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A note on p-valent functions

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1. Introduction.

The set of close-to-convex univalent functions introduced by Kaplan [**2**] and Umezawa [**3**] contains many familiar univalent ones, for instance the starlike functions, the functions convex in one direction, the functions starlike with respect to symmetrical points [**5**], and the functions with derivative of positive real part in the unit circle. It, however, does not contain the spiral-like ones.

Recently a wider sufficient condition for univalence which includes the weakest sufficient condition for spiral-likeness has been given by Ogawa [**1**], and it has been extended to the case of p-valence at the same time. His main theorem for p-valence may be stated without loss of equivalency as follows.

THEOREM A. *Let* $f(z)=z^p+\cdots$ *be regular in* $|z|\leqq r$, *and let* $f(z)f'(z)\neq 0$ *for* $0<|z|\leqq r$. *If* $f(z)$ *satisfies the condition*

$$\int_C [d\arg df(z)+kd\arg f(z)]>-\pi$$

for all arcs C *on* $|z|=r$, *where* k *is a real constant such that* $k>-(1+1/2p)$, *then* $f(z)$ *is* p-*valent in* $|z|\leqq r$.

The purpose of this note is to extend or improve Theorem A and some of other results in his paper [**1**].

2. Fundamental results.

LEMMA 1. *Let* $f(z)=z^p+\cdots$, $\varphi(z)$ *be regular in* $|z|\leqq r$ *and* $|z|<+\infty$ *respectively, and let* $f'(z)\neq 0$ *for* $0<|z|\leqq r$. *If neither* $f(z)$ *nor* $\varphi'(\log f(z))$ *vanishes on* $|z|=r$ *and the number of valence of* $f(z)$ *in* $|z|\leqq r$ *is larger than* p, *then there exists at least one arc* C *on* $|z|=r$ *such that*

$$\int_C d\arg d\varphi(\log f(z))\leqq -\pi\,. \tag{2.1}$$

PROOF. As shown by Ogawa [**1**, p. 434], under our assumption on $f(z)$ there exists such a loop C_w on the image curve of $|z|=r$ under $w=f(z)$ as neither passes nor surrounds the origin and satisfies the inequality

$$\int_{C_w} d \arg dw \leqq -\pi .$$

We now consider an arbitrary branch of $\log w$, and denote by C_ζ the image arc of C_w under $\zeta = \log w$. Then from the property of C_w stated above, we see that C_ζ is also a loop such that

$$\int_{C_\zeta} d \arg d\zeta \leqq -\pi .$$

Therefore because of the assumption that $\varphi'(\zeta) \neq 0$ on C_ζ, we have

$$\int_{C_\zeta} d \arg \varphi'(\zeta) = -2\pi n(0) ,$$

where $n(0)$ is the number of zeros of $\varphi'(\zeta)$ inside C_ζ. Consequently

$$\int_{C_\zeta} d \arg d\varphi(\zeta) = \int_{C_\zeta} d \arg \varphi'(\zeta) + \int_{C_\zeta} d \arg d\zeta \leqq -\pi ,$$

whence (2.1) holds for an arbitrary branch of $\log f(z)$, if we denote by C the arc on $|z| = r$ corresponding to C_w.

From this lemma we have at once

THEOREM 1. *Let $f(z) = z^p + \cdots$, $\varphi(z)$ be regular in $|z| \leqq r$ and $|z| < +\infty$ respectively, and let $f'(z) \neq 0$ for $0 < |z| \leqq r$. If neither $f(z)$ nor $\varphi'(\log f(z))$ vanishes on $|z| = r$ and the inequality*

$$\int_C d \arg d\varphi(\log f(z)) > -\pi \tag{2.2}$$

holds for all arcs C on $|z| = r$, then $f(z)$ is p-valent in $|z| \leqq r$.

