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A NOTE ON p-ADIC SIMPLICIAL VOLUMES

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Abstract. We define and study generalizations of simplicial volume over arbitrary seminormed rings with a focus on p-adic simplicial volumes. We investigate the dependence on the prime and establish homology bounds in terms of p-adic simplicial volumes. As the main examples, we compute the weightless and p-adic simplicial volumes of surfaces. This is based on an alternative way to calculate classical simplicial volume of surfaces without hyperbolic straightening and shows that surfaces satisfy mod p and p-adic approximation of simplicial volume.

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1. Introduction. The simplicial volume of an oriented compact connected manifold is the ℓ^1-seminorm of the fundamental class in singular homology with $\mathbb{R}$-coefficients, which encodes topological information related to the Riemannian volume [**10**]. A number of variations of simplicial volume such as the *integral simplicial volume* or *weightless simplicial volume* over finite fields proved to be useful in Betti number, rank gradient, and torsion homology estimates [**8**, **14**, **15**, **16**, **20**].

In the present article, we will focus on p-adic simplicial volumes. The basic setup is as follows: If M is an oriented compact connected manifold and $(R, |\cdot|)$ is a seminormed ring (see Section 2.1), then the *simplicial volume of M with R-coefficients* is defined as the infimum

$$\|M, \partial M\|_R := \inf\left\{ \sum_{j=1}^{k} |a_j| \;\middle|\; \sum_{j=1}^{k} a_j \cdot \sigma_j \in Z(M, \partial M; R) \right\} \in \mathbb{R}_{\geq 0},$$

over the "ℓ^1-norms" of all relative fundamental cycles of M. For $\mathbb{R}$ or $\mathbb{Z}$ with the ordinary absolute value, one obtains the classical simplicial volume $\|M\|$ and the integral simplicial volume $\|M\|_{\mathbb{Z}}$. For a ring R with the trivial seminorm, this gives rise to the weightless simplicial volume $\|M\|_{(R)}$ [**16**]. For other seminormed rings, one obtains new, unexplored invariants. We prove a number of fundamental results that describe how these simplicial volumes for different seminormed rings are related.

Using the ring $\mathbb{Z}_p$ of p-adic integers or the field $\mathbb{Q}_p$ of p-adic numbers with the p-adic absolute value as underlying seminormed rings leads to p-adic simplicial volumes. The long-term hope is that $\|M, \partial M\|_{\mathbb{Z}_p}$ and $\|M, \partial M\|_{\mathbb{Q}_p}$ might contain refined information on p-torsion in the homology of M.

1.1. Dependence on the prime. Extending the corresponding result for $\mathbb{F}_p$-simplicial volumes [**16**, Theorem 1.2], we show that the p-adic simplicial volumes contain new information only for a finite number of primes:

THEOREM 1.1. *Let M be an oriented compact connected manifold. Then, for almost all primes p,*

$$\|M, \partial M\|_{(\mathbb{F}_p)} = \|M, \partial M\|_{\mathbb{Z}_p} = \|M, \partial M\|_{\mathbb{Q}_p} = \|M, \partial M\|_{(\mathbb{Q})}.$$

We prove this in Section 3.3. While the inequalities $\|M, \partial M\|_{(\mathbb{F}_p)} \leq \|M, \partial M\|_{\mathbb{Z}_p}$ and $\|M, \partial M\|_{\mathbb{Q}_p} \leq \|M, \partial M\|_{\mathbb{Z}_p}$ hold for all prime numbers p (see Corollary 2.10) and $\|\cdot\|_{(\mathbb{F}_p)}$ and $\|\cdot\|_{\mathbb{Z}_p}$ exhibit similar behavior, we are currently not aware of a single example where one of these inequalities is strict.

1.2. Homology estimates. The p-adic simplicial volumes provide upper bounds for the Betti numbers. The following result is given in Corollaries 3.3 and 3.5.

THEOREM 1.2. *Let M be an oriented compact connected manifold. Then for all primes p and all $n \in \mathbb{N}$, the Betti numbers satisfy*

$$b_n(M; \mathbb{F}_p) \leq \|M, \partial M\|_{\mathbb{Z}_p},$$
$$b_n(M; \mathbb{Q}) \leq \|M, \partial M\|_{\mathbb{Q}_p}.$$

The first inequality uses the well-known Poincaré duality argument [**17**, Example 14.28][**16**, Proposition 2.6]. The second inequality is based on the additional torsion estimate (Propsition 3.4)

$$\dim_{\mathbb{F}_p} p^m H_n(M; \mathbb{Z}/p^{m+1}\mathbb{Z}) \leq p^m \|p^m \cdot [M, \partial M]\|_{\mathbb{Z}_p}$$

and the observation that the right-hand side converges to $\|M, \partial M\|_{\mathbb{Q}_p}$ as m tends to infinity. This suggests the following question:

QUESTION 1.3. For which oriented compact connected manifolds M and which primes p is there a strict inequality

$$\|M, \partial M\|_{\mathbb{Q}_p} < \|M, \partial M\|_{\mathbb{Z}_p} \quad \text{or} \quad \|M, \partial M\|_{(\mathbb{F}_p)} < \|M, \partial M\|_{\mathbb{Z}_p}?$$

Is a strict inequality related to p-torsion in the homology of M ?

1.3. Surfaces and approximation. In Section 4, we compute the p-adic simplicial volumes for some examples. In particular, we compute the weightless simplicial volume of surfaces. Let Σ_g be the oriented closed connected surface of genus g. For $b \geq 1$, we write $\Sigma_{g,b}$ to denote the surface of genus g with b boundary components.

THEOREM 1.4. *Let R be an integral domain, equipped with the trivial absolute value. Then*

(1) $\|\Sigma_g\|_{(R)} = 4g - 2$ *for all* $g \in \mathbb{N}_{\geq 1}$ *and*
(2) $\|\Sigma_{0,1}\|_{(R)} = 1$ *and* $\|\Sigma_{g,b}\|_{(R)} = 3b + 4g - 4$ *for all* $g \in \mathbb{N}$ *and all* $b \in \mathbb{N}_{\geq 1}$ *with* $(g, b) \neq (0, 1)$.

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Using this result, we compute the $\mathbb{Z}_p$-simplicial volume of all surfaces (Corollary 4.6) and we show that surfaces satisfy mod p and p-adic approximation of simplicial volume (Remark 4.8). Moreover, Theorem 1.4 can also be used to compute the classical simplicial volume of surfaces in a way that avoids the use of hyperbolic straightening (Remark 4.7) (and instead is based on mapping degrees).

It should be noted that, in general, it is not clear how the classical simplicial volume and the p-adic simplicial volumes are related.

1.4. Non-values. Recent results show that classical simplicial volumes are right computable [13], which in particular allows to give explicit examples of real numbers that cannot occur as the simplicial volume of a manifold. Based on the same methods, we establish that also the p-adic simplicial volumes $\|M\|_{\mathbb{Z}_p}$ and $\|M\|_{\mathbb{Q}_p}$ are right computable; see Proposition 5.2.

2. Foundations. We introduce simplicial volumes with coefficients in rings with a submultiplicative seminorm, for example, an absolute value. In particular, we obtain p-adic versions of simplicial volume. Moreover, we establish some basic inheritance and comparison properties of such simplicial volumes similar to those already known in the classical or weightless case.

2.1. Simplicial volume. Let R be a commutative ring with unit. A *seminorm* on R is a function $|\cdot|\colon R \to \mathbb{R}_{\geq 0}$ with $|0| = 0$, $|1| = 1$ that is submultiplicative

$$|st| \leq |s||t|$$

and satisfies the triangle inequality

$$|s+t| \leq |s| + |t|$$

for all $s, t \in R$. If the seminorm is multiplicative, it is called an *absolute value*. A *seminormed ring* is pair $(R, |\cdot|)$, consisting of a commutative ring R with unit and a seminorm $|\cdot|$ on R. A seminormed ring $(R, |\cdot|)$ is a *normed ring* if $|\cdot|$ is an absolute value.

Seminormed rings give rise to a notion of simplicial volume:

DEFINITION 2.1 (Simplicial volume). Let $(R, |\cdot|)$ be a seminormed ring. Let M be an oriented compact connected d-manifold, and let $Z(M, \partial M; R) \subset C_d(M; R)$ be the set of all relative singular R-fundamental cycles of $(M, \partial M)$. Then the *simplicial volume of M with R-coefficients* is defined as

$$\|M, \partial M\|_R := \inf\left\{ \sum_{j=1}^{k} |a_j| \;\middle|\; \sum_{j=1}^{k} a_j \cdot \sigma_j \in Z(M, \partial M; R) \right\} \in \mathbb{R}_{\geq 0}.$$

EXAMPLE 2.2 (Classical simplicial volume). The usual norm on $\mathbb{R}$ is an absolute value on $\mathbb{R}$. The corresponding simplicial volume $\|\cdot\| := \|\cdot\|_{\mathbb{R}}$ is the classicial simplicial volume, introduced by Gromov [18, 10].

Similarly, the usual norm on $\mathbb{Z}$ is an absolute value on $\mathbb{Z}$. The corresponding simplicial volume is denoted by $\|\cdot\|_{\mathbb{Z}}$, the so-called *integral simplicial volume*. Integral simplicial volume admits lower bounds in terms of Betti numbers [17, Example 14.28], logarithmic homology torsion [20], and the rank gradient of the fundamental group [15].

