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A Note on Proper Affine Vector Fields in Bianchi Type IV Space-Times

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Abstract

Affine vector fields of Bianchi type IV space-times are investigated using holonomy and decomposability, the rank of the 6×6 Riemann matrix and direct integration technique. From the above study it follows that the Bianchi type IV space-times possesses only one case when it admits proper affine vector fields.

Keywords: Affine vector fields, holonomy and decomposability, direct integration technique.

1 INTRODUCTION

In this paper we investigate the existance of proper affine vector fields in Bianchi type IV space-times using holonomy and decomposability, the rank of the 6×6 Rieman matrix and direct integration techinques. Affine vector fields which preserve the geodesic structure and affine parameter of a space-time carries significant information and interest in the Einstein's theory of general relativity. It is therefor important to study this symmetry. Throughout M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (-, +, +, +). The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. Here, M is assumed non-flat in the sense that the curvature tensor does not vanish over any non-empty open subset of M.

A vector field X on M is called an affine vector field if it satisfies

$$X_{a;bc} = R_{abcd}\, X^{d}, \tag{1}$$

where $R_{abcd} = g_{af} R^{f}{}_{bcd} = g_{af}(\Gamma^{f}_{bd,c} - \Gamma^{f}_{bc,d} + \Gamma^{f}_{ce}\Gamma^{e}_{bd} - \Gamma^{f}_{ed}\Gamma^{e}_{bc})$. If one decomposes $X_{a;b}$ on M into its symmetric and skew-symmetric parts

$$X_{a;b} = \frac{1}{2}H_{ab} + G_{ab},\ (H_{ab}(\equiv X_{a;b} + X_{b;a}) = H_{ba},\ \ G_{ab} = -G_{ba}) \tag{2}$$

then equation (1) is equivalent to

$$(i)\ H_{ab;c} = 0 \quad (ii)\ G_{ab;c} = R_{abcd}\, X^{d} \quad (iii)\ G_{ab;c}X^{c} = 0. \tag{3}$$

The proof of the above equation (1) implies (3) or equations (3) implies (1) can be found in [2,3]. If $H_{ab} = 2cg_{ab}, c \in R,$ then the vector field X is called homothetic (and *Killing* if $c = 0$). The vector field X is said to be proper affine if it is not homothetic vector field and also X is said to be proper homothetic vector field if it is not Killing vector field on M [4].

2 Affine Vector Fields

Suppose that M is a simple connected space-time. Then the holonomy group of M is a connected Lie subgroup of the idenity component of the Lorentz group and is thus characterized by its subalgebra in the Lorentz algebra. These have been labeled into fifteen types $R_1 - R_{15}$ [1]. It follows from [4] that the only such space-times which could admit proper affine vector fields are those which admit nowhere zero covariantly constant second order symmetric tensor field H_{ab}. This forces the holonomy type to be either R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13} [4]. A study of the affine vector fields for the above holonomy type can be found in [4]. It follows from [5] that the rank of the 6×6 Riemann matrix of the above space-times which have holonomy type R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13} is at most three. Hence for studying affine vector fields we are interested in those cases when the rank of the 6×6 Riemann matrix is less than or equal to three.

3 Main Results

Consider the Bianchi type IV space-times in the usual coordinate system (t, x, y, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with line element [6]

$$ds^2 = -dt^2 + e^{-2z}\left[A(t)dx^2 + \left(z^2A(t) + B(t)\right)dy^2 + 2zA(t)dxdy\right] + C(t)dz^2 \tag{4}$$

where $A(t), B(t)$ and $C(t)$ are no where zero functions of t. The above space-time admits three linearly independent Killing vector fields which are

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad (x-y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \tag{5}$$

The non-zero independent components of the Riemann tensor are

$R_{0101} = -\frac{1}{4A}\left[2A\ddot{A} - \dot{A}^2\right] e^{-2z} \equiv \alpha_1$, $R_{0102} = -\frac{1}{4A}\left[2A\ddot{A} - \dot{A}^2\right] z\, e^{-2z} \equiv \alpha_7$,