Considering the two special cases where $\varphi(z) = z$ and $\varphi(z) = e^{(k+1)z}$ for a complex constant k, we have the following

COROLLARY 1. *Let $f(z) = z^p + \cdots$ be regular in $|z| \leqq r$, and let $f'(z) \neq 0$ for $0 < |z| \leqq r$. If $f(z)$ does not vanish on $|z| = r$ and satisfies the condition*

$$\int_C [d \arg df(z) + d \arg f(z)^k] > -\pi \tag{2.3}$$

for all arcs C on $|z| = r$, where k is a complex constant, then $f(z)$ is p-valent in $|z| \leqq r$.

REMARK 1. For all functions $f(z)$ satisfying the hypothesis of this corollary, we have

$$2\pi p(1 + \Re k) = \int_{|z|=r} [d \arg df(z) + d \arg f(z)^k] > -\pi ,$$

whence it is necessary that $\Re k > -(1 + 1/2p)$.

Corollary 1 is an extension of Theorem A.

COROLLARY 2. *Let $f(z) = z^p + \cdots$ be regular in $|z| \leqq r$, and let $f(z) \neq 0$ for $0 < |z| \leqq r$. If $f'(z)$ does not vanish on $|z| = r$ and the inequality*

$$(2.4)\qquad \int_0^{2\pi}\left|\Re\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)\right|d\theta<2\pi(1+p+p\Re k), \qquad z=re^{i\theta},$$

holds for a complex constant k such that $\Re k>-(1+1/2p)$, then $f(z)$ is p-valent in $|z|\leqq r$.

PROOF. We shall first show that $f'(z)$ does not vanish for $0<|z|<r$. If $f'(z)$ has at least one zero in $0<|z|<r$, then by the argument principle we have

$$\int_0^{2\pi}\Re\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)d\theta\geqq 2\pi(1+p+p\Re k), \qquad z=re^{i\theta},$$

which contradicts (2.4). Hence $f'(z)\neq 0$ for $0<|z|<r$. Now we have

$$\int_0^{2\pi}\Re\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)d\theta=2\pi(p+p\Re k), \qquad z=re^{i\theta}.$$

From this and (2.4) we see that there is no arc C on $|z|=r$ for which

$$\int_C\Re\left(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\right)d\theta\leqq -\pi, \qquad z=re^{i\theta}\in C,$$

holds. Hence by Corollary 1 $f(z)$ is p-valent in $|z|\leqq r$.

3. Sufficient conditions for p-valence related to Corollary 2.

The following theorem is a result given by Ogawa [**1**, 437].

THEOREM B. *Let a function $f(z)$ with the expansion $z^p+\sum_{n=p+1}^{\infty}a_nz^n$, $p>0$, at the origin be meromorphic in $|z|\leqq r$, and let m, k be real constants such that $m>1/2$, $k>-(1+1/2p)$, and furthermore $1/(k+1)$ is not equal to any integer. If $f(z)$ satisfies the condition*

$$(3.1)\qquad -m<1+\Re\frac{zf''(z)}{f'(z)}+k\Re\frac{zf'(z)}{f(z)}<\frac{m(1+2p+2kp)}{2m-1}, \qquad |z|\leqq r,$$

then $f(z)$ is regular and p-valent in $|z|\leqq r$.

In this section it will be shown that this theorem can be extended to the case where k is complex, without the restriction that $1/(k+1)$ is not equal to any integer.

LEMMA 2. *Let $h(\theta)$ be continuous for $0\leqq\theta\leqq 2\pi$, and let $\int_0^{2\pi}h(\theta)d\theta=2\pi A$. Suppose that*

$$-m<h(\theta)<m(A+B)/(2m+A-B), \qquad 0\leqq\theta\leqq 2\pi,$$

where m, B are positive constants such that $(A+B)/(2m+A-B)>0$, then

$$\int_0^{2\pi}|h(\theta)|d\theta<2\pi B.$$

This can be proved in the same way as used by Umezawa [**4**, 196-197] or

Ozaki [**6**, 52-53], and so the proof may be omitted.

LEMMA 3. *Let* $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+nq}z^{p+nq}$ *be regular in* $|z|\leqq r$ *and satisfy the condition*

$$-m<\Re\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{m(q+2p)}{2m-q} \qquad |z|\leqq r, \tag{3.2}$$

for a positive constant m *larger than* $q/2$, *then* $f(z)$ *is* p-*valent and close-to-convex in* $|z|\leqq r$.

