Operations Research
Vol. 54, No. 3, May–June 2006, pp. 602–604
ISSN 0030-364X | EISSN 1526-5463 | 06 | 5403 | 0602
DOI 10.1287/opre.1060.0282

TECHNICAL NOTE

A Note on Probability Distributions with Increasing Generalized Failure Rates

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Distributions with an increasing generalized failure rate (IGFR) have useful applications in pricing and supply chain contracting problems. We provide alternative characterizations of the IGFR property that lead to simplify verifying whether the IGFR condition holds. We also relate the limit of the generalized failure rate and the moments of a distribution.

Subject classifications: probability distributions.
Area of review: Manufacturing, Service, and Supply Chain Operations.
History: Received December 2003; revision received April 2005; accepted April 2005.

1. Introduction

Let X be a nonnegative random variable with distribution Φ, and let $\bar{\Phi}(\xi) = 1 - \Phi(\xi)$. We assume that Φ has density ϕ. Let (α, β) for $0 \leqslant \alpha < \beta \leqslant \infty$ be the support of X. $h(\xi) = \phi(\xi)/\bar{\Phi}(\xi)$ is the failure rate of X. X has an increasing failure rate (IFR) or, equivalently, Φ is an IFR distribution if $h(\xi)$ is weakly increasing for all ξ such that $\Phi(\xi) < 1$. Lariviere and Porteus (2001) define the *generalized failure rate* of X as

$$g(\xi) = \xi h(\xi).$$

X has an increasing generalized failure rate (IGFR) and Φ is an IGFR distribution if $g(\xi)$ is weakly increasing for all ξ such that $\Phi(\xi) < 1$. Decreasing failure rate (DFR) or decreasing generalized failure rate (DGFR) distributions can be defined analogously. Clearly, if X is IFR it is also IGFR, but the reverse need not hold; many DFR distributions are IGFR.

Pricing a service illustrates the use of IGFR distributions. One customer arrives per period, and service takes one period. The cost of service is zero. Customers privately observe their valuations, which are independent and identically distributed according to $\Phi(\xi)$. A firm posting price p then faces demand $D(p) = \bar{\Phi}(p)$ and sets p to maximize revenue, $\pi(p) = pD(p)$. An optimal price p^* must solve

$$\pi'(p^*) = D(p^*) + p^* D'(p^*) = \bar{\Phi}(p^*)(1 - g(p^*)) = 0.$$

The uniqueness of p^* depends on the generalized failure rate $g(\xi)$. In particular, if $g(\xi)$ is increasing, it can equal one at only a single point, and the unique p^* must solve $g(p^*) = 1$.

This analysis assumes $\lim_{\xi \downarrow \alpha} g(\xi) \leqslant 1$ and $\lim_{\xi \uparrow \beta} g(\xi) > 1$. Fortunately, $g(\xi)$ has an economic interpretation that offers guidance when these assumptions fail: $g(p) = -pD'(p)/D(p)$, so the generalized failure rate is also the elasticity of demand. If $\lim_{\xi \downarrow \alpha} g(\alpha) > 1$, demand is always elastic, and one is best off charging $p^* = \alpha$ and serving all customers. If $\lim_{\xi \uparrow \beta} g(\xi) < 1$, demand is always inelastic, and a price increase always increases profits. Hence, $p^* = \beta$. Realistic problems must have $g(\xi) > 1$ for all $\xi > y$ for some finite y. Section 3 gives a mild condition guaranteeing the existence of such a y.

The above is a simplification of Ziya et al. (2004a) and has a direct analogue in the supply chain contracting literature (Lariviere and Porteus 2001). Ziya et al. (2004b) compares the IGFR assumption with other commonly imposed conditions for unimodality.

The IGFR property is useful, but often difficult to verify. Here, we examine alternative characterizations of IGFR distributions that simplify checking that X is IGFR. We then relate the limit of the generalized failure rate to the moments of a distribution.

