}}{\Gamma(1-s)^{r_2}} \cos\left(\frac{\pi s}{2}\right)^{r_1-1} y^{-s} ds, \tag{4.1}$$

Combining (3.7) with (4.1), we see that

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)\Omega_{\mathbb{K}}(ny)}{n} = \sqrt{D_{\mathbb{K}}} 2^{r_2 - d} \pi^{r_1 + r_2 - d} \frac{1}{2\pi i} \int_{(c)} \frac{\zeta_{\mathbb{K}}(1+s)\zeta_{\mathbb{K}}(1-s)}{\cos\left(\frac{\pi s}{2}\right)} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right)^{-s} ds.$$
(4.2)

We now shift the line of integration from c > 1 to $-3 < \mu = \text{Re}(s) < -1$ and consider the positively oriented contour formed by the line segments $[\mu - iT, c - iT]$, [c - iT, c + iT], $[c + iT, \mu + iT]$, $[\mu + iT, \mu - iT]$. Note that the integrand has double pole at s = 0 and simple poles at s = 1 and s = -1. Furthermore, using Lemma 2.1 and (3.5), it can be shown that the integrals along the horizontal segments vanish as $T \to \infty$. By applying Cauchy's residue theorem along with (4.2), we obtain

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)\Omega_{\mathbb{K}}(ny)}{n} = \sqrt{D_{\mathbb{K}}} 2^{r_2 - d} \pi^{r_1 + r_2 - d} \left(R_0 + R_1 + R_{-1} + I(y)\right), \tag{4.3}$$

where

$$I(y) := \frac{1}{2\pi i} \int_{(\mu)} \frac{\zeta_{\mathbb{K}}(1+s)\zeta_{\mathbb{K}}(1-s)}{\cos\left(\frac{\pi s}{2}\right)} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right)^{-s} ds.$$
(4.4)

We can evaluate the residues in (4.3) as

$$R_{0} = H^{2} \log \left(\frac{y D_{\mathbb{K}}}{(2\pi)^{d}} \right)$$

$$R_{1} = -\frac{2(2\pi)^{d}}{\pi y D_{\mathbb{K}}} \zeta_{\mathbb{K}}(2) \zeta_{\mathbb{K}}(0).$$

$$R_{-1} = \frac{2y D_{\mathbb{K}}}{\pi (2\pi)^{d}} \zeta_{\mathbb{K}}(0) \zeta_{\mathbb{K}}(2).$$
(4.5)

Employing the change of variable s = -w in (4.4), we have, for 1 < c = Re(s) < 3,

$$I(y) = \frac{1}{2\pi i} \int_{(c)} \frac{\zeta_{\mathbb{K}}(1-w)\zeta_{\mathbb{K}}(1+w)}{\cos\left(\frac{\pi w}{2}\right)} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right)^w ds$$
$$= \left(\sqrt{D_{\mathbb{K}}}2^{r_2-d}\pi^{r_1+r_2-d}\right)^{-1} \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n} \Omega_{\mathbb{K}}\left(\frac{n(2\pi)^{2d}}{yD_{\mathbb{K}}^2}\right),$$
(4.6)

where we used (4.2). Now equations (4.3) (4.5) and (4.6) together yield

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)\Omega_{\mathbb{K}}(ny)}{n} = \sqrt{D_{\mathbb{K}}} 2^{r_2 - d} \pi^{r_1 + r_2 - d} \left\{ H^2 \log\left(\frac{yD_{\mathbb{K}}}{(2\pi)^d}\right) - \frac{2(2\pi)^d}{\pi y D_{\mathbb{K}}} \zeta_{\mathbb{K}}(2)\zeta_{\mathbb{K}}(0) + \frac{2yD_{\mathbb{K}}}{\pi (2\pi)^d} \zeta_{\mathbb{K}}(0)\zeta_{\mathbb{K}}(2) \right\}$$

$$+\sum_{n=1}^{\infty}\frac{V_{\mathbb{K}}(n)}{n}\Omega_{\mathbb{K}}\left(\frac{n(2\pi)^{2d}}{yD_{\mathbb{K}}^{2}}\right).$$

Thus we have

$$\sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)\Omega_{\mathbb{K}}(ny)}{n} - \sum_{n=1}^{\infty} \frac{V_{\mathbb{K}}(n)}{n} \Omega_{\mathbb{K}}\left(\frac{n(2\pi)^{2d}}{yD_{\mathbb{K}}^{2}}\right) = \frac{\sqrt{D_{\mathbb{K}}}H^{2}}{2^{r_{1}}(2\pi)^{r_{2}}} \log\left(\frac{yD_{\mathbb{K}}}{(2\pi)^{d}}\right) + \frac{2}{\pi} \frac{\sqrt{D_{\mathbb{K}}}\zeta_{\mathbb{K}}(0)\zeta_{\mathbb{K}}(2)}{2^{r_{1}}(2\pi)^{r_{2}}} \left(\frac{yD_{\mathbb{K}}}{(2\pi)^{d}} - \frac{(2\pi)^{d}}{yD_{\mathbb{K}}}\right).$$

Letting $y = \frac{2^d \alpha}{D_{\mathbb{K}}}$ with $\alpha \beta = \pi^{2d}$ in the above equation we conclude our result.