PROOF. Since (3.2) holds, $f'(z)$ has no zeros in $0<|z|\leqq r$. Therefore it is sufficient to show that for all arcs C on $|z|=r$ $f(z)$ satisfies the inequality

$$\int_C d\arg df(z)>-\pi\,. \tag{3.3}$$

Now we have

$$\int_0^{2\pi}\Re\left(1+\frac{zf''(z)}{f'(z)}\right)d\theta=2\pi p, \qquad z=re^{i\theta}. \tag{3.4}$$

By virtue of Lemma 2, from this and (3.2) it follows that

$$\int_0^{2\pi}\left|\Re\left(1+\frac{zf''(z)}{f'(z)}\right)\right|d\theta<2\pi(p+q)\,, \qquad z=re^{i\theta}. \tag{3.5}$$

Next, suppose that there exists an arc $C':z=re^{i\theta}$, $\theta_1\leqq\theta\leqq\theta_2$, on $|z|=r$ such that $\int_{C'}d\arg df(z)\leqq-\pi$. For a non-negative integer n and a real number α in $0\leqq\alpha<2\pi/q$ suitably chosen, we have $\theta_2-\theta_1=\alpha+2n\pi/q$. On the other hand, from the expansion of $f(z)$, for an arbitrary real number β we find

$$\int_\beta^{\beta+2\pi/q}d\arg df(re^{i\theta})=\frac{2p}{q}\pi\,.$$

Hence

$$\int_{\theta_1}^{\theta_1+\alpha}d\arg df(re^{i\theta})=\int_{C'}d\arg df(z)-\frac{2np}{q}\pi\leqq-\pi\,.$$

Consequently, if we consider the q arcs $C_j:z=re^{i\theta}$, $\theta_1+2j\pi/q\leqq\theta\leqq\theta_1+\alpha+2j\pi/q$, $j=0,1,\cdots,q-1$, then these arcs do not overlap each other and satisfy

$$\int_{C_j}d\arg df(z)=\int_{\theta_1}^{\theta_1+\alpha}d\arg df(re^{i\theta})\leqq-\pi\,, \qquad j=0,1,\cdots,q-1\,.$$

This fact together with (3.4) yields

$$\int_0^{2\pi}\left|\Re\left(1+\frac{zf''(z)}{f'(z)}\right)\right|d\theta\geqq2\pi(p+q)\,, \qquad z=re^{i\theta},$$

which contradicts (3.5). Consequently for all arcs C on $|z|=r$, (3.3) holds. Thus the proof is completed.

LEMMA 4. *Let* $f(z)=z^p+\sum_{n=p-1}^{-\infty}a_nz^n$ *be regular in* $r\leqq|z|<+\infty$, *and let* p, q *be positive integers such that* $q>2p$. *If* $f(z)$ *satisfies the condition*

$$(3.6) \qquad -m<\Re\Big(1+\frac{zf''(z)}{f'(z)}\Big)<\frac{mq}{2m-q+2p}, \qquad |z|\geqq r,$$

for a real constant m larger than $(q-2p)/2$, *then* $f(z)$ *is at most* $(q-1)$-*valent in* $|z|\geqq r$ *and convex of order at most* $q-p-1$ *in one direction there.*

PROOF. Since (3.6) holds, $f'(z)$ has no zeros in $r\leqq|z|<+\infty$, and so we have

$$\int_0^{2\pi}\Re\Big(1+\frac{zf''(z)}{f'(z)}\Big)d\theta=2\pi p, \qquad z=re^{i\theta}.$$

By virtue of Lemma 2, from this and (3.6) it follows that

$$\int_0^{2\pi}\Big|\Re\Big(1+\frac{zf''(z)}{f'(z)}\Big)\Big|d\theta<2\pi(q-p), \qquad z=re^{i\theta}.$$

Therefore by Umezawa's lemma [**4**, p. 200], there exists such a straight line through the origin as is cut at most $2(q-p-1)$ times by the image curve of $|z|=r$ under $zf'(z)$, so that the image curve of $|z|=r$ under $f(z)$ is convex of order at most $q-p-1$ in one direction. Accordingly $f(z)$ is at most $[n(\infty)+q-p-1]$-valent in $|z|\geqq r$, where $n(\infty)$ denotes the number of poles of $f(z)$ in $|z|\geqq r$ [**4**, p. 199]. On the other hand obviously $n(\infty)=p$. Thus the lemma is proved.