EXAMPLE 2.3 (Weightless simplicial volume). Every non-trivial commutative unital ring R can be equipped with the *trivial* seminorm

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$$\begin{aligned} |\cdot|_{\mathrm{triv}} \colon R &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto 1 - \delta_{x,0}. \end{aligned}$$

The simplicial volume corresponding to the trivial seminorm will be called *weightless* and will be denoted by $\|M, \partial M\|_{(R)}$. The weightless simplicial volume over finite fields $R = \mathbb{F}_p$ has been studied before [**16**].

EXAMPLE 2.4 (p-adic simplicial volumes). Let p be a prime number. The ring of p-adic integers is denoted by $\mathbb{Z}_p$ and the field of p-adic numbers by $\mathbb{Q}_p$. The usual p-adic absolute value gives rise to two (possibly distinct) notions of p-adic simplicial volume, namely, $\|\cdot\|_{\mathbb{Z}_p}$ and $\|\cdot\|_{\mathbb{Q}_p}$, respectively.

LEMMA 2.5. *Let $(R, |\cdot|)$ be a seminormed ring and let $I \subset R$ be a proper ideal. We define*

$$|s + I|_{R/I} := \inf\{|s + i| \mid i \in I\}$$

for all $s + I \in R/I$. Then $|\cdot|_{R/I}$ is a seminorm on the quotient ring R/I provided $|1 + I|_{R/I} = 1$.

Proof. We verify the submultiplicativity. Indeed, for all $s, t \in R$, one obtains

$$\begin{aligned} |(s+I)(t+I)|_{R/I} &= \inf\{|st + i| \mid i \in I\} \leq \inf\{|(s+i)(t+j)| \mid i, j \in I\} \\ &\leq \inf\{|(s+i)||(t+j)| \mid i, j \in I\} \leq |s+I|_{R/I}|t+I|_{R/I}. \end{aligned}$$

The triangle inequality follows from a similar argument. □

EXAMPLE 2.6 (Seminorms on $\mathbb{Z}/p^m\mathbb{Z}$). There are two distinct seminorms on the rings $\mathbb{Z}/p^m\mathbb{Z}$ that will play a role in this article. Using Lemma 2.5, the p-adic absolute value $|\cdot|_p$ on $\mathbb{Z}_p$ induces a seminorm on $\mathbb{Z}/p^m\mathbb{Z}$. We will also denote this seminorm by $|\cdot|_p$; for $x \neq 0$, it is given by

$$|x|_p = p^{-r},$$

if x lies in $p^r\mathbb{Z}/p^m\mathbb{Z}$ but not in $p^{r+1}\mathbb{Z}/p^m\mathbb{Z}$. The corresponding simplicial volume will be denoted by $\|\cdot\|_{\mathbb{Z}/p^m\mathbb{Z}}$.

As in Example 2.3, the rings $\mathbb{Z}/p^m\mathbb{Z}$ can be equipped with the trivial seminorm, which induces the weightless simplicial volume $\|\cdot\|_{(\mathbb{Z}/p^m\mathbb{Z})}$. If $m = 1$, then the quotient seminorm and the trivial seminorm on $\mathbb{Z}/p\mathbb{Z}$ coincide.

2.2. Changing the seminorm. Let R be a commutative ring with unit. We denote by $\mathcal{S}(R)$ the set of all seminorms on R. We equip the space of all seminorms with the topology of pointwise convergence, for which a basis of open neighborhoods of a seminorm α is given by the sets

$$U_{\mathrm{pw}}(\varepsilon, F) := \left\{\beta \in \mathcal{S}(R) \;\middle|\; \forall_{x \in F} \;\; |\beta(x) - \alpha(x)| < \varepsilon\right\},$$

where $\varepsilon \in \mathbb{R}_{>0}$ and F is a finite subset of R. For every oriented compact connected manifold M, the simplicial volume defines a function

$$\|M, \partial M\|_{\bullet} \colon \mathcal{S}(R) \to \mathbb{R}_{\geq 0}.$$

PROPOSITION 2.7 (Upper semicontinuity). *Let M be an oriented compact connected manifold. The simplicial volume function $\|M, \partial M\|_{\bullet}$ is upper semicontinuous with respect to the topology of pointwise convergence.*

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Proof. Let $\alpha \in \mathcal{S}(R)$ and let $\varepsilon > 0$. Take a relative fundamental cycle $c = \sum_{j=1}^{k} a_j \sigma_j \in Z(M, \partial M; R)$ with $|c|_{\alpha,1} < \|M, \partial M\|_\alpha + \varepsilon/2$. Now every seminorm $\beta \in U_{\mathrm{pw}}(\varepsilon/2k, \{a_1, \ldots, a_k\})$ satisfies

$$\|M, \partial M\|_\beta \leq |c|_{\beta,1} \leq |c|_{\alpha,1} + \varepsilon/2 < \|M, \partial M\|_\alpha + \varepsilon,$$

and we deduce that the simplicial volume is upper semicontinuous with respect to the topology of pointwise convergence. □

2.3. Changing the coefficients.

PROPOSITION 2.8 (Monotonicity). *Let $(R, |\cdot|_R)$ and $(S, |\cdot|_S)$ be seminormed rings and let $f : R \longrightarrow S$ be a unital ring homomorphism that for some $\lambda > 0$ satisfies $|f(x)|_S \leq \lambda \cdot |x|_R$ for all $x \in R$. Then,*

$$\|M, \partial M\|_S \leq \lambda \cdot \|M, \partial M\|_R$$

holds for all oriented compact connected manifolds M.

Proof. As f is unital, the chain map $C_*(\mathrm{Id}_M; f) : C_*(M; R) \longrightarrow C_*(M; S)$ induced by f maps relative R-fundamental cycles to relative S-fundamental cycles of $(M, \partial M)$. Moreover, $\|C_*(\mathrm{Id}_M; f)\| \leq \lambda$ (whence $\|H_*(\mathrm{Id}_M; f)\| \leq \lambda$), because $\|f\| \leq \lambda$. Therefore,

$$\|M, \partial M\|_S = \big\|[M, \partial M]_S\big\|_S = \big\|H_*(\mathrm{Id}_M; f)([M, \partial M]_R)\big\|_S \leq \lambda \|M, \partial M\|_R,$$

as claimed. □

COROLLARY 2.9 (Universal integral bound). *Let R be a seminormed ring, and let M be an oriented compact connected manifold. Then,*

$$\|M, \partial M\|_R \leq \|M, \partial M\|_{\mathbb{Z}}.$$

Proof. It follows from the triangle inequality that the canonical unital ring homomorphism $\mathbb{Z} \longrightarrow R$ satisfies the hypotheses of Proposition 2.8 with the factor $\lambda = 1$. □

COROLLARY 2.10. *Let p be a prime number, and let M be an oriented compact connected manifold. Then, the following inequalities hold for all $m \geq 1$:*

(1) $\|M, \partial M\|_{(\mathbb{F}_p)} \leq \|M, \partial M\|_{\mathbb{Z}/p^m\mathbb{Z}} \leq \|M, \partial M\|_{\mathbb{Z}_p} \leq \|M, \partial M\|_{\mathbb{Z}}$,
(2) $\|M, \partial M\|_{\mathbb{Q}_p} \leq \|M, \partial M\|_{\mathbb{Z}_p} \leq \|M, \partial M\|_{\mathbb{Z}}$,
(3) $\|M, \partial M\|_{\mathbb{Z}/p^m\mathbb{Z}} \leq \|M, \partial M\|_{\mathbb{Z}/p^{m+1}\mathbb{Z}}$,
(4) $\|M, \partial M\|_{\mathbb{Z}/p^m\mathbb{Z}} \leq \|M, \partial M\|_{(\mathbb{Z}/p^m\mathbb{Z})} \leq p^{m-1} \|M, \partial M\|_{\mathbb{Z}/p^m\mathbb{Z}}$.

Proof. For the first three assertions, we only need to apply Proposition 2.8 with $\lambda = 1$ (and Corollary 2.9) to the canonical projections

$$\mathbb{Z}_p \longrightarrow \mathbb{Z}/p^{m+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^m\mathbb{Z} \longrightarrow \mathbb{F}_p$$

and to the canonical inclusion $\mathbb{Z}_p \longrightarrow \mathbb{Q}_p$. The last assertion follows from Proposition 2.8 and the inequalities

$$|\cdot|_p \leq |\cdot|_{\mathrm{triv}} \leq p^{m-1} |\cdot|_p$$

between the p-adic and the trivial seminorm on the ring $\mathbb{Z}/p^m\mathbb{Z}$. □

PROPOSITION 2.11 (Density). *Let $(R, |\cdot|_R)$ and $(S, |\cdot|_S)$ be seminormed rings, and let $f : R \longrightarrow S$ be a unital ring homomorphism with $|\cdot|_S$-dense image. If $|f(x)|_S = |x|_R$ for all $x \in R$, then*

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$$\|M, \partial M\|_R = \|M, \partial M\|_S$$

holds for all oriented compact connected manifolds M.

Proof. The inequality $\|M, \partial M\|_S \leq \|M, \partial M\|_R$ follows from Proposition 2.8. The converse inequality works as in the classical case, by approximating boundaries of chains [**21**, Lemma 2.9].

