

A note on quark and gluon energy-momentum tensors

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ABSTRACT: We discuss the constraints on quark and gluon energy-momentum tensors in QCD that follow from the requirement of Renormalisation-Group invariance of the traces of these operators. Our study covers the most general form of the latter traces, while the energy-momentum tensors themselves are only subjected to very mild constraints. We derive Renormalisation-Group equations for the two finite independent functions of the strong coupling constant and renormalisation scale of minimal subtraction which completely define the energy-momentum tensors. We demonstrate that previously proposed definitions of the renormalized quark and gluon energy-momentum tensors are special cases of our results assuming no explicit dependence on the renormalisation scale. Finally, we present $\overline{\text{MS}}$ -renormalised quark and gluon energy-momentum tensors at four-loop order.

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1 Introduction

The classical rank-two symmetric energy-momentum tensor (EMT) describes the density and flux of energy and momentum of a physical system. For a closed system, its four-vector components are the conserved Noether currents associated with spacetime-translation invariance up to separately conserved terms that yield the symmetric form. The density and flux of energy and momentum are the sources of the gravitational field in Einstein's field equations of General Relativity. The EMT is thus indispensable in order to describe the motion of gravitating systems.

The quantum EMT has attracted substantial attention after the discovery of the so-called trace anomaly [1–3] in Quantum Field Theories. This anomaly is closely connected to the broken scale invariance due to quantum corrections as reflected in Callan-Symanzik equations [4–6]. Following the early classical works on the trace anomaly in gauge field theories [7–13] and especially in Quantum Chromodynamics (QCD), there has been interest in the connection of the anomaly with the origin and the decomposition of the nucleon mass, see e.g. refs. [14–30] and references therein. Particular attention has been devoted to the proton mass, as this is unarguably one of the most fundamental and intriguingly quantities in particle physics. Interestingly, experiments proposed at the Jefferson Laboratory [31] and future Electron-Ion Collider [32], such as the near-threshold photo-production of J/ψ in electron-hadron/nucleon scattering [19, 25, 33, 34], as well as the currently envisaged EicC project [35] via the Υp near-threshold scattering, could shed light on the origin of the hadron/nucleon mass.

Since the first systematic decomposition of the EMT of QCD and the related decomposition of hadron masses [14, 15], commonly referred to as Ji's decomposition scheme, a multitude of alternatives have appeared in the literature [18, 20–24, 26, 27]. One reason for the different proposals is the fact that the partition of the trace of the EMT depends on the regularisation and renormalisation scheme, see e.g. refs. [15, 20, 23, 36]. In the present publication, we will also be mainly concerned with the operator renormalisation used to

define quark and gluon EMTs. Our starting point are refs. [20, 37], where QCD is regularised dimensionally, while not all operators are renormalised by minimal subtraction. Our goal is to obtain well-defined finite quark and gluon EMTs that have Renormalisation-Group (RG) invariant traces. This last property is particularly useful in view of applications to the proton mass, since perturbation theory can barely be used at the relevant renormalisation scale. RG invariance removes the large scale dependence otherwise present in the expectation values of the traces of the quark and gluon EMTs. Our work extends previous studies to the most general case assuming only non-renormalisation of the total EMT. We also discuss a minimal correction of the quark and gluon EMTs defined in the $\overline{\text{MS}}$ scheme to render their traces RG invariant. Finally, we extend the previous three-loop $\overline{\text{MS}}$ results for the quark and gluon contributions to the proton and pion mass [37] to the four-loop level and compare them with their RG-invariant counterparts.

The article is organized as follows. In the next section, we discuss the renormalisation of the physical operators occurring in the definition of the EMT. We follow the main argument of refs. [20, 37], but adapt the notation and simplify several steps of the derivations. Furthermore, we pay particular attention to the $\overline{\text{MS}}$ scheme in point 2.1. In section 3, we define the individual renormalised quark and gluon EMTs and discuss the general form of their RG-invariant traces. We also present a minimal modification of $\overline{\text{MS}}$ -scheme EMTs to obtain RG-invariant traces in point 3.1. Our four-loop results for expectation values of $\overline{\text{MS}}$ -scheme EMT traces in proton and pion states are presented in section 4 and compared to their RG-invariant counterparts. We conclude in section 5.