THEOREM 2. *Let a function* $f(z)$ *with the expansion* $z^p+\sum_{n=p+1}^{\infty}a_nz^n$, $p>0$, *at the origin be meromorphic in* $|z|\leqq r$, *and let m, k be a real constant and a complex one respectively such that* $m>1/2$, $\Re k>-(1+1/2p)$. *If* $f(z)$ *satisfies the condition*

$$(3.7) \qquad -m<\Re\Big(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\Big)<\frac{m(1+2p+2p\Re k)}{2m-1}, \qquad |z|\leqq r,$$

then $f(z)$ *is regular and p-valent in* $|z|\leqq r$, *and furthermore* $f'(z)$ *does not vanish for* $0<|z|\leqq r$.

PROOF. As shown by Ogawa [**1**, p. 437], in order that $f(z)$ has a zero of order q or a pole of order $-q$, $q<0$, in $0<|z|\leqq r$ under the condition (3.7), it is necessary that $q-1+kq=0$. Therefore we see that (1) when $1/(k+1)$ is not equal to any integer, $f(z)$ has neither zeros nor poles in $0<|z|\leqq r$, (2) when $1/(k+1)$ is equal to a positive integer, $f(z)$ has no poles in $0<|z|\leqq r$ but may have zeros of order $1/(k+1)$, and (3) when $1/(k+1)$ is equal to a negative integer, $f(z)$ has no zeros in $0<|z|\leqq r$ but may have poles of order $-1/(k+1)$.

In the case of (2), we set

$$g(z)=f(z^q)^{1/q}=z^p+\sum_{n=1}^{\infty}a_{p+nq}z^{p+nq}, \qquad q=\frac{1}{k+1}.$$

Then $g(z)$ is regular in $|z|\leqq r^{1/q}$ even if $f(z)$ has zeros in $0<|z|\leqq r$, and satisfies the equality

$$(3.8)\qquad 1+\frac{zg''(z)}{g'(z)}=q\Big(1+\frac{z^q f''(z^q)}{f'(z^q)}+k\frac{z^q f'(z^q)}{f(z^q)}\Big),\qquad |z|\leqq r^{1/q},$$

from which and (3.7) we have

$$-M<\Re\Big(1+\frac{zg''(z)}{g'(z)}\Big)<\frac{M(q+2p)}{2M-q},\qquad M=mq,\qquad |z|\leqq r^{1/q}.$$

Hence by Lemma 3 $g(z)$ is p-valent in $|z|\leqq r^{1/q}$, and so $g(z)$ can not vanish for $0<|z|\leqq r^{1/q}$. Consequently $f(z)$ has no zeros in $0<|z|\leqq r$.

In the case of (3), the function $g(z)$ defined above is regular in $r^{1/q}\leqq|z|<+\infty$ even if $f(z)$ has poles in $0<|z|\leqq r$, and satisfies (3.8) for $|z|\geqq r^{1/q}$. Since $q<0$, from (3.7) and (3.8) we have

$$-M<\Re\Big(1+\frac{zg''(z)}{g'(z)}\Big)<\frac{M|q|}{2M-|q|+2p},\qquad M=\frac{m(|q|-2p)}{2m-1},\qquad |z|\geqq r^{1/q},$$

where $|q|>2p$ because of $k>-(1+1/2p)$. Therefore by Lemma 4 $g(z)$ is at most $(|q|-1)$-valent in $|z|\geqq r^{1/q}$, and so $g(z)$ can not vanish for $r^{1/q}\leqq|z|<+\infty$ in view of the fact that if $g(a)=0$, $|a|\geqq r^{1/q}$, then for $\varepsilon=e^{2\pi i/|q|}$ we have $g(a\varepsilon^n)=0$, $n=0,1,\cdots,|q|-1$. Consequently $f(z)$ has no poles in $0<|z|\leqq r$.