2. Alternative Characterizations of IGFR Distributions

Let $\Phi_L(\xi)$, $\phi_L(\xi)$, and $h_L(\xi)$ denote the distribution, density, and failure rate of $X_L = \log(X)$. $\bar{\Phi}_L(\xi) = 1 - \Phi_L(\xi)$. Nonnegative random variables X_1 and X_2 with failure rates $h_1(\xi)$ and $h_2(\xi)$ are ordered in the *hazard rate order* with X_1 being smaller ($X_1 \preceq_{hr} X_2$) if $h_1(\xi) \geqslant h_2(\xi)$ for all $\xi \geqslant 0$ (Ross 1983). A function $f(\xi)$ is *log-concave* if $\log(f(\xi))$ is concave. A function $f(x, y)$ is *totally positive of order* 2 (TP_2) if for $x_1 < x_2$ and $y_1 < y_2$,

$$f(x_1, y_1) f(x_2, y_2) - f(x_1, y_2) f(x_2, y_1) \geqslant 0.$$

THEOREM 1. *The following statements are equivalent*:
1. *X is IGFR.*
2. *X_L is IFR.*
3. *$X \preceq_{hr} \lambda X$ for $\lambda \geqslant 1$.*
4. *$f(\xi, \theta) = \bar{\Phi}(\xi/\theta)$ is TP_2.*

PROOF. Because $\bar{\Phi}_L(\xi) = \bar{\Phi}(e^{\xi})$ and $\phi_L(\xi) = e^{\xi}\phi(e^{\xi})$, $h_L(\xi) = g(e^{\xi})$ and $h'_L(\xi) = e^{\xi}g'(e^{\xi})$, which establishes the equivalence of parts 1 and 2. To link parts 1 and 3, note that the generalized failure rate of λX is $g_\lambda(\xi) = g(\xi/\lambda)$. Finally, X_L is IFR if and only if $f_L(\xi, \theta) = \bar{\Phi}_L(\xi - \theta)$ is TP_2 (Barlow and Proschan 1965), which is equivalent to part 4. □

REMARK 1. Assuming that X_L is IFR is thus sufficient for revenue $\pi(p)$ to be unimodal. To reach this conclusion directly, let $p_L = \log(p)$ so the firm's objective becomes

$$\pi_L(p_L) = \pi(e^{p_L}) = e^{p_L}D(e^{p_L}) = e^{p_L}\bar{\Phi}(e^{p_L}).$$

Log-concavity of $\pi_L(p_L)$ is sufficient for unimodality, but $\pi_L(p_L)$ is log-concave if and only if $\bar{\Phi}_L(\xi)$ is log-concave, which is equivalent to X_L being IFR (Barlow and Proschan 1965).

REMARK 2. Keilson and Sumita (1982) show the equivalence of 2 and 3. Ma (1999) links 3 and 4 and applies the characterization given by 3 in a reliability setting. Part 3 implies that as the market scales up in pricing problems, a supplier selling to a newsvendor sells more (Lariviere and Porteus 2001), while a service provider charges more (Ziya et al. 2004a).

Most importantly, Theorem 1 allows us to exploit properties of IFR distributions. For example, if X_L is IFR, so is $\delta + \nu X_L$ for constants $\delta, \nu > 0$. Thus, if X is IGFR, so is δX^{ν} (Paul 2005). (The IGFR property is not generally preserved by shifts. One can show that if X is IGFR but DFR, there exists a δ such that $X + \delta$ is not IGFR.)

The theorem also simplifies verifying the IGFR property. Suppose that X has a lognormal distribution. It then has a nonmonotone failure rate and is not obviously IGFR. However, X_L is normally distributed and hence IFR; X must be IGFR. The following corollary presents two means of determining whether X is IGFR.

COROLLARY 1. *If either of the following conditions hold, then X is IGFR.*
1. *$\phi(e^{\xi})$ is log-concave.*
2. *$\xi\phi(\xi)$ is increasing.*

PROOF. Log-concavity of $\phi_L(\xi)$ or concavity of $\bar{\Phi}_L(\xi)$ implies that X_L is IFR (Barlow and Proschan 1965). The former reduces to $\phi(e^{\xi})$ being log-concave; the latter to $d\xi\phi(\xi)/d\xi > 0$. □

Note that neither condition requires a closed-form expression for the cumulative distribution. We apply the corollary to the beta distribution,

$$\phi(\xi) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\xi^{a-1}(1-\xi)^{b-1}$$

for $a, b > 0$ and $0 < \xi < 1$. The first part gives that a beta distribution is IGFR if $b \geqslant 1$ because

$$\frac{d^2\log(\phi(e^{\xi}))}{d\xi^2} = -(b-1)(e^{\xi/2} - e^{-\xi/2})^{-2}.$$

Clearly, $d\xi\phi(\xi)/d\xi > 0$ for $b < 1$, and hence all beta distributions are IGFR.