We briefly recall the argument. Let $d = \dim(M)$ and let $\varepsilon > 0$. Take a fundamental cycle $c \in Z(M, \partial M; S)$ with $|c|_{1,S} \leq \|M, \partial M\|_S + \varepsilon$ and some fundamental cycle $c' \in Z(M, \partial M; R)$. Then, $b = c - C_d(\mathrm{Id}_M; f)(c')$ is a boundary, that is, $b = \partial_{d+1}(x)$ for some $x \in C_{d+1}(M; S)$. As the image of f is dense, we find an element $x' \in C_{d+1}(M; R)$ that satisfies

$$|C_d(\mathrm{Id}_M; f)(x') - x|_{1,S} \leq \varepsilon.$$

Then, $c' + \partial_{d+1}(x')$ is a fundamental cycle in $Z(M, \partial M; R)$, which satisfies

$$\begin{aligned}\|M, \partial M\|_R \leq |c' + \partial_{d+1}(x')|_{1,R} &= |C_d(\mathrm{Id}_M; f)(c' + \partial_{d+1}(x'))|_{1,S} \\ &= \left|c - b + \partial_{d+1} C_d(\mathrm{Id}_M; f)(x')\right|_{1,S} \\ &= \left|c + \partial_{d+1}\left(C_d(\mathrm{Id}_M; f)(x') - x\right)\right|_{1,S} \\ &\leq \|M, \partial M\|_S + \varepsilon + (d+2)\varepsilon.\end{aligned}$$

Taking $\varepsilon \to 0$ proves the claim. □

COROLLARY 2.12. *Let p be a prime number, let $|\cdot|_p$ denote the p-adic absolute value on $\mathbb{Z}$ and $\mathbb{Q}$, and let M be an oriented compact connected manifold. Then,*

$$\begin{aligned}\|M, \partial M\|_{\mathbb{Z}_p} &= \|M, \partial M\|_{\mathbb{Z}, |\cdot|_p}, \\ \|M, \partial M\|_{\mathbb{Q}_p} &= \|M, \partial M\|_{\mathbb{Q}, |\cdot|_p}.\end{aligned}$$

Proof. By definition, $\mathbb{Z}$ is $|\cdot|_p$-dense in $\mathbb{Z}_p$ and $\mathbb{Q}$ is $|\cdot|_p$-dense in $\mathbb{Q}_p$. Therefore, we can apply Proposition 2.11. □

In fact, we have the following simultaneous approximation result:

COROLLARY 2.13. *Let M be an oriented compact connected manifold, and let T be a finite set of prime numbers.*

(1) For every $\varepsilon > 0$, there is a $c \in Z(M, \partial M; \mathbb{Z})$, such that for all $p \in T$

$$|c|_{1,p} \leq \|M, \partial M\|_{\mathbb{Z}_p} + \varepsilon.$$

(2) For every $\varepsilon > 0$, there is a $c \in Z(M, \partial M; \mathbb{Q})$ such that for all $p \in T$

$$|c|_{1,p} \leq \|M, \partial M\|_{\mathbb{Q}_p} + \varepsilon \quad \textit{and} \quad |c|_{1,\mathbb{R}} \leq \|M, \partial M\|_{\mathbb{R}} + \varepsilon.$$

Proof. For every prime $p \in T$, we pick (using Corollary 2.12) a relative fundamental cycle $c_p \in Z(M, \partial M; \mathbb{Z})$ that almost realizes the p-adic simplicial volume. The density of $\mathbb{Z}$ in $\prod_{p \in T} \mathbb{Z}_p$ [**19**, (3.4)] allows us to find integers $a_p \in \mathbb{Z}$ (for $p \in T$) such that a_p is close to 1 in the p-adic absolute value, but close to 0 in the q-adic absolute value for

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all $q \in T \setminus \{p\}$. The integer $r = 1 - \sum_{p \in T} a_p$ is close to 0 in the p-adic absolute values for all $p \in T$. Replacing a_{p_1} by $a_{p_1} + r$ for some $p_1 \in T$, we can arrange that $\sum_{p \in T} a_p = 1$. Then,

$$c = \sum_{p \in T} a_p c_p$$

is a relative fundamental cycle and approximates the p-adic simplicial volumes for all $p \in T$.

Assertion (2) follows from the same argument using the density of $\mathbb{Q}$ in the ring $\mathbb{R} \times \prod_{p \in T} \mathbb{Q}_p$. □

PROPOSITION 2.14. *Let M be an oriented compact connected manifold. If p is a prime number such that $\|M, \partial M\|_{\mathbb{Q}_p} < p$, then*

$$\|M, \partial M\|_{\mathbb{Q}_p} = \|M, \partial M\|_{\mathbb{Z}_p}.$$

If M is closed, $\dim M$ *is even, and $\|M\|_{\mathbb{Q}_p} < 2p$, then $\|M\|_{\mathbb{Q}_p} = \|M\|_{\mathbb{Z}_p}$.*

In particular: For almost all primes p, we have

$$\|M, \partial M\|_{\mathbb{Q}_p} = \|M, \partial M\|_{\mathbb{Z}_p}.$$

Proof. By Corollary 2.10, we only need to take care of the estimate

$$\|M, \partial M\|_{\mathbb{Q}_p} \geq \|M, \partial M\|_{\mathbb{Z}_p}.$$

If $\|M, \partial M\|_{\mathbb{Q}_p} < p$, then every relative fundamental cycle $\sum_{j=1}^{k} a_j \cdot \sigma_j \in C_*(M; \mathbb{Q}_p)$ with norm less than p satisfies $|a_j|_p < p$ for all $j \in \{1, \ldots, k\}$ and so $a_j \in \mathbb{Z}_p$.

Suppose that M is closed, $\dim M$ is even and that $\|M\|_{\mathbb{Q}_p} < 2p$. We claim that every fundamental cycle $c = \sum_{j=1}^{k} a_j \cdot \sigma_j \in C_*(M; \mathbb{Q}_p)$ with $|c|_{1,p} < 2p$ lies in $C_*(M; \mathbb{Z}_p)$. Indeed, suppose that, say, $a_1 \notin \mathbb{Z}_p$, then $|a_1|_p = p$ and $|a_j|_p \leq 1$ for all $j > 1$. We multiply c with p to observe that the simplex σ_1 is a cycle modulo p; this is impossible, since an even-dimensional simplex has an odd number of faces, which are summed up with alternating signs.

For all primes p, we have $\|M, \partial M\|_{\mathbb{Q}_p} \leq \|M, \partial M\|_{\mathbb{Z}}$ (Corollary 2.10). Therefore, each prime $p > \|M, \partial M\|_{\mathbb{Z}}$ satisfies the hypothesis of the first part. □

We will continue the investigation of the relation between the $\mathbb{Z}_p$-version and the $\mathbb{Q}_p$-version with slightly different methods in Section 3.3.

2.4. Scaling the fundamental class. The definition of simplicial volume $\|\cdot\|_R$ with coefficients in a seminormed ring R clearly can be extended to all homology classes in singular homology $H_*(\cdot; R)$ with R-coefficients.

PROPOSITION 2.15. *Let p be a prime number, and let M be an oriented compact connected manifold. Then, the sequence $(p^m \cdot \|p^m \cdot [M, \partial M]_{\mathbb{Z}_p}\|_{\mathbb{Z}_p})_{m \in \mathbb{N}}$ is monotonically decreasing and*

$$\|M, \partial M\|_{\mathbb{Q}_p} = \lim_{m \to \infty} p^m \cdot \|p^m \cdot [M, \partial M]_{\mathbb{Z}_p}\|_{\mathbb{Z}_p}.$$

Proof. If $m \in \mathbb{N}_{>0}$ and $c \in C_{\dim M}(M; \mathbb{Z}_p)$ is a relative cycle that represents $p^m \cdot [M, \partial M]_{\mathbb{Z}_p}$, then $p \cdot c$ represents $p^{m+1} \cdot [M, \partial M]_{\mathbb{Z}_p}$ and hence

$$p^{m+1} \cdot \|p^{m+1} \cdot [M, \partial M]\|_{\mathbb{Z}_p} \leq p^{m+1} \cdot |p \cdot c|_1 = p^m \cdot |c|_1.$$

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In addition, the chain $p^{-m} \cdot c$ is a relative $\mathbb{Q}_p$-fundamental cycle of $(M, \partial M)$ and so $\|M, \partial M\|_{\mathbb{Q}_p} \leq p^m \cdot |c|_{1,p}$. Taking the infimum over all such c implies monotonicity of the sequence and shows that

$$\|M, \partial M\|_{\mathbb{Q}_p} \leq p^m \cdot \|p^m \cdot [M, \partial M]_{\mathbb{Z}_p}\|_{\mathbb{Z}_p}.$$

Conversely, let $\varepsilon \in \mathbb{R}_{>0}$ and let $c \in Z(M, \partial M; \mathbb{Q}_p)$ with $|c|_{1,p} \leq \|M, \partial M\|_{\mathbb{Q}_p} + \varepsilon$. The relative cycle c has only finitely many coefficients; hence, there exists an $r \in \mathbb{N}$ such that all coefficients of c lie in $p^{-r} \cdot \mathbb{Z}_p$. Thus, for all $m \in \mathbb{N}_{\geq r}$, the relative cycle $p^m \cdot c$ represents $p^m \cdot [M, \partial M]_{\mathbb{Z}_p}$, which yields

$$\|p^m \cdot [M, \partial M]_{\mathbb{Z}_p}\|_{\mathbb{Z}_p} \leq p^{-m} \cdot |c|_{1,p} \leq p^{-m} \cdot \|M, \partial M\|_{\mathbb{Q}_p} + p^{-m} \cdot \varepsilon,$$

for all $m \in \mathbb{N}_{\geq r}$. Taking $\varepsilon \to 0$ then proves the claim. □