$R_{0113} = \frac{1}{2C}\left[A\dot{C} - \dot{A}C\right]e^{-2z} \equiv \alpha_8$,

$R_{0123} = \frac{1}{4BC}\left[2\dot{A}BC - AC\dot{B} - AB\dot{C} + 2B\left(A\dot{C} - \dot{A}C\right)z\right]e^{-2z} \equiv \alpha_9$,

$R_{0202} = -\frac{1}{4AB}\left[2AB\ddot{B} - A\dot{B}^2 + \left(2A\ddot{A} - \dot{A}^2\right)Bz^2\right]e^{-2z} \equiv \alpha_2$,

$R_{0213} = \frac{1}{4C}\left[\dot{A}C - A\dot{C} + 2\left(A\dot{C} - \dot{A}C\right)z\right]e^{-2z} \equiv \alpha_{10}$,

$R_{0223} = \frac{1}{4BC}\left[2B^2\dot{C} - 2B\dot{B}C + \left(3\dot{A}BC - A\dot{B}C - 2AB\dot{C}\right)z + 2\left(AB\dot{C} - \dot{A}BC\right)z^2\right]e^{-2z} \equiv \alpha_{11}$,

$R_{0303} = \frac{1}{4C}\left[\dot{C}^2 - 2C\ddot{C}\right] \equiv \alpha_3$, $R_{1212} = \frac{1}{4C}\left[\dot{A}\dot{B}C + A^2 - 4AB\right]e^{-4z} \equiv \alpha_4$,

$R_{1313} = \frac{1}{4B}\left[A^2 - 4AB + \dot{A}B\dot{C}\right]e^{-2z} \equiv \alpha_5$, $R_{0312} = \frac{1}{4B}\left[A\dot{B} - \dot{A}B\right]e^{-2z} \equiv \alpha_{12}$,

$R_{1323} = \frac{1}{4B}\left[4AB - 4zAB + z\dot{A}B\dot{C} + zA^2\right]e^{-2z} \equiv \alpha_{13}$,

$R_{2323} = \frac{1}{4B}\left[-3AB + 8zAB - 4z^2AB - 4B^2 + z^2\dot{A}B\dot{C} + B\dot{B}\dot{C} + A^2z^2\right]e^{-2z} \equiv \alpha_6$.

Writing the curvature tensor with components R_{abcd} at p as a 6×6 symmetric matrix [7]

$$R_{abcd} = \begin{pmatrix} \alpha_1 & \alpha_7 & 0 & 0 & \alpha_8 & \alpha_9 \\ \alpha_7 & \alpha_2 & 0 & 0 & \alpha_{10} & \alpha_{11} \\ 0 & 0 & \alpha_3 & \alpha_{12} & 0 & 0 \\ 0 & 0 & \alpha_{12} & \alpha_4 & 0 & 0 \\ \alpha_8 & \alpha_{10} & 0 & 0 & \alpha_5 & \alpha_{13} \\ \alpha_9 & \alpha_{11} & 0 & 0 & \alpha_{13} & \alpha_6 \end{pmatrix} \tag{6}$$