Acknowledgements. The first author is an INSPIRE Faculty at IISER Kolkata funded by the grant DST/INSPIRE/04/2021/002753 of DST, Govt. of India. Partial work of this project was carried out when the second author was a post-doctoral fellow at the Institute of Mathematics, Academia Scinica, Taiwan and wants to thank the institute for the academic support. The third author's research was supported by the Fulbright-Nehru Postdoctoral Fellowship Grant 2846/FNPDR/2022. All three authors sincerely thank the respective funding agencies for their support.

References

- [1] R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, Astérisque **61** (1979), 11–13.
- [2] R. Apéry, Interpolation de fractions continues et irrationalité de certaines constantes, in: Bull. Section des Sci., Tome III, Bibliothéque Nationale, Paris, 1981, 37–63.
- [3] B.C. Berndt, Y. Lee, and J. Sohn, The formulas of Koshliakov and Guinand in Ramanujan's lost notebook, Surveys in Number Theory, Series: Developments in Mathematics, Vol. 17, K. Alladi, ed., Springer, New York, 2008, pp. 21–42.
- [4] B. C. Berndt and A. Straub, Ramanujan's formula for $\zeta(2n+1)$, Exploring the Riemann zeta function, 13–34, Springer, Cham, 2017.
- [5] O. Bordelles, Arithmetic tales, London: Springer, 2012.
- [6] D. M. Bradley, Series acceleration formulas for Dirichlet series with periodic coefficients, Ramanujan J. 6 (2002), 331–346.
- [7] E. T. Copson, Theory of Functions of a Complex Variable, Oxford University Press, Oxford, 1935.
- [8] A. Dixit and R. Gupta, Koshliakov zeta functions I. Modular relations, Adv. Math. 393 (2021), Paper No. 108093.
- [9] A. Dixit, R. Gupta, R. Kumar and B. Maji, Generalized Lambert series, Raabe's cosine transform and a generalization of Ramanujan's formula for $\zeta(2m+1)$, Nagoya Math. J. **239** (2020), 232–293.
- [10] A. Dixit, A. Kesarwani and R. Kumar, Explicit transformations of certain Lambert series, Res. Math. Sci. 9 34 (2022).
- [11] A. Dixit and R. Kumar, Applications of the Lipschitz summation formula and A generalization of Raabe's cosine transform, submitted for publication, 2022.
- [12] A. Dixit and B. Maji, Generalized Lambert series and arithmetic nature of odd zeta values, Proc. Roy. Soc. Edinburg Sect. A: Mathematics, 150, Issue 2, 741–769, 2020.
- [13] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 1 (New York, 1953). Zentralblatt MATH, 51.
- [14] J. Franke, Ramanujan identities of higher degree, Research in Number Theory 4 (2018), 1–19.
- [15] A. Gupta and B. Maji, On Ramanujan's formula for $\zeta(1/2)$ and $\zeta(2m+1)$, J. Math. Anal. Appl. 507 (2022), 125738.
- [16] H. Iwaniec and E. Kowalski, Analytic Number Theory, Amer. Math. Soc. Colloquium Publ. 53, Amer. Math. Soc., Providence, RI, (2004).
- [17] P. Kirschenhofer and H. Prodinger, On some applications of formulae of Ramanujan in the analysis of algorithms, Mathematika 38 (1991), 14–33.
- [18] H. Klingen, Über die Werte der Dedekindschen Zeta funktionen, Math. Ann. 145 (1962), 265–272.
- [19] Y. Komori, K. Matsumoto and H. Tsumura, Barnes multiple zeta-functions, Ramanujan's formula, and relevant series involving hyperbolic functions, J. Ramanujan Math. Soc. 28 No. 1 (2013), 49–69.
- [20] S. Lang, Algebraic number theory, Addison-Wesley: Reading, MA, (1970).
- [21] M. Lerch, Sur la fonction $\zeta(s)$ pour valeurs impaires de l'argument, J. Sci. Math. Astron. pub. pelo Dr. F. Gomes Teixeira, Coimbra 14 (1901), 65–69.
- [22] Y. L. Luke, Special Functions and Their Approximations v. 1, Academic press, (1969).
- [23] F. Oberhettinger, Tables of Mellin Transforms (Springer-Verlag, New York, 1974).
- [24] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.

- [25] C. L. Siegel, Berechnung von Zetafunktionen an ganzzahligen Stellen, Nachr. Akad. Wiss. Gött., 10, 87–102 (1969).
- [26] N. M. Temme, Special functions: An introduction to the classical functions of mathematical physics, Wiley-Interscience Publication, New York, 1996.
- [27] G. N. Watson, A Treatise on the Theory of Bessel Functions, second ed., Cambridge University Press, London, 1944.

[28] D. Zagier, Hyperbolic manifolds and special values of Dedekind zeta-functions, Invent. Math. 83 (1986), 285–301.

[29] V. Zudilin, One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(5)$ is irrational, (Russian) Uspekhi Mat. Nauk **56** (2001), no. 4(340), 149–150; translation in Russian Math. Surveys **56** (2001), no. 4, 774–776.

DEPARTMENT OF MATHEMATICS AND STATISTICS, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH KOLKATA, MOHANPUR-741246, WEST BENGAL, INDIA.

Email address: soumya.tatan@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT TYLER, TX 75799, U.S.A. *Email address*: rgupta@uttyler.edu

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, U.S.A. *Email address*: rjk60310psu.edu