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A Note on Changhee Polynomials and Numbers

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Abstract. Recently, Changhee polynomials and numbers are introduced in [6]. Some interesting identities and properties of those polynomials are derived from umbral calculus(see [6]). In this paper, we consider Witt-type formula for the Changhee numbers and polynomials and derive some new interesting identities and properties of those polynomials and numbers from the Witt-type formula which are related to special polynomials.

1. Introduction

Let p be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by

$|p|_p = \frac{1}{p}$. Let $\mathbb{C}(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in \mathbb{C}(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [9, 15])}. \tag{1.1}$$

Let $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \text{ (see [3,5-10])}. \tag{1.2}$$

It is well known that the Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ (see [1-14])}. \tag{1.3}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers. The Changhee polynomials are defined by the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}, \text{ (see [6])}. \tag{1.4}$$

When $x = 0$, $Ch_n = Ch_n(0)$ are called the Changhee numbers, (see [6]).

The Stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l, \text{(see [6])}. \tag{1.5}$$

Recently, Changhee numbers and polynomials are introduced in [6]. Many interesting identities of those numbers and polynomials arise from umbral calculus (see [6]). In this paper, we consider Witt-type formula for the Changhee numbers and polynomials and derive some new interesting identities and properties of those polynomials and numbers from the Witt-type formula which are related to special polynomials.

2. Changhee numbers and polynomials

Let us take $f(x) = (1+t)^x$ for $|t|_p < p^{-\frac{1}{p-1}}$. Then, from (1.2), we have

$$\int_{\mathbb{Z}_p} (1+t)^x d\mu_{-1}(x) = \frac{2}{2+t} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!}. \tag{2.1}$$

Thus, by (2.1), we get

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x)_n d\mu_{-1}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!}. \tag{2.2}$$

By comparing the coefficients on both sides of (2.2), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} (x)_n d\mu_{-1}(x) = Ch_n.$$

From (1.4) and (2.1), we have

$$\int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_{-1}(y) = \frac{2(1+t)^x}{2+t} = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}. \tag{2.3}$$

Thus, by (2.3), we get

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}. \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y) = Ch_n(x).$$

From (2.1) and (2.2), we note that

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}(x) t^n = \frac{2}{2+t} = \sum_{n=0}^{\infty} (-\frac{1}{2})^n t^n. \tag{2.5}$$

Thus, by comparing the coefficients on the both sides, we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}(x) = (-1)^n (\frac{1}{2})^n.$$

Replacing t by $e^t - 1$ in (2.1), we get

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} Ch_n \frac{1}{n!} (e^t - 1)^n = \sum_{n=0}^{\infty} Ch_n \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \tag{2.6}$$

By (1.3) and (2.6), we get

$$\sum_{m=0}^{\infty} E_m \frac{t^m}{m!} = \frac{2}{e^t+1} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} Ch_n S_2(m,n) \frac{t^m}{m!}. \tag{2.7}$$

By comparing the coefficients on the both sides of (2.7), we obtain the following theorem.

Theorem 2.4. *For $m \geq 0$, we have*

$$E_m = \sum_{n=0}^{m} Ch_n S_2(m,n),$$

where $S_2(m,n)$ is the Stirling number of the second kind.

By Theorem 2.2, we easily get

$$Ch_n(x) = \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y) = \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} (x+y)^l d\mu_{-1}(y) = \sum_{l=0}^{n} S_1(n,l) E_l(x). \tag{2.8}$$

Therefore, by (2.8), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$Ch_n(x) = \sum_{l=0}^{n} S_1(n,l) E_l(x).$$

where $S_1(n,l)$ is the Stirling number of the first kind.

Let us consider the Changhee numbers of order k as follows:

$$Ch_n^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 x_2 \dots x_k)_n d\mu_{-1}(x_1) \dots d\mu_{-1}(x_k), \tag{2.9}$$

where $k \in \mathbb{N}$ and $n \in \mathbb{Z}_{\geq 0}$.

From (2.9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} Ch_n^{(k)} \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \frac{(x_1 \dots x_k)_n}{n!} t^n d\mu_{-1}(x_1) \dots d\mu_{-1}(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x_1 \dots x_k} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_k). \end{aligned} \tag{2.10}$$

Therefore, by (2.10), we obtain the generating function of $Ch_n^{(k)}$ as follows.

Theorem 2.6. *The generating function of $Ch_n^{(k)}$ is given by*

$$\sum_{n=0}^{\infty} Ch_n^{(k)} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x_1 \dots x_k} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_k).$$

From (2.10), we have

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 \dots x_k}{n} d\mu_{-1}(x_1) \dots d\mu_{-1}(x_k) = \frac{Ch_n^{(k)}}{n!}. \tag{2.11}$$

By (2.9), we get

$$\begin{aligned} Ch_n^{(k)} &= \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} x_1^l \ldots x_k^l d\mu_{-1}(x_1) \ldots d\mu_{-1}(x_k) \\ &= \sum_{l=0}^{n} S_1(n,l)(E_l)^k. \end{aligned} \tag{2.12}$$

where $(E_l)^k = \underbrace{E_l \times E_l \times \cdots \times E_l}_{k-times}$.