2 Renormalisation of physical-basis operators

This section follows refs. [20, 37] to a large extent, although we modify the notation and attempt to simplify the argumentation.

The physical symmetric EMT of QCD, obtained according to the Belinfante-Rosenfeld [38–40] method, is given by:

$$\Theta^{\mu\nu} \equiv -F^{a\mu}{}_{\rho} F^{a\nu\rho} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma}^a F^{a\rho\sigma} + \frac{i}{4} \sum_q \bar{q} \left(\gamma^\mu \overleftrightarrow{D}^\nu + \gamma^\nu \overleftrightarrow{D}^\mu \right) q, \quad (2.1)$$

with field-strength tensor:

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c, \quad (2.2)$$

and covariant derivative:

$$\overleftrightarrow{D}_\mu \equiv D_\mu - \overleftarrow{D}_\mu, \quad D_\mu = \partial^\mu - i g_s A_\mu^a T^a. \quad (2.3)$$

A_μ^a is the gluon field, q is a quark field of mass m_q , g_s is the strong coupling constant, T^a are hermitian $\text{SU}(N_c)$, $N_c = 3$, generators, and f^{abc} are real structure constants normalised according to $[T^a, T^b] = i f^{abc} T^c$. Depending on the quantisation, the actual symmetric EMT derived from the quantised Lagrangian with gauge-fixing terms contains additional terms that do not contribute in matrix elements of physical on-shell states. In linear gauges, for

instance, these additional terms amount to a BRS (Becchi-Rouet-Stora) variation of a local operator [10, 11] (see also application at the three-loop order in ref. [41]). In the present work, we will only be concerned with eq. (2.1) which is necessary for physical applications.

The EMT eq. (2.1) is conveniently expressed in a basis of local operators:

$$\mathcal{O}_1^{\mu\nu} \equiv -F^{a\mu}{}_{\rho} F^{a\nu\rho}, \quad \mathcal{O}_2^{\mu\nu} \equiv g^{\mu\nu} F_{\rho\sigma}^a F^{a\rho\sigma} \equiv g^{\mu\nu} \mathcal{O}_F, \quad (2.4)$$

$$\mathcal{O}_3^{\mu\nu} \equiv \frac{i}{4} \sum_q \bar{q} \left(\gamma^\mu \overleftrightarrow{D}^\nu + \gamma^\nu \overleftrightarrow{D}^\mu \right) q, \quad \mathcal{O}_4^{\mu\nu} \equiv g^{\mu\nu} \sum_q m_q \bar{q} q \equiv g^{\mu\nu} \mathcal{O}_m, \quad (2.5)$$

$$\Theta = \mathcal{O}_1 + \frac{1}{4} \mathcal{O}_2 + \mathcal{O}_3, \quad (2.6)$$

where we have suppressed Lorentz indices in the last equation as we will also do below. The fourth operator, \mathcal{O}_4 , has been introduced to accommodate the renormalisation of \mathcal{O}_2 . Indeed, there is [10, 11, 42, 43]:

$$[\mathcal{O}_F]_R = Z_{FF} [\mathcal{O}_F]_B + Z_{Fm} [\mathcal{O}_m]_B, \quad [\mathcal{O}_m]_R = [\mathcal{O}_m]_B, \quad (2.7)$$

where $[\cdot]_R$ denotes a renormalised operator with finite matrix elements, while $[\cdot]_B$ is defined by replacing fields, masses and coupling constants by their bare counterparts. In eq. (2.7) and in the remainder of this publication, we neglect operators vanishing by the equations-of-motion and BRS-exact operators assuming Faddeev-Popov quantisation. The renormalisation constants Z_{FF} and Z_{Fm} are related in the $\overline{\text{MS}}$ scheme with spacetime dimension $D \equiv 4 - 2\epsilon$ to the fundamental anomalous dimensions of the theory [13]:

$$Z_{FF} = \left(1 - \frac{\beta}{\epsilon} \right)^{-1}, \quad Z_{Fm} = -\frac{4\gamma_m}{\epsilon} Z_{FF}, \quad (2.8)$$

where β and γ_m are defined through:

$$\frac{d \ln a_s}{d \ln \mu^2} \equiv -\epsilon + \beta, \quad a_s \equiv \frac{\alpha_s}{4\pi} = \frac{g_s^2}{16\pi^2}, \quad \frac{d \ln m}{d \ln \mu^2} \equiv \gamma_m. \quad (2.9)$$