As mentioned in section 2, the space-times which can admit proper affine vector fields have holonomy type R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13} and the rank of the 6×6 Riemann matrix is at most three. Therefore we are only interested in those cases when the rank of the 6×6 Riemann matrix is less than or equal to three. In general for any 6×6 symmetric matrix there exist total fourty one possibilities when the rank of the 6×6 symmetric matrix is less or equal to three, that is, twenty possibilities for rank three, fifteen possibilities for rank two and six possibilities for rank one. Suppose the rank of the 6×6 Riemann matrix is one. Then there is only one non-zero row or column in (6). If we set five rows or columns identically zero in (6) then there exist six possibilities when the rank of the 6×6 Riemann matrix is one. All these six possibilities give us contradiction. For example consider the case when the rank of the 6×6 Riemann matrix is one i.e. $\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=\alpha_7=\alpha_8=\alpha_9=\alpha_{10}=\alpha_{11}=\alpha_{12}=\alpha_{13}=0$ and $\alpha_1\neq 0.$ Subsituting the above information back in equation (6) one has $\alpha_1=0$ which gives contradiction (here we assume that $\alpha_1\neq 0$). So this case is not possible. Similarly if one proceeds further one finds that there exists only one possibility when the rank of the 6×6 Riemann matrix is three or less which is: $2A\ddot{A}-\dot{A}^2=0,$ $A\dot{C}-\dot{A}C=0,$ $\dot{A}B-A\dot{B}=0,$ $2B\ddot{B}-\dot{B}^2=0,$ $B\dot{C}-\dot{B}C=0,$ $2C\ddot{C}-\dot{C}^2=0$ and the rank of the 6×6 Riemann matrix is three.

In this case we have $B(t)=a\,A(t),$ $C(t)=b\,A(t)$ and $B(t)=(a/b)C(t),$ where $a,b\in R\setminus\{0\}.$ Equations $2A\ddot{A}-\dot{A}^2=0,$ $2B\ddot{B}-\dot{B}^2=0$ and $2C\ddot{C}-\dot{C}^2=0\Rightarrow$ $A=(e_1\,t+e_2)^2,$ $B=(b_1\,t+b_2)^2$ and $C=(f_1\,t+f_2)^2,$ where $e_1,e_2,b_1,b_2,f_1,f_2\in R(e_1\neq 0,b_1\neq 0,f_1\neq 0).$ It follows from the above calculation that $a\,e_1=b_1,$ $b\,e_1=f_1,$ $a\,e_2=b_2$ and $b\,e_2=f_2.$ The sub case when $e_1=0$ (which implies $b_1=0$ and $f_1=0$) will be considered latter. Here, the rank of the 6×6 Riemann matrix is three and there exists a unique (up to a multiple) no where zero timelike vector field $t_a=t_{,a}$ solution of equation (3) and $t_{a;b}\neq 0.$ The line element in this case takes the form

$$ds^2=-dt^2+(e_1\,t+e_2)^2\{e^{-2z}\left[dx^2+\left(z^2+a\right)dy^2+2zdxdy\right]+b\,dz^2\}. \tag{7}$$

Substituting the above information into affine equations (1) and after tedious and lengthy calculation one finds that affine vector fields in this case are

$$X^0=c_4t,\ X^1=\left(x-y\right)c_1+c_2,\ X^2=c_1y+c_3,\ X^3=c_1, \tag{8}$$

$c_1,c_2,c_3,c_4\in R.$ One can write the above equation (8) after subtracting the killing vector fields as

$$X=\left(t,0,0,0\right). \tag{9}$$

Clearly, in this case the above space-times (7) admit proper affine vector field.

Now consider the sub case when $e_1=0.$ The above space-time (10) becomes

$$ds^2=-dt^2+e_2^2\{e^{-2z}\left[dx^2+\left(z^2+a\right)dy^2+2zdxdy\right]+b\,dz^2\}. \tag{10}$$

The above space-time is 1+3 decomposable and belongs to curavture class C. In this sub case there exists a nowhere zero timelike vector field $t_a=t_{,a}$ such that $t_{a;b}=0.$ From the Ricci identity $R^a{}_{bcd}\,t_a=0.$ Affine vector fields in this case [4] are

$$X=(tc_4+c_5)\frac{\partial}{\partial t}+X', \tag{11}$$

where $c_4,c_5\in R$ and X' is a homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant $t.$ The completion of this sub case needs to find a homothetic vector fields in the induced geometry of the submanifolds of constant $t.$ The induced metric $g_{\alpha\beta}$ (where $\alpha,\beta=1,2,3$) with non zero components is given by

$$g_{11}=e_2^2\,e^{-2z},\ g_{12}=e_2^2\,z\,e^{-2z},g_{22}=e_2^2\left(z^2+a\right)e^{-2z},g_{33}=e_2^2\,b. \tag{12}$$