2.5. The degree estimate.

PROPOSITION 2.16. *Let $(R, |\cdot|_R)$ be a seminormed ring, and let $f \colon M \longrightarrow N$ be a continuous map between oriented compact connected manifolds of the same dimension. If* $\deg f \in R^\times \cup \{0\}$ *(where $R^\times$ denotes the set of multiplicative units of R), then*

$$|\deg f|_R \cdot \|N, \partial N\|_R \leq \|M, \partial M\|_R.$$

Proof. If $\deg f = 0$, the assertion is obvious. Therefore, we may assume that $\deg f \in R^\times$. If $c \in Z(M, \partial M; R)$, then, by definition of the mapping degree, $1/\deg f \cdot C_*(f; R)(c) \in Z(N, \partial N; R)$. Therefore,

$$|\deg f|_R \cdot \|N, \partial N\|_R \leq |\deg f|_R \cdot \frac{1}{|\deg f|_R} \cdot \big|C_*(f; R)(c)\big|_{1,R} \leq |c|_{1,R}.$$

We can now take the infimum over all c. □

COROLLARY 2.17. *Let p be a prime number, and let $M \longrightarrow N$ be a continuous map between oriented compact connected manifolds of the same dimension.*

(1) Then, $|\deg f|_p \cdot \|N, \partial N\|_{\mathbb{Q}_p} \leq \|M, \partial M\|_{\mathbb{Q}_p}$.
(2) If p is coprime to $\deg f$*, then* $\|N, \partial N\|_{\mathbb{Z}_p} \leq \|M, \partial M\|_{\mathbb{Z}_p}$.

Proof. This is an immediate consequence of Proposition 2.16. □

PROPOSITION 2.18. *Let R be a seminormed ring, and let $f \colon M \longrightarrow N$ be a finite ℓ-sheeted covering map of oriented compact connected manifolds of the same dimension. Then,*

$$\|M, \partial M\|_R \leq \ell \cdot \|N, \partial N\|_R.$$

Proof. Let $c \in Z(N, \partial N; R)$, say $c = \sum_{j=1}^k a_j \cdot \sigma_j$. Then, the transfer

$$\tau(c) := \sum_{j=1}^k a_j \cdot \tau(\sigma_j) \in C_{\dim M}(M; R)$$

is a relative R-fundamental cycle of $(M, \partial M)$. Since $\tau(\sigma_j)$ is a sum of ℓ distinct singular simplices, the triangle inequality implies the claim. □

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3. Poincaré duality and homological estimates. We will now use Poincaré duality to establish Betti number estimates, we will use the semi-simplicial sets associated with fundamental cycles to study the dependence on the primes, and we will derive a simple product estimate. Variations of these arguments have been used before [11, p. 301][20, Section 3.2] in related situations.

3.1. Poincaré duality. Let M be an oriented compact connected d-manifold, let R be a ring with unit, and let $c = \sum_{j=1}^{k} a_j\sigma_j \in Z(M, \partial M; R)$. By Poincaré–Lefschetz duality [12, 3.43], for each $n \in \mathbb{N}$, the cap product map

$$\begin{aligned} H^{d-n}(M, \partial M; R) &\longrightarrow H_n(M; R) \\ [f] &\longmapsto [f] \cap [c] = \pm\left[\sum_{j=1}^{k} a_j \cdot f(\sigma_j|_{[n,\dots,d]}) \cdot \sigma_j|_{[0,\dots,n]}\right] \end{aligned} \tag{3.1}$$

is an R-isomorphism. There is an analogous duality between $H^{d-n}(M; R)$ and $H_n(M, \partial M; R)$.

3.2. The semi-simplicial set generated by a fundamental cycle. Let M be an oriented compact connected d-manifold, and let $c = \sum_{j=1}^{k} a_j\sigma_j \in C_d(M; R)$ be a relative fundamental cycle in reduced form. For each n, we define X_n to be the set of all n-dimensional faces of the simplices $\sigma_1, \dots, \sigma_k$. The ordinary face maps endow $X = (X_n)_{n\in\mathbb{N}}$ with the structure of a semi-simplicial set, which will be called the *semi-simplicial set generated by c*. The semi-simplicial subset of simplices of X contained in the boundary ∂M will be denoted ∂X. There is a canonical continuous map of pairs

$$\rho_c\colon (X^{\text{top}}, \partial X^{\text{top}}) \to (M, \partial M)$$

from the geometric realization X^{top} of X into M. The cycle c defines a relative homology class in $H_d(X, \partial X; R)$, which will be denoted by $[X, \partial X]$.

Lemma 3.1. *Let X be the semi-simplicial set generated by a relative fundamental cycle c of an oriented compact connected manifold M. The maps*

$$\begin{aligned} &H_n(\rho_c; R)\colon H_n(X; R) \to H_n(M; R), \\ &H_n(\rho_c; R)\colon H_n(X, \partial X; R) \to H_n(M, \partial M; R) \end{aligned}$$

are surjective for all n. The maps

$$\begin{aligned} &H^n(\rho_c; R)\colon H^n(M; R) \to H^n(X; R), \\ &H^n(\rho_c; R)\colon H^n(M, \partial M; R) \to H^n(X, \partial X; R) \end{aligned}$$

are injective for all $n \in \mathbb{N}$.

Proof. We write $c = \sum_{j=1}^{k} a_j\sigma_j \in C_d(M; R)$. In view of Poincaré duality (as in (3.1)), every class in $H_n(M; R)$ (respectively, in $H_n(M, \partial M; R)$) can be represented by a cycle supported on the faces $\sigma_1|_{[0,\dots,n]}, \dots, \sigma_k|_{[0,\dots,n]}$. This proves surjectivity, since all these cycles lie in the image of $C_n(\rho_c; R)$.

The injectivity of $H^n(\rho_c; R)\colon H^n(M; R) \to H^n(X; R)$ follows from the surjectivity statement using the commutative diagram

$$\begin{array}{ccc} H^n(M;R) & \xrightarrow{[M,\partial M]\cap} & H_{d-n}(M,\partial M;R) \\ \big\downarrow{\scriptstyle H^n(\rho_c;R)} & & {\scriptstyle H_{d-n}(\rho_c;R)}\big\uparrow \\ H^n(X;R) & \xrightarrow{[X,\partial X]\cap} & H_{d-n}(X,\partial X;R), \end{array}$$

where the cap product with $[M, \partial M]$ is an isomorphism by duality. The last assertion follows similarly, interchanging relative and absolute homology. □

3.3. Dependence on the prime.

Proof of Theorem 1.1. Let M be an oriented compact connected d-manifold. We want to show that

$$\|M, \partial M\|_{(\mathbb{F}_p)} = \|M, \partial M\|_{\mathbb{Z}_p} = \|M, \partial M\|_{\mathbb{Q}_p} = \|M, \partial M\|_{(\mathbb{Q})}$$

holds for almost all prime numbers p. The argument given here is based on the idea of the corresponding result for weightless simplicial volumes [16, Theorem 1.2], reformulated in the language of semi-simplicial sets. We recall that the equality $\|M, \partial M\|_{\mathbb{Z}_p} = \|M, \partial M\|_{\mathbb{Q}_p}$ holds for almost all primes by Proposition 2.14.

By Corollary 2.9, the inequality $\|M, \partial M\|_{(\mathbb{F}_p)} \leq \|M, \partial M\|_{\mathbb{Z}}$ holds for all primes. In particular, the weightless simplicial $\mathbb{F}_p$-volume is always attained on a relative cycle with at most $k := \|M, \partial M\|_{\mathbb{Z}}$ simplicies.

Say $d := \dim_M$. There are only finitely many distinct isomorphism classes of pairs $(X, \partial X)$ consisting of a d-dimensional semi-simplicial set X generated by at most k simplices of dimension d and a semi-simplicial subset ∂X; we write S_k^d for a set of representatives of these isomorphism classes. Let $(X, \partial X) \in S_k^d$. The boundary map $\partial_d\colon C_d(X, \partial X; \mathbb{Z}) \to C_{d-1}(X, \partial X; \mathbb{Z})$ is a linear map between free $\mathbb{Z}$-modules of finite rank and, as such, has a finite number of elementary divisors. In particular, there is a cofinite set W of primes that do not divide any elementary divisor of a boundary map ∂_d of an $(X, \partial X) \in S_k^d$.

Let $p \in W$ and let $\mathbb{Z}_{(p)}$ denote the localization of $\mathbb{Z}$ at the prime ideal $(p) \subseteq \mathbb{Z}$. Let c be a relative fundamental cycle in $C_d(M; \mathbb{F}_p)$ that realizes the mod p simplicial volume. We will show that c lifts to a relative cycle $\tilde{c} \in C_d(M; \mathbb{Z}_{(p)})$ supported on the same set of simplices; by Proposition 2.8, this yields

$$\|M, \partial M\|_{(\mathbb{Q})} \leq \|M, \partial M\|_{(\mathbb{Z}_{(p)})} \leq \|M, \partial M\|_{(\mathbb{F}_p)}.$$

Using that $\|M, \partial M\|_{(\mathbb{F}_p)} \leq \|M, \partial M\|_{(\mathbb{Q})}$ holds for almost all primes [16, Proof of Theorem 1.2], the equality of the two weightless terms for almost all primes follows. We observe that $\mathbb{Z}_{(p)}$ is a dense subring of $\mathbb{Z}_p$ and that the inequality $|\cdot|_p \leq |\cdot|_{\text{triv}}$ holds on $\mathbb{Z}_{(p)}$; using Propositions 2.11 and 2.8, we obtain

$$\|M, \partial M\|_{\mathbb{Z}_p} = \|M, \partial M\|_{\mathbb{Z}_{(p)}, |\cdot|_p} \leq \|M, \partial M\|_{(\mathbb{Z}_{(p)})} \leq \|M, \partial M\|_{(\mathbb{F}_p)}$$

for all $p \in W$. By Corollary 2.10, this implies equality.