The four operators $\mathcal{O}_{1,\dots,4}$ mix under renormalisation:

$$[\mathcal{O}_i]_R \equiv Z_{ij} [\mathcal{O}_j]_B. \quad (2.10)$$

Our main **assumption** that will influence the quark and gluon EMTs defined in the next section can be stated as follows:

The renormalisation constants Z_{ij} are defined in the $\overline{\text{MS}}$ scheme for all operators but the mixing of $\mathcal{O}_{2,4}$ into $\mathcal{O}_{1,3}$. The Laurent expansion of Z_{ij} , $i = 1, 3$, $j = 2, 4$ may contain terms proportional to ϵ^k , $k \geq 0$.

We will discuss this assumption further in section 5. The renormalisation-constant matrix has the structure:

$$Z = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ 0 & Z_{FF} & 0 & Z_{Fm} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.11)$$

where we have taken into account:

1. $Z_{ij} = 0$ for $i = 2, 4$ and $j = 1, 3$ because $\mathcal{O}_{2,4}$ are proportional to scalar operators, while $\mathcal{O}_{1,3}$ are (reducible) rank-two tensor operators;
2. $Z_{4j} = 0$ for $j = 1, 2, 3$ because the canonical dimension of \mathcal{O}_4 is 3, less than those of $\mathcal{O}_{1,2,3}$;
3. $Z_{22} = Z_{FF}$, $Z_{24} = Z_{Fm}$, $Z_{44} = 1$ because $\mathcal{O}_{2,4}^{\mu\nu} = g^{\mu\nu} \mathcal{O}_{F,m}$.

The renormalisation constants are subject to a constraint following from the non-renormalisation [7, 9–11, 44] of the EMT. Since we have allowed for non- $\overline{\text{MS}}$ renormalisation constants, the equality of the renormalised and bare EMTs is not guaranteed anymore. However, we further **assume** that this equality holds:

$$[\Theta]_{\text{R}} \equiv [\Theta]_{\text{B}}. \tag{2.12}$$

As a consequence, we obtain the constraints:

$$\begin{aligned} Z_{13} &= 1 - Z_{33}, & Z_{31} &= 1 - Z_{11}, \\ Z_{32} &= \frac{1}{4} - \frac{1}{4}Z_{FF} - Z_{12}, & Z_{34} &= -\frac{1}{4}Z_{Fm} - Z_{14}. \end{aligned} \tag{2.13}$$

It turns out that the renormalisation constants that we have allowed to be defined in a non- $\overline{\text{MS}}$ scheme, Z_{ij} , $i = 1, 3$, $j = 2, 4$, may be further constrained by considering a different operator basis containing traceless rank-two operators that correspond to an irreducible representation of the Lorentz group. With the following notation for the trace of a rank-two tensor operator:

$$\text{Tr}(\mathcal{A}) \equiv \mathcal{A}^\rho{}_\rho, \tag{2.14}$$

there is:

$$\text{Tr}(\mathcal{O}_1) = -\mathcal{O}_F, \quad \text{Tr}(\mathcal{O}_2) = D\mathcal{O}_F, \quad \text{Tr}(\mathcal{O}_3) = \mathcal{O}_m, \quad \text{Tr}(\mathcal{O}_4) = D\mathcal{O}_m, \tag{2.15}$$

where the space-time dimension is denoted as $D = 4 - 2\epsilon$ in dimensional regularization.

Let us define:

$$\tilde{\mathcal{O}}_{1,3}^{\mu\nu} \equiv \mathcal{O}_{1,3}^{\mu\nu} - \frac{g^{\mu\nu}}{D} \text{Tr}(\mathcal{O}_{1,3}). \tag{2.16}$$

These operators are by construction traceless:

$$\text{Tr}(\tilde{\mathcal{O}}_{1,3}) = 0. \tag{2.17}$$

We extend the set by rescaling the remaining operators so that:

$$\text{Tr}(\tilde{\mathcal{O}}_2) = \mathcal{O}_F, \quad \text{Tr}(\tilde{\mathcal{O}}_4) = \mathcal{O}_m. \tag{2.18}$$