The above consideration concludes that for every k such that $\Re k>-(1+1/2p)$, $f(z)$ is regular in $|z|\leqq r$ and does not vanish for $0<|z|\leqq r$. From this fact and (3.7) we see further that $f'(z)\neq 0$ for $0<|z|\leqq r$.

Now we have

$$\int_0^{2\pi}\Re\Big(1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)}\Big)d\theta=2\pi(p+p\Re k),\qquad z=re^{i\theta},$$

which together with (3.7) yields (2.4) in virtue of Lemma 2. Hence by Corollary 2 $f(z)$ is p-valent in $|z|\leqq r$. Thus the proof is completed.

Considering the four special cases where $m\to 1/2$, $m\to+\infty$, $m=1+p+p\Re k$, and $m=1$, we have the following

COROLLARY 3. *In Theorem* 2, *let $f(z)$ satisfy for $|z|\leqq r$ one of the following conditions instead of* (3.7), *then we have the same conclusion.*

$$(3.9)\qquad \Re F(z)>-\frac{1}{2},\qquad F(z)=1+\frac{zf''(z)}{f'(z)}+k\frac{zf'(z)}{f(z)},$$

$$(3.10)\qquad \Re F(z)<\frac{1}{2}+p+p\Re k,$$

$$(3.11)\qquad |\Re F(z)|<1+p+p\Re k,$$

$$(3.12)\qquad |\Re(F(z)-p-pk)|<1+p+p\Re k.$$

4. Another sufficient condition for p-valence related to Corollaries 1, 3.

Another result due to Ogawa [**1**, p. 435] may be stated without loss of equivalency as follows.

THEOREM C. *Let $f(z)=z^p+\cdots$ be regular in $|z|\leqq r$, and let $f(z)f'(z)\neq 0$ for $0<|z|\leqq r$. On the other hand, let $\varphi(z)$ be regular and uni- or multivalently convex in $|z|\leqq r$. If there exists the relation*

$$\Re\left[e^{i\alpha}\frac{f'(z)f(z)^k}{\varphi'(z)}\right]>0, \qquad |z|\leqq r, \tag{4.1}$$

for real constants α and k, then $f(z)$ is p-valent in $|z|\leqq r$.

Since $f'(z)f(z)^k$ has the expansion $pz^{p-1+kp}+\cdots$ at the origin, in order that (4.1) holds, $p-1+kp$ must be equal to the number of multiplicity of the zero at the origin of $\varphi'(z)$. Therefore Theorem C is effective only for special numbers k such that $k=-1+q/p$, where q denotes positive integers. In this section we shall extend this theorem to a wider one which is effective for any complex number k such that $\Re k\geqq -1$. In passing, we here remark that his fourth theorem [**1**, p. 435] is included in his third one, namely Theorem C.

We first consider a function $\varphi(z)=z^p+\cdots$ which is regular in $|z|\leqq r$ and satisfies the condition

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}+k\frac{z\varphi'(z)}{\varphi(z)}\right)\geqq 0, \qquad |z|\leqq r, \tag{4.2}$$

for a complex constant k such that $\Re k\geqq -1$. Then by Corollary 3 $\varphi(z)$ is p-valent in $|z|\leqq r$ and $\varphi'(z)$ does not vanish for $0<|z|\leqq r$. We denote by $S(k)$ the set of such functions $\varphi(z)$.

In particular, for real k's we notice that (1) when $k=-1$, $S(k)$ consists of only one function z^p, (2) when $-1<k\leqq 0$, $S(k)$ consists of p-valently convex functions, (3) when $k>0$, $S(k)$ consists of p-valently starlike functions, and (4) when $k_1<k_2$, $S(k_1)\subset S(k_2)$. This will be shown below.