Thus, it remains to show that c admits a lift: consider the semi-simplicial set X generated by c. Since $p \in W$, all elementary divisors of the boundary map $\partial_d\colon C_d(X, \partial X; \mathbb{Z}_{(p)}) \to C_{d-1}(X, \partial X; \mathbb{Z}_{(p)})$ are 1. In other words, after a suitable choice of bases, the $\mathbb{Z}_{(p)}$-linear map ∂_d can be represented by a diagonal matrix with entries

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0 and 1. Based on this description, it is an elementary observation that every element in the kernel of $\bar{\partial}_d \colon C_d(X, \partial X; \mathbb{F}_p) \to C_{d-1}(X, \partial X; \mathbb{F}_p)$ lifts to an element in the kernel of ∂_d [16]. To conclude, we note that a class in $H_d(M, \partial M; \mathbb{Z}_{(p)})$ is a fundamental class if and only if it reduces to a fundamental class in $H_d(M, \partial M; \mathbb{F}_p)$. □

3.4. Betti number estimates.

PROPOSITION 3.2. *Let M be an oriented compact connected manifold, let $(R, |\cdot|_R)$ be a seminormed principal ideal domain with $|x|_R \geq 1$ for all $x \in R \setminus \{0\}$, and let $n \in \mathbb{N}$. Then,*

$$\operatorname{rk}_R H_n(M; R) \leq \|M, \partial M\|_R.$$

Proof. We proceed as in the closed case [9, Lemma 4.1]: Let $d := \dim M$ and let $c \in Z(M, \partial M; R)$, say $c = \sum_{j=1}^k a_j \cdot \sigma_j$. By Poincaré–Lefschetz duality, the cap product map given in (3.1) is an R-isomorphism. Hence, $H_n(M; R)$ is a subquotient of an R-module that is generated by k elements. Therefore, $\operatorname{rk}_R H_n(M; R) \leq k$. Moreover, the condition on $|\cdot|_R$ implies that $k \leq |c|_{1,R}$. Taking the infimum over all c gives the desired estimate. □

COROLLARY 3.3. *If $p \in \mathbb{N}$ is prime and M is an oriented compact connected manifold, then, for all $n \in \mathbb{N}$, we have*

$$b_n(M; \mathbb{F}_p) \leq \|M, \partial M\|_{(\mathbb{F}_p)} \leq \|M, \partial M\|_{\mathbb{Z}_p}.$$

Proof. The first inequality follows from Proposition 3.2, the second inequality is contained in Corollary 2.10. □

Here is a refined p-torsion estimate of the same spirit:

PROPOSITION 3.4. *Let M be an oriented compact connected manifold and let $n, m \in \mathbb{N}$. Then,*

$$\dim_{\mathbb{F}_p} p^m H_n(M; \mathbb{Z}/p^{m+1}\mathbb{Z}) \leq \|p^m \cdot [M, \partial M]\|_{(\mathbb{Z}/p^{m+1}\mathbb{Z})}.$$

Proof. Let us first note that $p^m H_n(M; \mathbb{Z}/p^{m+1}\mathbb{Z})$ indeed carries a canonical $\mathbb{F}_p$-vector space structure.

We now proceed as in the proof of Proposition 3.2 and pick a cycle $c = \sum_{j=1}^k a_j \cdot \sigma_j$ that represents $p^m[M, \partial M]$. We may assume that $k = \|p^m[M, \partial M]\|_{(\mathbb{Z}/p^{m+1}\mathbb{Z})}$.

By Poincaré–Lefschetz duality (see (3.1)), the cap product with c yields a surjection

$$H^{d-n}(M, \partial M; \mathbb{Z}/p^{m+1}\mathbb{Z}) \longrightarrow p^m H_n(M; \mathbb{Z}/p^{m+1}\mathbb{Z}),$$

and thus every homology class on the right-hand side can be represented by a chain on the simplices $\sigma_1|_{[0,\dots,n]}, \dots, \sigma_k|_{[0,\dots,n]}$, that is, the right-hand side is isomorphic to a subquotient of $(\mathbb{Z}/p^{m+1}\mathbb{Z})^k$. Since every subquotient of $(\mathbb{Z}/p^{m+1}\mathbb{Z})^k$ can be generated by at most k elements, we deduce that

$$\dim_{\mathbb{F}_p} p^m H_n(M; \mathbb{Z}/p^{m+1}\mathbb{Z}) \leq k = \|p^m \cdot [M, \partial M]\|_{(\mathbb{Z}/p^{m+1}\mathbb{Z})}. \qquad \square$$

COROLLARY 3.5. *If $p \in \mathbb{N}$ is prime and M is an oriented compact connected manifold, then, for all $n \in \mathbb{N}$, we have*

$$b_n(M; \mathbb{Q}) \leq \|M, \partial M\|_{\mathbb{Q}_p}.$$

Proof. It follows from the universal coefficient theorem that

$$H_n(M; \mathbb{Z}/p^{m+1}\mathbb{Z}) \cong (\mathbb{Z}/p^{m+1}\mathbb{Z})^{b_n(M;\mathbb{Q})} \oplus T_{m+1},$$

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for a finite abelian group T_{m+1} of exponent at most p^{m+1}. In particular, we observe that $\dim_{\mathbb{F}_p} p^m H_n(M; \mathbb{Z}/p^{m+1}\mathbb{Z}) \geq b_n(M; \mathbb{Q})$.

Now Proposition 3.4 and Corollary 2.10 show that

$$\begin{aligned} b_n(M; \mathbb{Q}) &\leq \|p^m[M, \partial M]\|_{(\mathbb{Z}/p^{m+1}\mathbb{Z})}, \\ &\leq p^m \|p^m[M, \partial M]\|_{\mathbb{Z}/p^{m+1}\mathbb{Z}}, \\ &\leq p^m \|p^m[M, \partial M]\|_{\mathbb{Z}_p}. \end{aligned}$$

As m tends to ∞, we apply Proposition 2.15 to complete the proof. □

3.5. Maximality of the fundamental class. Similarly to the weightless case [**16**, Propositions 2.6, 2.10], also in the p-adic case, the fundamental class has maximal norm, which in particular leads to a basic estimate for products.

PROPOSITION 3.6 (maximality of the fundamental class). *Let $(R, |\cdot|)$ be a seminormed ring that satisfies $|x| \leq 1$ for all $x \in R$, let M be an oriented compact connected manifold, let $n \in \mathbb{N}$, and let $\alpha \in H_n(M; R)$ or $\alpha \in H_n(M, \partial M; R)$. Then,*

$$\|\alpha\|_{1,R} \leq \|M, \partial M\|_R.$$

Proof. The proof works as in the weightless case [**16**, Proposition 2.6]: Let $\alpha \in H_n(M; R)$ (the other case works in the same way); moreover, let $\varphi \in H^{d-n}(M, \partial M; R)$ be Poincaré dual to α, that is, $\varphi \cap [M]_R = \alpha$.

Let $f \in C^{d-n}(M; R)$ be a relative cocycle representing φ, and let $c = \sum_{j=1}^k a_j \sigma_j \in Z(M, \partial M; R)$. Then, the explicit Poincaré duality formula (3.1) shows that

$$z := \pm \sum_{j=1}^k a_j \cdot f(\sigma_j|_{[n,\ldots,d]}) \cdot \sigma_j|_{[0,\ldots,n]}$$

is a cycle representing α. In particular, the hypothesis on the seminorm on R implies that

$$\|\alpha\|_{1,R} \leq |z|_{1,R} \leq \sum_{j=1}^k |a_j| \cdot \left|f(\sigma_j|_{[n,\ldots,d]})\right| \leq \sum_{j=1}^k |a_j| = |c|_{1,R}.$$

Taking the infimum over all c proves that $\|\alpha\|_{1,R} \leq \|M, \partial M\|_R$. □

COROLLARY 3.7 (Product estimate). *Let $(R, |\cdot|)$ be a seminormed ring that satisfies $|x| \leq 1$ for all $x \in R$, and let M and N be oriented closed connected manifolds. Then,*

$$\max(\|M\|_R, \|N\|_R) \leq \|M \times N\|_R \leq \binom{\dim M + \dim N}{\dim M} \cdot \|M\|_R \cdot \|N\|_R.$$

Proof. The upper bound is the usual homological cross product argument [**2**, Theorem F.2.5]. Let $x \in N$. Then, the inclusion $M \longrightarrow M \times \{x\} \longrightarrow M \times N$ and the projection $M \times N \longrightarrow M$ show that $H_*(M; R)$ embeds isometrically into $H_*(M \times N; R)$ [**16**, proof of Proposition 2.10]. We can then apply Proposition 3.6 to M and N. □

In particular, Proposition 3.6 and Corollary 3.7 apply to $\mathbb{Z}_p$:

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COROLLARY 3.8. *Let p be a prime and let M and N be oriented closed connected manifolds. Then,*

$$\max\bigl(\|M\|_{\mathbb{Z}_p}, \|N\|_{\mathbb{Z}_p}\bigr) \leq \|M \times N\|_{\mathbb{Z}_p} \leq \binom{\dim M + \dim N}{\dim M} \cdot \|M\|_{\mathbb{Z}_p} \cdot \|N\|_{\mathbb{Z}_p}.$$

It might be tempting to go for a duality principle between singular homology with $\mathbb{Q}_p$-coefficients and bounded cohomology with $\mathbb{Q}_p$-coefficients. However, one should be aware that the considered ℓ^1-norm on the singular chain complex is an archimedean construction (the ℓ^1-norm) of a non-archimedean norm (the norm on $\mathbb{Q}_p$). In this mixed situation, no suitable version of the Hahn–Banach theorem can hold.