The operators $\tilde{\mathcal{O}}_{1,\dots,4}$ are related to $\mathcal{O}_{1,\dots,4}$ by:

$$\tilde{\mathcal{O}}_i \equiv M_{ij} \mathcal{O}_j, \quad M \equiv \begin{pmatrix} 1 & 1/D & 0 & 0 \\ 0 & 1/D & 0 & 0 \\ 0 & 0 & 1 & -1/D \\ 0 & 0 & 0 & 1/D \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & D \end{pmatrix}, \tag{2.19}$$

while their renormalisation takes the form:

$$[\tilde{\mathcal{O}}_i]_{\text{R}} \equiv \tilde{Z}_{ij} [\tilde{\mathcal{O}}_j]_{\text{B}}, \quad (2.20)$$

with the renormalisation constants defined in the $\overline{\text{MS}}$ scheme:

$$\tilde{Z} = \begin{pmatrix} Z_{11} & 0 & Z_{13} & 0 \\ 0 & Z_{FF} & 0 & Z_{Fm} \\ Z_{31} & 0 & Z_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.21)$$

This structure follows from:

1. $\tilde{Z}_{ij} = 0$ for $i = 1, 3$ and $j = 2, 4$ because $\tilde{\mathcal{O}}_{2,4}$ are proportional to scalar operators, while $\tilde{\mathcal{O}}_{1,3}$ are irreducible rank-two tensor operators;
2. $\tilde{Z}_{ij} = Z_{ij}$ for $i, j = 1, 3$ because $\mathcal{O}_{1,3}$ contain the irreducible operators $\tilde{\mathcal{O}}_{1,3}$ with unit coefficient;
3. $\tilde{Z}_{ij} = Z_{ij}$ for $i, j = 2, 4$ because $\tilde{\mathcal{O}}_{2,4} = 1/D \mathcal{O}_{2,4}$.

With the help of $\tilde{\mathcal{O}}_{1,\dots,4}$, it is possible to define another set of renormalised operators that are equal to $\mathcal{O}_{1,\dots,4}$ at tree-level:

$$[\mathcal{O}'_i]_{\text{R}} \equiv M_{ij}^{-1} [\tilde{\mathcal{O}}_j]_{\text{R}}. \quad (2.22)$$

The difference between the two bases is:

$$[\mathcal{O}_i]_{\text{R}} - [\mathcal{O}'_i]_{\text{R}} = \left(\mathbb{1} - M^{-1} \tilde{Z} M Z^{-1} \right)_{ij} [\mathcal{O}_j]_{\text{R}}. \quad (2.23)$$

Explicit calculation confirms the expectation that the difference vanishes for $i = 2, 4$, while it is proportional to $[\mathcal{O}_{2,4}]_{\text{R}}$ for $i = 1, 3$:

$$[\mathcal{O}_{2,4}]_{\text{R}} - [\mathcal{O}'_{2,4}]_{\text{R}} = 0, \quad (2.24)$$

$$[\mathcal{O}_{1,3}]_{\text{R}} - [\mathcal{O}'_{1,3}]_{\text{R}} = \frac{1}{D} \left(x_{1,3} [\mathcal{O}_2]_{\text{R}} + y_{1,3} [\mathcal{O}_4]_{\text{R}} \right), \quad (2.25)$$

where:

$$x_1 = \frac{Z_{FF} - Z_{11} + DZ_{12}}{Z_{FF}}, \quad y_1 = (1 - x_1) Z_{Fm} + Z_{13} + DZ_{14}, \quad (2.26)$$

$$x_3 = \frac{-Z_{31} + DZ_{32}}{Z_{FF}}, \quad y_3 = -1 - x_3 Z_{Fm} + Z_{33} + DZ_{34}.$$

Due to eqs. (2.13), only two of the constants $x_{1,3}, y_{1,3}$ are independent:

$$x_1 + x_3 = \frac{\epsilon}{2} \left(1 - Z_{FF}^{-1} \right) = \frac{\beta}{2}, \quad y_1 + y_3 = \frac{\epsilon}{2} Z_{Fm} Z_{FF}^{-1} = -2\gamma_m, \quad (2.27)$$

where we have also used eqs. (2.8) to obtain the right-hand sides of the relations. The constants $x_{1,3}, y_{1,3}$ vanish at tree-level and are regular at $\epsilon = 0$:

$$x_{1,3}, y_{1,3} = \mathcal{O}(a_s), \quad x_{1,3}, y_{1,3} \equiv x_{1,3}^{(0)}, y_{1,3}^{(0)} + \mathcal{O}(\epsilon), \quad x_{1,3}^{(0)}, y_{1,3}^{(0)} \in \mathbb{R}. \quad (2.28)$$