Set $h(z)=1+z\varphi''(z)/\varphi'(z)+kz\varphi'(z)/\varphi(z)$. By the minimum principle for harmonic functions, we see from (4.2) that $\Re h(z)$ vanishes at a point in $|z|<r$ only when $\Re h(z)\equiv 0$, which occurs for and only for $k=-1$ because of $\Re h(0)=p+kp$. And then a brief calculation shows that $\varphi(z)=z^p$.

Next we consider the case $\Re h(z)\not\equiv 0$. Suppose that the image curve C of $|z|=\rho(<r)$ under $w=z\varphi'(z)/\varphi(z)$ cuts or touches the imaginary axis. Then there exists a point of either intersection or contact w_0 such that

$$[d\arg w(z)]_{w=w_0}\leqq 0, \quad |z|=\rho, \quad w_0=z_0\varphi'(z_0)/\varphi(z_0), \tag{4.3}$$

because C does not surround the origin from the fact that $z\varphi'(z)/\varphi(z)$ has no zeros in $|z|\leqq\rho$. Rewriting (4.3), we have

$$1+\Re\frac{z_0\varphi''(z_0)}{\varphi'(z_0)}-\Re\frac{z_0\varphi'(z_0)}{\varphi(z_0)}\leqq 0.$$

Since $\Re[z_0\varphi'(z_0)/\varphi(z_0)]=0$, this together with (4.2) yields $\Re h(z_0)=0$, which however can not occur in the present case. Consequently $\Re[z\varphi'(z)/\varphi(z)]\neq 0$ for $|z|<r$, so that

$$\Re\frac{z\varphi'(z)}{\varphi(z)}>0\,, \qquad |z|<r\,. \tag{4.4}$$

Hence (3) is true. Furthermore from (4.2) and (4.4) we see that (2) and (4) are also true.

REMARK 2. (4. 2) means that

$$d\arg d\varphi(z)+d\arg\varphi(z)^k\geqq 0\,, \qquad |z|=\rho\leqq r\,.$$

We now have the following

THEOREM 3. *Let $f(z)=z^p+\cdots$ be regular in $|z|\leqq r$, and let $\varphi(z)$ be a function of $S(k)$ for a complex constant k such that $\Re k\geqq -1$. If there exists the relation*

$$\Re\Big[e^{i\alpha}\frac{f'(z)}{\varphi'(z)}\Big(\frac{f(z)}{\varphi(z)}\Big)^k\Big]>0\,, \qquad |z|\leqq r\,, \tag{4.5}$$

where α is a real constant and $(f(z)/\varphi(z))^k$ denotes all values at z obtained by the analytic continuation from its one functional element assigned at the origin, then $f(z)$ is p-valent in $|z|\leqq r$.

PROOF. We shall first show that $f(z)f'(z)$ does not vanish for $0<|z|\leqq r$. In order that $f(z)$ has a zero of order q in $0<|z|\leqq r$ under the condition (4.5), it is necessary that $q-1+kq=0$. Therefore $f(z)$ may have zeros in $0<|z|\leqq r$ only when $1/(k+1)$ is equal to a positive integer. And then, as in the proof of Theorem 2, the function

$$g(z)=f(z^q)^{1/q}=z^p+\sum_{n=1}^{\infty}a_{p+nq}z^{p+nq}\,, \qquad q=\frac{1}{k+1}\,,$$

is regular in $|z|\leqq r^{1/q}$ whether $f(z)$ has zeros in $0<|z|\leqq r$ or not, and it follows from (4.5) that

$$\Re[e^{i\alpha}g'(z)/z^{q-1}\varphi'(z^q)\varphi(z^q)^{-1+1/q}]>0\,, \qquad |z|\leqq r^{1/q}\,.$$

Hence $g'(z)\neq 0$ for $0<|z|\leqq r^{1/q}$. Furthermore for all arcs C' on $|z|=r^{1/q}$ we have

$$\int_{C'} d\arg dg(z)>-\pi\,,$$

since $d\arg d\varphi(z^q)+d\arg\varphi(z^q)^k\geqq 0$, $|z|=r^{1/q}$, $k=-1+1/q$. Consequently $g(z)$ is p-valently close-to-convex in $|z|\leqq r^{1/q}$, and so $g(z)$ can not vanish for $0<|z|\leqq r^{1/q}$. This deduces that $f(z)$ has no zeros in $0<|z|\leqq r$ even if $1/(k+1)$ is equal to a positive integer. Therefore from (4.5) we see further that $f'(z)$ also does not vanish for $0<|z|\leqq r$.