4. Basic examples.

4.1. Spheres, tori, projective spaces.

EXAMPLE 4.1 (spheres). Let $d \in \mathbb{N}$. It is known that [**14**]

$$\|S^d\|_{\mathbb{Z}} = \begin{cases} 1 & \text{if } d \text{ is odd} \\ 2 & \text{if } d \text{ is even.} \end{cases}$$

Now let p be a prime number.

- If d is odd, then $\|S^d\|_{\mathbb{Z}_p} = \|S^d\|_{\mathbb{Q}_p} = 1$: The Betti number estimate gives $1 = b_0(S^d; \mathbb{Q}) \leq \|S^d\|_{\mathbb{Q}_p} \leq \|S^d\|_{\mathbb{Z}_p}$ (Corollaries 3.5, 2.10). Moreover, we also have $\|S^d\|_{\mathbb{Z}_p} \leq \|S^d\|_{\mathbb{Z}} = 1$.
- If d is even, then $\|S^d\|_{\mathbb{Z}_p} = \|S^d\|_{\mathbb{Q}_p} = 2$. We have

  $$\|S^d\|_{\mathbb{Q}_p} \leq \|S^d\|_{\mathbb{Z}_p} \leq \|S^d\|_{\mathbb{Z}} = 2.$$

  Thus, it suffices to show that $\|S^d\|_{\mathbb{Q}_p} \geq 2$. In view of Corollary 2.12, we only need to show that $\|S^d\|_{\mathbb{Q}, |\cdot|_p} \geq 2$. Let $c \in Z(S^d; \mathbb{Q})$. *Assume* for a contradiction that $|c|_{1,p} < 2$. Then, we can write c in the form

  $$c = \sum_{j \in J} \frac{a_j}{m} \cdot \sigma_j + \sum_{k \in K} p \cdot \frac{b_k}{m} \cdot \tau_k,$$

  where J and K are finite sets, the coefficients a_j, b_k, and m are integral and where p does *not* divide m or the a_j with $j \in J$. Then,

  $$m \cdot c = \sum_{j \in J} a_j \cdot \sigma_j + p \cdot \sum_{k \in K} b_k \cdot \tau_k$$

  is a cycle in $C_d(S^d; \mathbb{Z})$, representing $m \cdot [M]_{\mathbb{Z}}$. Because p does not divide m, we obtain that $J \neq \emptyset$. (If J were empty, then $\sum_{k \in K} b_k \cdot \tau_k$ would also be a cycle and whence $m \cdot [M]_{\mathbb{Z}}$ would be divisible by p, which is impossible).

  Let ϱ denote the constant singular $(d-1)$-simplex on the one-point space $\bullet$. Applying the chain map induced by the constant map $S^d \longrightarrow \bullet$ to the equation $\partial(m \cdot c) = 0$ shows that

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$$0 = \sum_{j \in J} a_j \cdot \varrho + p \cdot \sum_{k \in K} b_k \cdot \varrho,$$
$$= \Big(\sum_{j \in J} a_j + p \cdot \sum_{k \in K} b_k\Big) \cdot \varrho,$$

holds in $C_{d-1}(\bullet; \mathbb{Z})$. Therefore, we obtain

$$0 = \sum_{j \in J} a_j + p \cdot \sum_{k \in K} b_k.$$

Because $J \neq \emptyset$ and p does not divide any of the a_j with $j \in J$, we see that J contains at least two elements i, j. In particular,

$$|c|_{1,p} \geq |a_i|_p + |a_j|_p \geq 1 + 1 = 2,$$

which contradicts our assumption $|c|_{1,p} < 2$. Thus, $\|S^d\|_{\mathbb{Q},|\cdot|_p} \geq 2$.

COROLLARY 4.2. *Let p be a prime number, and let M be an oriented closed connected (non-empty) manifold.*

(1) Then $\|M, \partial M\|_{\mathbb{Q}_p} \geq 1$.
(2) If $\dim M$ is even, then $\|M, \partial M\|_{\mathbb{Q}_p} \geq 2$.

Proof. As M is non-empty and closed, there exists a map $M \longrightarrow S^{\dim M}$ of degree 1. We can now apply the degree estimate (Corollary 2.17) and the computation for spheres (Example 4.1). □

EXAMPLE 4.3 (The torus). Let p be a prime number. Then the 2-torus T^2 satisfies

$$\|T^2\|_{\mathbb{Z}_p} = \|T^2\|_{\mathbb{Q}_p} = 2.$$

On the one hand, we can easily represent the fundamental class of T^2 by two singular triangles; on the other hand, Corollary 4.2 gives the lower bound.

EXAMPLE 4.4 (Projective spaces). Let $d \in \mathbb{N}$ be odd and let p be a prime number.

Case $p > 2$: If $p > 2$, then $\|\mathbb{R}P^d\|_{\mathbb{Z}_p} = \|\mathbb{R}P^d\|_{\mathbb{Q}_p} = 1$: From Corollary 4.2, we obtain $\|\mathbb{R}P^d\|_{\mathbb{Z}_p} \geq \|\mathbb{R}P^d\|_{\mathbb{Q}_p} \geq 1$. Furthermore, the double covering $S^d \longrightarrow \mathbb{R}P^d$ and the computation for spheres (Example 4.1) show that

$$\|\mathbb{R}P^d\|_{\mathbb{Z}_p} \leq \|S^d\|_{\mathbb{Z}_p} = 1 \quad \text{and} \quad \|\mathbb{R}P^d\|_{\mathbb{Q}_p} \leq \|S^d\|_{\mathbb{Q}_p} = 1$$

using $p > 2$ in Corollary 2.17.

It should be noted that $\|\mathbb{R}P^d\|_{\mathbb{Z}} = 2$ [**14**, Proposition 4.4][**16**, Example 2.7].

Case $p = 2$: We have $\|\mathbb{R}P^d\|_{\mathbb{Z}_2} = 2$, because $\|\mathbb{R}P^d\|_{(\mathbb{F}_2)} = 2$ [**16**, Example 2.7] and $\|\mathbb{R}P^d\|_{\mathbb{Z}} = 2$ [**14**, Proposition 4.4] as well as $\|\cdot\|_{(\mathbb{F}_2)} \leq \|\cdot\|_{\mathbb{Z}_2} \leq \|\cdot\|_{\mathbb{Z}}$ (Corollary 2.10).

In addition, we claim that $\|\mathbb{R}P^d\|_{\mathbb{Q}_2} = 2$. We know that $\|\mathbb{R}P^d\|_{\mathbb{Q}_2} \leq \|\mathbb{R}P^d\|_{\mathbb{Z}_2} = 2$. *Assume* for a contradiction that $\|\mathbb{R}P^d\|_{\mathbb{Q}_2} < 2$. Then, Proposition 2.14 implies that $\|\mathbb{R}P^d\|_{\mathbb{Q}_2} = \|\mathbb{R}P^d\|_{\mathbb{Z}_2} = 2$, which yields a contradiction.

4.2. Surfaces. Recall that Σ_g denotes the oriented closed connected surface of genus g and $\Sigma_{g,b}$ denotes the surface of genus g with $b \geq 1$ boundary components.

Proof of Theorem 1.4. The case $\Sigma_0 \cong S^2$ is already contained in Example 4.1.

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We first prove the inequalities "$\geq$." Let M be Σ_g or $\Sigma_{g,b}$ and let K denote the field of fractions of R. We endow K with the trivial absolute value. Using the inequality $\|M, \partial M\|_{(R)} \geq \|M, \partial M\|_{(K)}$ from Proposition 2.8, we see that it is sufficient to establish the lower bound for the field K.

Let $c = \sum_{j=1}^{k} a_j\sigma_j \in C_2(M; K)$ be a fundamental cycle of minimal norm, that is, $|c|_1 = k$ is minimal. Consider the semi-simplicial set X generated by c and its chain complex

$$C_0(X, \partial X; K) \xleftarrow{\partial_1} C_1(X, \partial X; K) \xleftarrow{\partial_2} C_2(X, \partial X; K) = C_2(X; K).$$

We observe that $\dim_K C_2(X; K) = k$, and we claim that the kernel of ∂_2 is 1-dimensional; that is, it is the line spanned by c. Assume for a contradiction that $\dim_K \ker(\partial_2) \geq 2$. In this case, the relative fundamental cycles supported on $\{\sigma_1, \ldots, \sigma_k\}$ form an affine subspace of dimension at least 1 in $C_2(M; K)$. Using elementary linear algebra, we deduce that there is a fundamental cycle supported on a proper subset of $\{\sigma_1, \ldots, \sigma_k\}$, which contradicts the minimality of k. In particular, $\dim_K B_1(X, \partial X; K) = k - 1$.