Furthermore, they provide a parameterisation of Z_{ij} , $i = 1, 3$, $j = 2, 4$:

$$\begin{aligned} Z_{12} &= \frac{1}{D} (Z_{11} + (x_1 - 1)Z_{FF}), & Z_{14} &= \frac{1}{D} (y_1 - Z_{13} + (x_1 - 1)Z_{Fm}), \\ Z_{32} &= \frac{1}{D} (Z_{31} + x_3 Z_{FF}), & Z_{34} &= \frac{1}{D} (1 + y_3 - Z_{33} + x_3 Z_{Fm}). \end{aligned} \quad (2.29)$$

2.1 $\overline{\text{MS}}$ scheme constraints

The case of Z_{ij} , $i = 1, 3$, $j = 2, 4$ defined in the $\overline{\text{MS}}$ scheme provides further simplifications. Defining:

$$Z_{ij}(a_s, \epsilon) \equiv \delta_{ij} + \sum_{l=1}^{\infty} \frac{Z_{ij}^{(l)}(a_s)}{\epsilon^l}. \quad (2.30)$$

Eqs. (2.26) yield at $\mathcal{O}(\epsilon^0)$:

$$x_1^{\overline{\text{MS}}} = -2Z_{12}^{(1)}, \quad y_1^{\overline{\text{MS}}} = -2Z_{14}^{(1)}, \quad x_3^{\overline{\text{MS}}} = -2Z_{32}^{(1)}, \quad y_3^{\overline{\text{MS}}} = -2Z_{34}^{(1)}. \quad (2.31)$$

The same equations (2.26) taken at $\mathcal{O}(\epsilon^k)$, $k < 0$ provide constraints on Z_{ij} , $i = 1, 3$, $j = 2, 4$ that determine these constants uniquely. The results may be expressed through the elements of the anomalous-dimension matrix of $[\tilde{\mathcal{O}}_{1,3}]_{\text{R}}$:

$$\frac{d}{d \ln \mu^2} \begin{pmatrix} [\tilde{\mathcal{O}}_1]_{\text{R}} \\ [\tilde{\mathcal{O}}_3]_{\text{R}} \end{pmatrix} = \begin{pmatrix} \gamma_{11} & -\gamma_{33} \\ -\gamma_{11} & \gamma_{33} \end{pmatrix} \begin{pmatrix} [\tilde{\mathcal{O}}_1]_{\text{R}} \\ [\tilde{\mathcal{O}}_3]_{\text{R}} \end{pmatrix}. \quad (2.32)$$

Notice that we have exploited eqs. (2.13) to simplify the form of the anomalous-dimension matrix and express the scale dependence of the operators with the help of only two independent anomalous dimensions, γ_{11}, γ_{33} . With the expansions:

$$\beta \equiv -\sum_{l=1}^{\infty} a_s^l \beta_{l-1}, \quad \gamma_m \equiv \sum_{l=1}^{\infty} a_s^l \gamma_m^{(l)}, \quad \gamma_{ij} \equiv \sum_{l=1}^{\infty} a_s^l \gamma_{ij}^{(l)}, \quad (2.33)$$

there is for instance [20, 37]:

$$\begin{aligned} x_1^{\overline{\text{MS}}} &= \frac{a_s}{2} \left(-\beta_0 + \gamma_{11}^{(1)} \right) \\ &+ \frac{a_s^2}{8} \left(-4\beta_1 + (\beta_0 - \gamma_{11}^{(1)} - \gamma_{33}^{(1)}) \gamma_{11}^{(1)} + 2\gamma_{11}^{(2)} \right) + \mathcal{O}(a_s^3) \\ &= a_s \left(-\frac{11}{6} C_A \right) + \mathcal{O}(a_s^2), \\ y_1^{\overline{\text{MS}}} &= \frac{a_s}{2} \left(-4\gamma_m^{(1)} + \gamma_{33}^{(1)} \right) \\ &+ \frac{a_s^2}{8} \left(-16\gamma_m^{(2)} + 8\gamma_{11}^{(1)} \gamma_m^{(1)} - (\beta_0 + \gamma_{11}^{(1)} + \gamma_{33}^{(1)}) \gamma_{33}^{(1)} + 2\gamma_{33}^{(2)} \right) + \mathcal{O}(a_s^3) \\ &= a_s \left(\frac{14}{3} C_F \right) + \mathcal{O}(a_s^2). \end{aligned} \quad (2.34)$$