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, $k \geq 1$, we have*

$$Ch_n^{(k)} = \sum_{l=0}^{n} S_1(n,l)(E_l)^k,$$

where $S_1(m,n)$ is the Stirling number of the first kind.

Let us define the Changhee polynomials of order $k(\in \mathbb{N})$ as follows:

$$Ch_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 x_2 \ldots x_k + x)_n d\mu_{-1}(x_1) \ldots d\mu_{-1}(x_k). \tag{2.13}$$

Then the generating function of $Ch_n^{(k)}(x)$ is given by

$$\begin{aligned} \sum_{n=0}^{\infty} Ch_n^{(k)}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x_1 \ldots x_k + x}{n} t^n d\mu_{-1}(x_1) \ldots d\mu_{-1}(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x_1 \ldots x_k + x} d\mu_{-1}(x_1) \ldots d\mu_{-1}(x_k). \end{aligned} \tag{2.14}$$

Now, we observe that

$$
\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1\cdots x_k+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\
&=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}((x_1\cdots x_k)+x)^m d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\frac{(\log(1+t))^m}{m!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1\cdots x_k+x)^m d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)S_1(n,m)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1\cdots x_k)^{m-l}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)x^l\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}(E_{m-l})^k x^l\right\}\frac{t^n}{n!}.
\end{aligned}
\tag{2.15}
$$

By (2.14) and (2.15), we get

$$
Ch_n^{(k)}(x)=\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}(E_{m-l})^k x^l, \tag{2.16}
$$

where $(E_{m-l})^k=\underbrace{E_{m-l}\times\cdots\times E_{m-l}}_{k-times}$.

For $n\in\mathbb{Z}_{\geq 0}$, the rising factorial sequence is defined by

$$
x^{(n)}=x(x+1)\cdots(x+n-1). \tag{2.17}
$$

Let us define the Changhee numbers of the second kind as follows:

$$
\widehat{Ch_n}=\int_{\mathbb{Z}_p}(-x)_n d\mu_{-1}(x),\ (n\in\mathbb{Z}_{\geq 0}). \tag{2.18}
$$

It is easy to check that

$$
x^{(n)}=(-1)^n(-x)_n=\sum_{l=0}^{n}S_1(n,l)(-1)^{n-l}x^l. \tag{2.19}
$$

Thus, by (2.18) and (2.19), we get

$$
\begin{aligned}
\widehat{Ch_n}=\int_{\mathbb{Z}_p}(-x)_n d\mu_{-1}(x)&=\sum_{l=0}^{n}S_1(n,l)(-1)^l\int_{\mathbb{Z}_p}x^l d\mu_{-1}(x)\\
&=\sum_{l=0}^{n}S_1(n,l)(-1)^l E_l.
\end{aligned}
\tag{2.20}
$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$\widehat{Ch}_n = \sum_{l=0}^{n} S_1(n,l)(-1)^l E_l.$$

Let us consider the generating function of the Changhee numbers of the second kind as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{Ch}_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (-x)_n d\mu_{-1}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x+n-1}{n} (-1)^n t^n d\mu_{-1}(x) \\ &= \int_{\mathbb{Z}_p} (1+t)^{-x} d\mu_{-1}(x).\end{aligned} \tag{2.21}$$

By (1.2), we get

$$\int_{\mathbb{Z}_p} (1+t)^{-x} d\mu_{-1}(x) = \frac{2(1+t)}{2+t}. \tag{2.22}$$

From (2.21) and (2.22), we note that the generating function of the Changhee numbers of the second is given by

$$\sum_{n=0}^{\infty} \widehat{Ch}_n \frac{t^n}{n!} = \frac{2(1+t)}{2+t}. \tag{2.23}$$

By (2.23), we see that

$$\begin{aligned}\frac{2}{e^{-t}+1} &= \sum_{n=0}^{\infty} \frac{\widehat{Ch}_n}{n!} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{t^n}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} S_2(m,n) \widehat{Ch}_n \right) \frac{t^m}{m!}.\end{aligned} \tag{2.24}$$

Therefore, by (2.24), we obtain the following theorem.