Next, for all arcs C on $|z|=r$ (4.5) gives

$$\int_C [d\arg df(z)+d\arg f(z)^k]>-\pi\,.$$

Hence by Corollary 1 $f(z)$ is p-valent in $|z|\leqq r$.

REMARK 3. In Theorem C, let $\varphi(z)=a_0+a_qz^q+a_{q+1}z^{q+1}+\cdots$, then as we stated before, it is necessary that $p+kp=q$. And then the function $\varphi_1(z)=\left(\frac{1}{a_q}\int_0^z\varphi'(z)dz\right)^{1/(k+1)}=z^p+\cdots$ is regular in $|z|\leqq r$ and satisfies the condition (4.2). Furthermore from (4.1) there exists the relation

$$\Re\left[e^{i\alpha}\frac{f'(z)}{\varphi_1'(z)}\left(\frac{f(z)}{\varphi_1(z)}\right)^k\right]>0, \qquad |z|\leqq r,$$

for a suitable branch of $(f(z)/\varphi_1(z))^k$. Consequently Theorem C is included in our Theorem 3.

COROLLARY 4. *Let D be a domain which contains the origin and is bounded by a smooth Jordan curve C such that*

$$d\arg dz+d\arg z^k\geqq 0, \qquad z\in C, \tag{4.6}$$

where k is a complex constant whose real part is not smaller than -1. If $f(z)=z+\cdots$ is regular in $\bar{D}$ and satisfies the condition

$$\Re\left[e^{i\alpha}f'(z)\left(\frac{f(z)}{z}\right)^k\right]>0, \qquad z\in\bar{D},$$

where α is a real constant and $(f(z)/z)^k$ denotes all values at z obtained by the analytic continuation from its one functional element assigned at the origin, then $f(z)$ is univalent in $\bar{D}$.

This is an immediate consequence of Theorem 3, and is an extension of the well known sufficient condition for univalence due to Noshiro and Wolff. We here remark that (1) when $k=-1$, D is a disk with centre $z=0$, (2) when $-1<k\leqq 0$, D is a convex domain, (3) when $k>0$, D is a starlike domain with respect to $z=0$, and (4) when k increases, the set of domains D satisfying (4.6) expands monotonously, provided k is real.

Finally we shall give a coefficient theorem for the function of Theorem 3.

THEOREM 4. *In Theorem 3, suppose that k is real and $r=1$, then*

$$f(z)^{k+1}\ll[z^p/(1-z)^{2p}]^{k+1}. \tag{4.7}$$

Accordingly if $1/(k+1)$ is a positive integer, then we have

$$f(z)\ll z^p/(1-z)^{2p}.$$

PROOF. From (4.2) we have

$$1+\frac{z\varphi''(z)}{\varphi'(z)}+k\frac{z\varphi'(z)}{\varphi(z)}\ll p(k+1)\frac{1+z}{1-z},$$

which yields

$$\log\frac{\varphi'(z)}{pz^{p-1}}\left(\frac{\varphi(z)}{z^p}\right)^k\ll\log\left(\frac{1}{1-z}\right)^{2p(k+1)},$$

so that

$$\varphi'(z)\varphi(z)^k\ll pz^{p-1+kp}/(1-z)^{2p(k+1)}.$$

On the other hand, from (4.5) we have

$$\frac{f'(z)}{\varphi'(z)}\left(\frac{f(z)}{\varphi(z)}\right)^k \ll \frac{1+z}{1-z}.$$

Hence

$$f'(z)f(z)^k \ll pz^{p-1+kp}(1+z)/(1-z)^{2p(k+1)+1},$$

from which (4.7) follows. The latter half is evident.

Theorems A and B can be extended also to the case that $f(z)$ has an expansion of the form $z^q+\cdots, q<p$, this however will be discussed on another occasion.

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