A vector field X' is a homothetic vector field if it satisfies $L_{X'} g_{\alpha\beta} = 2\phi g_{\alpha\beta}$, for all $\alpha, \beta = 1,2,3,$ where $\phi \in R.$ One can expand the homothetic equation and using (12) to get

$$X^3 - X^1_{,1} - z X^2_{,1} = -\phi, \tag{13}$$

$$(1-2z)X^3 + zX^1_{,1} + (z^2 + a)X^2_{,1} + X^1_{,2} + zX^2_{,2} = 2\phi z, \tag{14}$$

$$bX^3_{,1} + e^{-2z} X^1_{,3} + z e^{-2z} X^2_{,3} = 0, \tag{15}$$

$$(z - z^2 - a)X^3 + z X^1_{,2} + (z^2 + a)X^2_{,2} = \phi(z^2 + a), \tag{16}$$

$$b X^3_{,2} + z e^{-2z} X^1_{,3} + (z^2 + a)e^{-2z} X^2_{,3} = 0, \tag{17}$$

$$X^3_{,3} = \phi. \tag{18}$$

Equation (18) gives $X^3 = \phi z + E^1(x,y),$ where $E^1(x,y)$ is a function of integration. Multiply equation (15) with z and subtracting from equation (17) and using the above value of X^3 and upon integration we get

$$X^2 = \frac{b}{a}\int \left(zE^1_x(x,y) - E^1_y(x,y)\right)e^{2z} dz + E^2(x,y),$$

where $E^2(x,y)$ is a function of integration. Substituting the above information in equation (15) and upon integration one has

$$X^1 = -\frac{b}{6a}\left[3a + 3z^2 + 2z^3\right]E^1_x(x,y)\, e^{2z} - \left[3z + 3z^2\right]E^1_y(x,y) + E^3(x,y),$$

where $E^3(x,y)$ is a function of integration. In order to determine homothetic vector field we need to calculate $E^1(x,y),$ $E^2(x,y)$ and $E^3(x,y).$ If one proceeds further one finds that $\phi = 0$ which implies that no proper homothetic vector field exists in the induced geometry of the submanifolds of constant $t.$ Hence homothetic vector fields are Killing vector fields which are

$$X^1 = (x - y)c_1 + c_2,\ X^2 = c_1 y + c_3,\ X^3 = c_1, \tag{19}$$

where $c_1, c_2, c_3 \in R.$ Affine vector fields in this case are (using equation (19) in (11)) given in equation (8). Clearly, in this case the above space-times (10) admit proper affine vector field.

SUMMARY

In this paper an attempt is made to explore all the possibilities when the Bianchi type IV space-times admit proper affine vector fields. An approach is adopted to study proper affine vector fields in the above space-times using holonomy and decomposability, the rank of the 6×6 Riemann matrix and direct integration technique. From the above study there exists only one case when the above space-times (4) admit proper affine vector fields. These space-times are (7) and (10). Proper affine vector field is given in equation (9).

References

[1] J. F. Schell, classification of four dimensional Riemannian spaces, J. Math. Physics, **2** (1961), 202-206.

[2] G. S. Hall, symmetries and curvature structure in general relativity, World Scientific, 2004.

[3] G. Shabbir, a note on proper affine vector fields in non-static spherically symmetric space-times, Acta Physics Polonica B, **40** (2009), 3-13.

[4] G. S. Hall, D. J. Low and J. R. Pulham, affine collineations in general relativity and their fixed point structure, J. Math. Physics, **35** (1994), 5930-5944.

[5] G. S. Hall and W. Kay, curvature structure in general relativity, J. Math. Physics, **29** (1988), 420-427.

[6] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselears and E. Herlt, Exact Solutions of Einstein's Field Equations, Cambridge University Press, 2003.

[7] G. Shabbir, proper projective symmetry in plane symmetric static space-times, Classical Quantum Gravity, **21** (2004), 339-347.

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