The k different simplices in X_2 have at most $3k$ distinct faces. Moreover, since $H_2(M, \partial M; K) \to H_1(\partial M; K)$ maps the relative fundamental class of M to a fundamental class of ∂M, the boundary of c touches every connected component of ∂M at least once. In other words, at most $3k - b$ faces of the simplices in X_2 are not contained in ∂M. As c is a relative cycle, every face of X_2 that is not contained in the boundary occurs at least twice. We conclude that $2 \dim_K C_1(X, \partial X; K) \leq 3k - b$.

By Lemma 3.1, we have the inequality

$$\dim_K H_1(X, \partial X; K) \geq \dim_K H_1(M, \partial M; K) = 2g + b - 1 + \delta_{b,0},$$

and the following calculation completes the first part of the proof

$$\begin{aligned} k - b + 2 &= 3k - b - 2(k - 1) \\ &\geq 2 \dim_K C_1(X, \partial X; K) - 2 \dim_K B_1(X, \partial X; K) \\ &\geq 2 \dim_K H_1(X, \partial X; K) \geq 4g + 2b - 2 + 2\delta_{b,0}. \end{aligned}$$

Moreover, in the pathological case of $\Sigma_{0,1}$, we have $\|\Sigma_{0,1}\|_{(R)} \geq 1$ by Proposition 3.2.

In order to show that the lower bound is sharp, we construct explicit relative fundamental cycles with the desired number of 2-simplices; this is done in Proposition 4.5 below for the integral simplicial volume. As integral simplicial volume is an upper bound for $\|\cdot\|_{(R)}$ (Proposition 2.8), this suffices to complete the proof. □

PROPOSITION 4.5. *Let $g \in \mathbb{N}$ and $b \in \mathbb{N}$. Then (with respect to the standard archimedean absolute value on $\mathbb{Z}$)*

(1) $\|\Sigma_g\|_{\mathbb{Z}} = 4g - 2$ if $g \geq 1$ and
(2) $\|\Sigma_{0,1}\|_{\mathbb{Z}} = 1$ and $\|\Sigma_{g,b}\|_{\mathbb{Z}} = 3b + 4g - 4$ for all $g \in \mathbb{N}$ and all $b \in \mathbb{N}_{\geq 1}$ with $(g, b) \neq (0, 1)$.

Proof. The lower bounds follow from the already established part of Theorem 1.4 and Proposition 2.8. Therefore, it suffices to establish the upper bounds: Let $g, b \in \mathbb{N}$. In the following pictures, boundary components are dashed and holes are shaded with stripes.

- In the case $g = 0$, $b = 1$, a single 2-simplex suffice (Figure 1, left).
- In the case $g = 0$ and $b = 2$, two 2-simplices suffices (Figure 1, right).

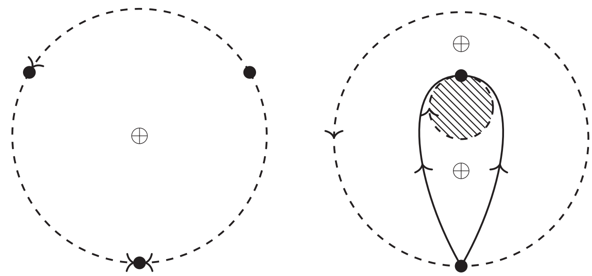

Figure 1. Left: Genus 0, with a single boundary component. Right: Genus 0, with two boundary components; this relative fundamental cycle consists of two singular 2-simplices.

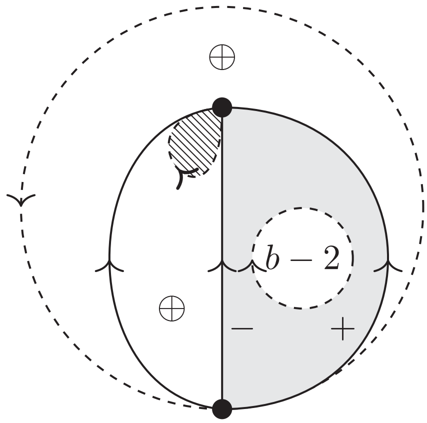

Figure 2. Genus 0, with $b \in \mathbb{N}_{>2}$ boundary components; there are $(b-2)$ triple building blocks.

- If $g = 0$ and $b \geq 3$, then Figure 2 shows how to construct a fundamental cycle consisting of

$$1 + 1 + 3(b-2) = 3b - 4$$

2-simplices. Here, we use $(b-2)$ triple building blocks of Figure 3, each consisting of three 2-simplices (Figure 4).
- If $g \geq 1$ and $b = 0$, we use the classical decomposition of the $4g$-gon (whose edges will be identified according to the labels) into $4g - 2$ simplices with the signs and orientations indicated in Figure 5.
- If $g \geq 1$ and $b \geq 1$, we can use the construction of Figure 6 with $(b-1)$ triple building blocks, where the upper part of the polygon is decomposed as in the closed higher genus case (Figure 5). Hence,

$$4g - 2 + 1 + 3(b-1) = 4g + 3b - 4,$$

2-simplices suffice. □

Corollary 4.6. *Let $g \in \mathbb{N}_{\geq 1}$ and let $p \in \mathbb{N}$ be prime.*

(1) Then $\|\Sigma_g\|_{\mathbb{Z}_p} = 4g - 2$.
(2) If $p > 2g - 1$, then $\|\Sigma_g\|_{\mathbb{Q}_p} = 4g - 2$.

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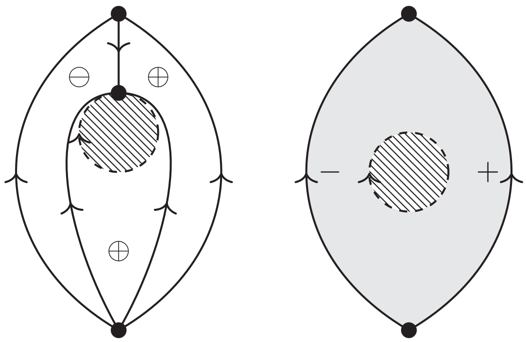

Figure 3. Left: The triple building block with a hole at the center; it consists of three singular 2-simplices with the indicated signs. Right: the graphical abbreviation we will use for this block; the signs at the edges are the signs resulting from the signs of the simplices on the left.

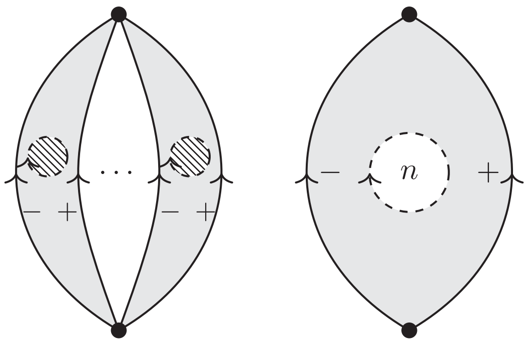

Figure 4. Left: Combining n triple building blocks "linearly" into a new building block, which consists of $3n$ simplices. Right: The abbreviation for the construction on the left-hand side.

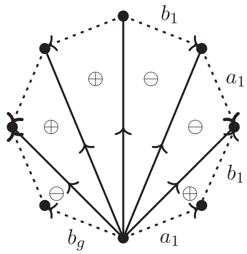

Figure 5. Genus at least 1, empty boundary. The dotted edges will be glued as specified by the labels. For higher genus, we just increase the number of ⊖ ⊖ ⊕⊕-blocks.

Proof. Ad 1. From Theorem 1.4 and Proposition 4.5, we obtain

$$4g - 2 \leq \|\Sigma_g\|_{(\mathbb{F}_p)} \leq \|\Sigma_g\|_{\mathbb{Z}_p} \leq \|\Sigma_g\|_{\mathbb{Z}} \leq 4g - 2$$

and thus the claimed equality.

Ad 2. This follows from the first part and Proposition 2.14. □

REMARK 4.7 (a new computation of ordinary simplicial volume of surfaces). Let $g \in \mathbb{N}_{\geq 2}$. Then the arguments above show that we can prove the identity $\|\Sigma_g\| = 4g - 4$ for the classical simplicial volume without hyperbolic straightening: From Proposition 4.5, we

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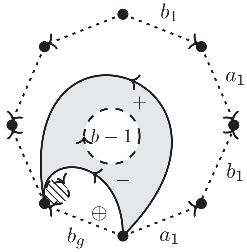

Figure 6. Genus at least 1 with $b \in \mathbb{N}_{\geq 1}$ boundary components; the shaded block consists of $(b-1)$ triple building blocks (containing $(b-1)$ boundary components). The dotted edges will be glued as specified by the labels. The upper part of the polygon is decomposed as in Figure 5.

know that $\|\Sigma_g\|_{\mathbb{Z}} = 4g-2$ (we proved this via semi-simplicial sets of cycles, without using hyperbolic straightening).