The quadratic Casimir color constants are defined as usual: $C_A = N_c$, $C_F = (N_c^2 - 1)/(2N_c)$ with the number of colors in the fundamental representation $N_c = 3$ in QCD.

3 Quark and gluon EMTs with Renormalisation-Group-invariant traces

Let us now define renormalised quark and gluon EMTs by decomposing the EMT according to the fields occurring in the operators similarly to refs. [20, 37]:

$$[\Theta]_{\text{R}} = \left[\mathcal{O}_1 + \frac{1}{4} \mathcal{O}_2 \right]_{\text{R}} + [\mathcal{O}_3]_{\text{R}} \equiv [\Theta_g]_{\text{R}} + [\Theta_q]_{\text{R}}, \quad (3.1)$$

$$[\Theta_g]_{\text{R}} = \left[\mathcal{O}_1 + \frac{1}{4} \mathcal{O}_2 \right]_{\text{R}}, \quad [\Theta_q]_{\text{R}} = [\mathcal{O}_3]_{\text{R}}, \quad (3.2)$$

where the renormalised operators on the r.h.s. are defined by eqs. (2.10), (2.11) and (2.29). The constants $x_{1,3}, y_{1,3}$ that enter the definitions may in principle have a non-trivial dependence on ϵ . In the $\overline{\text{MS}}$ scheme, they are pure numbers, see eqs. (2.31). According to eqs. (2.25):

$$[\Theta_{g,q}]_{\text{R}} = [\Theta_{g,q}]_{\text{R}}^{\overline{\text{MS}}} + \frac{1}{D} \left((x_{1,3} - x_{1,3}^{\overline{\text{MS}}}) [\mathcal{O}_2]_{\text{R}} + (y_{1,3} - y_{1,3}^{\overline{\text{MS}}}) [\mathcal{O}_4]_{\text{R}} \right), \quad (3.3)$$

and the differences $x_{1,3} - x_{1,3}^{\overline{\text{MS}}}$ and $y_{1,3} - y_{1,3}^{\overline{\text{MS}}}$ only influence physical observables at $\epsilon = 0$. Hence, as far as applications are concerned, we may assume:

$$x_{1,3}, y_{1,3} \in \mathbb{R}. \quad (3.4)$$

In this section, we work exclusively with renormalised operators. In consequence, we set $\epsilon = 0$ in all expressions.

Our goal is to obtain a decomposition of the EMT for which both Θ_g and Θ_q have RG-invariant traces. This allows to attach a physical meaning to matrix elements of these traces. The RG invariance of the trace of Θ itself is a consequence of eq. (2.12). Using eqs. (2.15), (2.7) and (2.8), we recover the well-known result:

$$\text{Tr}([\Theta]_{\text{R}}) = \text{Tr}(\Theta) = \frac{\beta}{2} [\mathcal{O}_F]_{\text{R}} + (1 - 2\gamma_m) [\mathcal{O}_m]_{\text{R}}. \quad (3.5)$$

The crucial observation is that the traces of the quark and gluon EMTs will also be expressed through $[\mathcal{O}_F]_{\text{R}}$ and $[\mathcal{O}_m]_{\text{R}}$ which may be combined linearly to yield two RG-invariant operators:

$$\frac{d}{d \ln \mu^2} \text{Tr}(\Theta) = 0, \quad \frac{d}{d \ln \mu^2} [\mathcal{O}_m]_{\text{R}} = 0. \quad (3.6)$$