Theorem 1 also leads to a preservation property.

COROLLARY 2. *Suppose that both X_1 and X_2 are IGFR. Then, X_1X_2 is IGFR.*

3. The Moments of IGFR Distributions

IFR distributions have finite moments of all orders (Barlow and Proschan 1965). This result is derived through a comparison to the exponential distribution, which has a constant failure rate. Analogously, IGFR distributions must be compared to the Pareto ($\phi(\xi) = kS^k\xi^{-k-1}$ for $k > 0$ and $\xi \geqslant S > 0$), for which $g(\xi) = k$ for $\xi \geqslant S$. If X_k is a Pareto random variable with parameter k, the nth moment of X_k is defined only if $k > n$.

THEOREM 2. *Suppose that X is IGFR with support (α, ∞) and that $\lim_{\xi\to\infty} g(\xi) = \kappa$, where κ is possibly infinite. $\mathbb{E}[X^n]$ for $n > 0$ is then finite if and only if $\kappa > n$.*

PROOF. If $\kappa > n$, there exists a $y < \infty$ such that $g(\xi) > n + \varepsilon$ for all $\xi \geqslant y$, where $0 < \varepsilon < \kappa - n$. Note that $\mathbb{E}[X^n] = \mathbb{E}[X^n \mid X \leqslant y]\Phi(y) + \mathbb{E}[X^n \mid X > y]\bar{\Phi}(y)$. The first term is finite because $\mathbb{E}[X^n \mid X \leqslant y] < y^{n+1}$. Define X_y as X conditional on X being greater than y. Letting $h_y(\xi)$ be the failure rate of X_y, we have $h_y(\xi) = h(\xi)$ for $\xi > y$. Further, $h(\xi) > (n+\varepsilon)/\xi$ for $\xi > y$ from the definition of y. Because $\bar{\Phi}(\xi) = \exp[-\int_0^{\xi} h(s)\,ds]$ (Ross 1983), X_y is stochastically smaller than a random variable whose failure rate is $(n+\varepsilon)/\xi$ for $\xi > y$. That is the failure rate for a Pareto random variable with parameters $(y, n+\varepsilon)$. Because the nth moment of the Pareto is finite, the nth moment of X_y is also finite.

Now suppose that X is IGFR and that $\mathbb{E}[X^n] < \infty$ for $n > \kappa$. Then, there exists a Pareto random variable with parameters $(z, \kappa + \varepsilon)$ for $0 < \varepsilon < n - \kappa$ and $z > 0$ such that this Pareto random variable is stochastically smaller than X conditional on X being greater than z. However, the nth moment of a Pareto random variable with parameters $(z, \kappa+\varepsilon)$ is undefined, and hence the nth moment of X is undefined, which yields the desired contradiction. □

The theorem generalizes Lemma 2 of Lariviere and Porteus (2001). Recall that our motivating pricing problem required the generalized failure rate to exceed one at a finite point for there to be a finite solution to the firm's pricing problem. We now see that it is sufficient to assume that the distribution of reservation values has a finite mean. Further, Theorem 2 allows us to relate IGFR distributions to a condition used by Van Mieghem and Dada (1999). For a pricing problem, they require that (a) $h(\xi) - (n+1)/\xi$ have at most one zero, and (b) $\lim_{\xi\downarrow 0} \xi h(\xi) < n + 1$, where n is a positive integer. This is satisfied by any strictly IGFR distribution with a finite $(n+1)$st moment.

Acknowledgments

The author is grateful for the comments of Serguei Netessine, Jan Van Mieghem, and two referees.

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