(a) We have $\|\Sigma_g\| \leq 4g-4$: For the sake of completeness, we recall Gromov's argument [10]. For each $k \in \mathbb{N}$, there exists a k-sheeted covering $\Sigma_{g_k} \longrightarrow \Sigma_g$, where $g_k = k \cdot g - k + 1$. Hence, we obtain (Proposition 2.16)

$$\|\Sigma_g\| \leq \inf_{k\in\mathbb{N}} \frac{\|\Sigma_{g_k}\|_{\mathbb{Z}}}{k} = \inf_{k\in\mathbb{N}} \frac{4\cdot(k\cdot g - k + 1) - 4}{k} = 4g-4.$$

(b) We have $\|\Sigma_g\| \geq 4g-4$: Because $\mathbb{Q}$ is dense in $\mathbb{R}$, we have (Proposition 2.11)

$$\|\Sigma_g\| = \|\Sigma_g\|_{\mathbb{Q}} = \inf\left\{ \frac{\|m\cdot[\Sigma_g]_{\mathbb{Z}}\|_{\mathbb{Z}}}{m} \;\middle|\; m \in \mathbb{N}_{>0} \right\}.$$

Let $m \in \mathbb{N}_{>0}$ and let $c = \sum_{j=1}^{k} a_j\sigma_j \in C_2(\Sigma_g;\mathbb{Z})$ be a cycle with $[c] = m\cdot[\Sigma_g]_{\mathbb{Z}}$ and $a_1,\dots,a_k \in \{-1,1\}$ as well as $|c|_1 = k$. Because c is a cycle, we can find a matching of the edges (and their signs) in the simplices of c such that the associated semi-simplicial set is a two-dimensional pseudo-manifold. As no proper singularities at the vertices can occur in dimension 2, this pseudo-manifold leads to a manifold, whence a surface. In other words, there is an oriented compact surface Σ (which we may assume to be connected) and a continuous map $f\colon \Sigma \longrightarrow \Sigma_g$ with

$$H_2(f;\mathbb{Z})([\Sigma]_{\mathbb{Z}}) = [c] = m\cdot[\Sigma_g]_{\mathbb{Z}} \in H_2(\Sigma_g;\mathbb{Z}) \quad \text{and} \quad \|\Sigma\|_{\mathbb{Z}} \leq k = |c|_1.$$

Smoothly approximating f and looking at the corresponding harmonic representative show that [3, p. 264]

$$m = |\deg f| \leq \frac{g(\Sigma)-1}{g-1}.$$

Therefore, we obtain

$$\frac{|c|_1}{m} \geq \frac{\|\Sigma\|_{\mathbb{Z}}}{m} \geq \frac{\big(4g(\Sigma)-2\big)\cdot(g-1)}{g(\Sigma)-1} \geq 4g-4.$$

Taking the infimum over all such cycles c proves the estimate.

4.3. On mod *p* and *p*-adic approximation of simplicial volume. Let M be an oriented compact connected manifold, and let $F(M)$ denote the set of all (isomorphism classes

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of) finite connected coverings of M. Moreover, let R be a seminormed ring. Then, the *stable R-simplicial volume* of M is defined by

$$\|M, \partial M\|_R^\infty := \inf_{(p\colon N\to M)\in F(M)} \frac{\|N, \partial N\|_R}{|\deg p|}.$$

Similar to the case of Betti numbers or logarithmic torsion, one might wonder for which manifolds M and which coefficients R, we have $\|M, \partial M\| = \|M, \partial M\|_R^\infty$. This question has been studied for $\mathbb{Z}$-coefficients [**8**, **9**, **15**, **5**, **6**, **7**] and to a much lesser degree for $\mathbb{F}_p$-coefficients [**16**]. However, it was, for instance, not even known whether the simplicial volume of surfaces satisfies mod p approximation. With the methods developed in the previous section, we can solve this problem for surfaces:

Remark 4.8 (Surfaces). The simplicial volume of surfaces satisfies integral, mod p, and p-adic approximation by the corresponding normalised simplicial volumes of finite coverings: Let $g \in \mathbb{N}_{\geq 1}$ and let p be a prime. Then, we have

$$\|\Sigma_g\| = 4g - 4 = \|\Sigma_g\|_\mathbb{Z}^\infty = \|\Sigma_g\|_{(\mathbb{F}_p)}^\infty = \|\Sigma_g\|_{\mathbb{Z}_p}^\infty.$$

The first two equalities are contained in Remark 4.7. By Proposition 4.5, Theorem 1.4, and Corollary 4.6, the integral, the weightless $\mathbb{F}_p$, and the $\mathbb{Z}_p$ simplicial volumes are equal for all closed surfaces; this implies the equality of the stable simplicial volumes.

Remark 4.9 (3-manifolds). Let M be an oriented closed connected aspherical 3-manifold. Then, it is known that [**7**]

$$\|M\|_\mathbb{Z}^\infty = \frac{\operatorname{hypvol}(M)}{v_3} = \|M\|.$$

Hence, by Corollary 2.10, we also have

$$\|M\|_{(\mathbb{F}_p)}^\infty \leq \|M\|_{\mathbb{Z}_p}^\infty \leq \|M\|_\mathbb{Z}^\infty = \frac{\operatorname{hypvol}(M)}{v_3}$$

for all primes p.

In the context of homology torsion growth, it would be interesting to determine whether $\|M\|_{(\mathbb{F}_p)}^\infty = \|M\|_{\mathbb{Z}_p}^\infty = \|M\|$ holds for all oriented closed connected aspherical 3-manifolds. This seems to be open already in the hyperbolic case.

Remark 4.10 (mod p Singer conjecture). Let p be an odd prime. Avramidi, Okun, Schreve established that the $\mathbb{F}_p$-Singer conjecture fails (in all high enough dimensions) [**1**], that is, there exist oriented closed connected aspherical d-manifolds M with residually finite fundamental group and an $n \neq d/2$ such that the associated $\mathbb{F}_p$-Betti number gradient in dimension n is non-zero.

By Corollary 3.3, also the mod p and the p-adic simplicial volume gradients are non-zero. In particular, the stable integral simplicial volume of M is non-zero.

It would be interesting to determine whether the classical simplicial volume of M is zero or not.

Remark 4.11 (Estimates for groups). The Betti number estimates from Section 3.4 give corresponding estimates between the homology gradients and the stable integral simplicial volumes over the given seminormed ring. These can be turned into homology gradient estimates for groups as follows: If G is a (discrete) group that admits a finite model X of the classifying space BG, we can embed X into a high-dimensional Euclidean

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space $\mathbb{R}^N$ and then thicken the image of the embedding to a compact manifold M with boundary, which is homotopy equivalent to X, whence has the same homology (gradients) as the group G. One can then study the behavior of simplicial volumes of M to get upper bounds for homology gradients of G.

5. Non-values. Analogously to the case of classical simplicial volume [13], we have:

THEOREM 5.1. *Let $p \in \mathbb{N}$ be prime, and let $A \subset \mathbb{N}$ be a set that is recursively enumerable but not recursive. Then, there is* no *oriented closed connected manifold M whose simplicial volume $\|M\|_{\mathbb{Q}_p}$ equals*

$$2 - \sum_{n \in \mathbb{N} \setminus A} 2^{-n}.$$

The same statement also holds for $\|M\|_{\mathbb{Z}_p}$.

The proof is based on the following notion: a real number $x \in \mathbb{R}$ is *right computable* if the set $\{a \in \mathbb{Q} \mid x < a\}$ is recursively enumerable [23]. For example, all algebraic numbers are (right-)computable [4, Section 6] and there are only countably many right computable real numbers.

Proof of Theorem 5.1. The numbers in this theorem are known to be *not* right computable: This can be easily derived from known properties of (right-)computable numbers and Specker sequences [22]. Therefore, this theorem is a direct consequence of the observation in Proposition 5.2 below. □

PROPOSITION 5.2. *Let $p \in \mathbb{N}$ be prime, and let M be an oriented closed connected manifold. Then, the real numbers $\|M\|_{\mathbb{Q}_p}$ and $\|M\|_{\mathbb{Z}_p}$ are right computable.*

Proof. We proceed as in the corresponding proof for ordinary simplicial volume [13, Theorem E]: The same combinatorial argument as for the integral norm $\|\cdot\|_{1,\mathbb{Z}}$ [13, Lemma 4.4] shows that the set

$$S := \bigl\{(m, a) \in \mathbb{N} \times \mathbb{Q} \bigm| \|m \cdot [M]_{\mathbb{Z}}\|_{\mathbb{Z},|\cdot|_p} < a\bigr\} \subset \mathbb{N} \times \mathbb{Q}$$

is recursively enumerable. In particular,

$$\bigl\{a \in \mathbb{Q} \bigm| \|[M]_{\mathbb{Z}}\|_{\mathbb{Z},|\cdot|_p} < a\bigr\} = \mathrm{pr}_2\bigl(S \cap \bigl(\{1\} \times \mathbb{Q}\bigr)\bigr)$$

is recursively enumerable, which means that $\|M\|_{\mathbb{Z}_p} = \|[M]_{\mathbb{Z}}\|_{\mathbb{Z},|\cdot|_p}$ (Corollary 2.12) is right computable.

Moreover, in combination with Proposition 2.15 and Corollary 2.12, we obtain that

$$\begin{aligned}\bigl\{a \in \mathbb{Q} \bigm| \|M\|_{\mathbb{Q}_p} < a\bigr\} &= \Bigl\{a \in \mathbb{Q} \Bigm| \exists_{m \in \mathbb{N}_{>0}} \ \|p^m \cdot [M]_{\mathbb{Z}}\|_{\mathbb{Z},|\cdot|_p} < \frac{a}{p^m}\Bigr\} \\ &= \Bigl\{a \in \mathbb{Q} \Bigm| \exists_{m \in \mathbb{N}_{>0}} \ \Bigl(p^m, \frac{a}{p^m}\Bigr) \in S\Bigr\}.\end{aligned}$$

Because S is recursively enumerable, this set is also recursively enumerable. Hence, $\|M\|_{\mathbb{Q}_p}$ is right computable. □

The same arguments also can be used to show the corresponding results for oriented compact connected manifolds with boundary.

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