In terms of these invariant operators:

$$\begin{aligned} \text{Tr}([\Theta_g]_{\text{R}}) &= x_1 [\mathcal{O}_F]_{\text{R}} + y_1 [\mathcal{O}_m]_{\text{R}} \\ &= \frac{2x_1}{\beta} \text{Tr}(\Theta) + \left(y_1 - \frac{2x_1}{\beta} (1 - 2\gamma_m) \right) [\mathcal{O}_m]_{\text{R}}, \\ \text{Tr}([\Theta_q]_{\text{R}}) &= x_3 [\mathcal{O}_F]_{\text{R}} + (1 + y_3) [\mathcal{O}_m]_{\text{R}} \\ &= \frac{2x_3}{\beta} \text{Tr}(\Theta) + \left(1 + y_3 - \frac{2x_3}{\beta} (1 - 2\gamma_m) \right) [\mathcal{O}_m]_{\text{R}}. \end{aligned} \quad (3.7)$$

Because the scale dependence of a_s is $\mathcal{O}(a_s^2)$, the scale dependence of the coefficients of the invariant operators on the r.h.s. is $\mathcal{O}(a_s^2)$ independently of the particular choice of the constants $x_{1,3}, y_{1,3}$. Hence, the scale dependence of the traces is always $\mathcal{O}(a_s^2)$. Let us now state the conditions that lead to RG-invariant traces:

if the constants $x_{1,3}, y_{1,3}$ that define the renormalised EMTs eqs. (3.2) through the renormalisation constants eqs. (2.29) satisfy:

$$\frac{d}{d \ln \mu^2} \left(\frac{x_1}{\beta} \right) = 0, \quad \frac{dy_1}{d \ln \mu^2} = -\frac{4x_1}{\beta} \frac{d\gamma_m}{d \ln \mu^2} = -4x_1 a_s \frac{d\gamma_m}{da_s}, \quad (3.8)$$

$$x_3 = \frac{\beta}{2} - x_1, \quad y_3 = -2\gamma_m - y_1, \quad (3.9)$$

then the traces of the renormalised quark and gluon EMTs are RG invariant:

$$\frac{d}{d \ln \mu^2} \text{Tr}([\Theta_{g,q}]_R) = 0. \quad (3.10)$$

The constants $x_{1,3}, y_{1,3}$ that yield RG-invariant quark and gluon EMT traces are thus some very general series in $a_s(\mu_0)$ starting from $\mathcal{O}(a_s(\mu_0))$, with their dependence on the scale μ^2 determined by solving eqs. (3.8). This introduces an explicit dependence on $\ln \mu^2/\mu_0^2$ given that:

$$\frac{d}{d \ln \mu^2} = \frac{\partial}{\partial \ln \mu^2} + \beta a_s \frac{\partial}{\partial a_s}. \quad (3.11)$$

The form of $x_{1,3}, y_{1,3}$ is substantially constrained, if we further assume that they do not explicitly depend on the scale μ , but are at most functions of a_s . Indeed, in this case:

$$x_1 = \frac{\beta}{2} k, \quad x_3 = \frac{\beta}{2} (1 - k), \quad k \in \mathbb{R}. \quad (3.12)$$

The general solution to the second of eqs. (3.8) is $y_1 = -2\gamma_m k + c$, $c \in \mathbb{R}$. However, since y_1 is $\mathcal{O}(a_s)$, c must vanish:

$$y_1 = -2\gamma_m k, \quad y_3 = -2\gamma_m (1 - k). \quad (3.13)$$

This yields the traces:

$$\text{Tr}([\Theta_g]_R) = k (\text{Tr}(\Theta) - [\mathcal{O}_m]_R) = k \left(\frac{\beta}{2} [\mathcal{O}_F]_R - 2\gamma_m [\mathcal{O}_m]_R \right), \quad (3.14)$$

$$\text{Tr}([\Theta_q]_R) = (1 - k) (\text{Tr}(\Theta) - [\mathcal{O}_m]_R) + [\mathcal{O}_m]_R.$$

Let us finally note that eqs. (3.14) cover, in particular, various special cases proposed previously in the literature. For instance, $k = 1$ corresponds to the D2 scheme of ref. [22], while $k = 0$ corresponds to the D3 scheme of ref. [23].

3.1 Perturbative correction of non-RG-invariant traces

Suppose that we are given a decomposition of the EMT eq. (3.2) characterised by some $x_{1,3}, y_{1,3}$ that are functions of a_s only but do not match at least one of eqs. (3.12), (3.13).