Theorem 2.9. *For $m \geq 0$, we have*

$$E_m = (-1)^m \sum_{n=0}^{m} S_2(m,n) \widehat{Ch}_n.$$

Let us define the Changhee polynomials of the second kind as follows:

$$\widehat{Ch}_n(x) = \int_{\mathbb{Z}_p} (-y-x)_n d\mu_{-1}(y). \tag{2.25}$$

Then, by (2.25), we see that the generating function of $\widehat{Ch}_n(x)$ is given by

$$\sum_{n=0}^{\infty}\widehat{Ch}_n(x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(-y-x)_n d\mu_{-1}(y)\frac{t^n}{n!}$$
$$=\int_{\mathbb{Z}_p}(1+t)^{-y-x}d\mu_{-1}(y) \tag{2.26}$$
$$=2\left(\frac{1+t}{2+t}\right)(1+t)^{-x},\ |t|_p<p^{-\frac{1}{p-1}}.$$

By (2.25), we get

$$\widehat{Ch}_n(x)=\sum_{l=0}^{n}S_1(n,l)(-1)^l\int_{\mathbb{Z}_p}(x+y)^l d\mu_{-1}(y)$$
$$=\sum_{l=0}^{n}S_1(n,l)(-1)^l E_l(x). \tag{2.27}$$

Therefore, by (2.27), we obtain the following theorem.

Theorem 2.10. *For $n\geq 0$, we have*

$$\widehat{Ch}_n(x)=\sum_{l=0}^{n}S_1(n,l)(-1)^l E_l(x).$$

From (2.26), we note that

$$\frac{2}{e^t+1}e^{t(1-x)}=\sum_{n=0}^{\infty}\widehat{Ch}_n(x)\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}$$
$$=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{Ch}_n(x)S_2(m,n)\right)\frac{t^m}{m!}. \tag{2.28}$$

Therefore, by (2.28), we obtain the following theorem.

Theorem 2.11. *For $m\geq 0$, we have*

$$E_m(1-x)=\sum_{n=0}^{m}\widehat{Ch}_n(x)S_2(m,n).$$

$$(-1)^n\frac{Ch_n}{n!}=(-1)^n\int_{\mathbb{Z}_p}\binom{x}{n}d\mu_{-1}(x)=\int_{\mathbb{Z}_p}\binom{-x+n-1}{n}d\mu_{-1}(x)$$
$$=\sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\binom{-x}{m}d\mu_{-1}(x) \tag{2.29}$$
$$=\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{1}{m!}Ch_m,$$

and

$$(-1)^n \frac{\widehat{Ch}_n}{n!} = (-1)^n \int_{\mathbb{Z}_p} \binom{-x}{n} d\mu_{-1}(x) = \int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_{-1}(x)$$
$$= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{x}{m} d\mu_{-1}(x) \tag{2.30}$$
$$= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{1}{m!} Ch_m.$$

Therefore, by (2.29) and (2.30), we obtain the following theorem.

Theorem 2.12. *For $m \geq 0$, we have*

$$(-1)^n \frac{Ch_n}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{Ch}_n}{m!},$$

and

$$(-1)^n \frac{\widehat{Ch}_n}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{1}{m!} Ch_m.$$

For $k \in \mathbb{N}$, let us consider the Changhee numbers of order k as follows:

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 x_2 \cdots x_k)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \widehat{Ch}_n^{(k)}, \tag{2.31}$$

where $n \in \mathbb{Z}_{\geq 0}$.

Then the generating function of $\widehat{Ch}_n^{(k)}$ is given by

$$\sum_{n=0}^{\infty} \widehat{Ch}_n^{(k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 \cdots x_k)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \frac{t^n}{n!}$$
$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{-x_1 \cdots x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.32}$$

Now, we observe that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{-x_1 \cdots x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)$$
$$= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-1)^m (x_1 \cdots x_k)^m d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_n) \frac{(\log(1+t))^m}{m!}$$
$$= \sum_{m=0}^{\infty} (-1)^m (E_m)^k \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \tag{2.33}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} (-1)^m (E_m)^k S_1(n,m) \right) \frac{t^n}{n!}.$$

From (2.32) and (2.33), we have

$$\widehat{Ch_n}^{(k)} = \sum_{m=0}^{n} (-1)^m (E_m)^k S_1(n,m), \tag{2.34}$$

where $(E_m)^k = \underbrace{E_m \times \cdots \times E_m}_{k-times}$.

Now, we define the Changhee polynomials of the second kind of order k as follows:

$$\widehat{Ch_n}^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 \cdots x_k - x)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.35}$$

Then the generating function of $\widehat{Ch_n}^{(k)}(x)$ are given by

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{Ch_n}^{(k)}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 \cdots x_k - x)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{-x_1 \cdots x_k - x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).\end{aligned} \tag{2.36}$$

Proceeding similarly to (2.15), we have

$$\widehat{Ch_n}^{(k)}(x) = \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m} \binom{m}{l} (E_{m-l})^k x^l,$$

where $(E_{m-l})^k = \underbrace{E_{m-l} \times \cdots \times E_{m-l}}_{k-times}$.

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