An example would be the $\overline{\text{MS}}$ constants eqs. (2.34). In this case, the traces eqs. (3.7) are not RG invariant:

$$\frac{d}{d \ln \mu^2} \begin{pmatrix} \text{Tr}([\Theta_g]_R) \\ \text{Tr}([\Theta_q]_R) \end{pmatrix} = \begin{pmatrix} \gamma_{gg}^{\text{tr}} & -\gamma_{qq}^{\text{tr}} \\ -\gamma_{gg}^{\text{tr}} & \gamma_{qq}^{\text{tr}} \end{pmatrix} \begin{pmatrix} \text{Tr}([\Theta_g]_R) \\ \text{Tr}([\Theta_q]_R) \end{pmatrix} \equiv \gamma^{\text{tr}} \begin{pmatrix} \text{Tr}([\Theta_g]_R) \\ \text{Tr}([\Theta_q]_R) \end{pmatrix}, \quad \gamma^{\text{tr}} \neq 0, \quad (3.15)$$

where we have used the invariance of the trace of the EMT, eq. (3.6). As noted above, the matrix elements of γ^{tr} are always $\mathcal{O}(a_s^2)$, independently of the particular $x_{1,3}, y_{1,3}$. One can, nevertheless, perform a rotation of the traces to obtain RG-invariant expressions:

$$\begin{pmatrix} \text{Tr}([\Theta'_g]_R) \\ \text{Tr}([\Theta'_q]_R) \end{pmatrix} \equiv F \begin{pmatrix} \text{Tr}([\Theta_g]_R) \\ \text{Tr}([\Theta_q]_R) \end{pmatrix}, \quad \frac{dF}{d \ln \mu^2} = -F \gamma^{\text{tr}}. \quad (3.16)$$

Assuming that F is perturbative, i.e. $F(a_s = 0) = \mathbf{1}$, it can be uniquely determined by solving the above RG equation for any given $x_{1,3}, y_{1,3}$. Furthermore, the result of the rotation must necessarily match the form of eqs. (3.14) with:

$$k = \left. \frac{2x_1}{\beta} \right|_{a_s=0}. \quad (3.17)$$

Indeed, there is for instance:

$$\text{Tr}([\Theta_g]_R) - \text{Tr}([\Theta'_g]_R) = \left(\frac{2x_1}{\beta} - k \right) \text{Tr}(\Theta) + \left(y_1 - \frac{2x_1}{\beta} (1 - 2\gamma_m) + k \right) [\mathcal{O}_m]_R. \quad (3.18)$$

However, $F = \mathbf{1} + \mathcal{O}(a_s)$ implies:

$$\frac{2x_1}{\beta} - k = \mathcal{O}(a_s), \quad y_1 - \frac{2x_1}{\beta} (1 - 2\gamma_m) + k = \mathcal{O}(a_s), \quad (3.19)$$

which yields eq. (3.17).

The above procedure applied to the D1 scheme of ref. [21] yields the D2 scheme of ref. [22]. In the present publication we make the following proposal:

Let the quark and gluon EMTs, $[\Theta_q]_R^{\text{RGltr}}$ and $[\Theta_g]_R^{\text{RGltr}}$, with RG-invariant traces be defined by perturbatively correcting the respective tensors defined in the $\overline{\text{MS}}$ scheme. This corresponds to setting in eqs. (3.14):

$$k = \frac{\beta_0|_{n_f=0}}{\beta_0} = \left(1 - \frac{2}{33} n_f \right)^{-1}, \quad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f, \quad (3.20)$$

where we have used eq. (2.34), and n_f in β_0 is the number of massless quarks and the color-trace normalization factor is denoted by T_F . $[\Theta_{g,q}]_R^{\text{RGltr}}$ are defined by eqs. (3.2) with the renormalised operators defined in eqs. (2.10), (2.11) with non- $\overline{\text{MS}}$ -scheme renormalisation constants eqs. (2.29) with $x_{1,3}, y_{1,3}$ from eqs. (3.12) and (3.13).

4 Example applications at four-loop order

The expectation value of the trace of the EMT in a single-particle state is proportional to the square of the mass, m , of the particle:

$$\langle mj; \mathbf{p}\lambda | \text{Tr}(\Theta) | mj; \mathbf{p}\lambda \rangle = 2m^2 \quad \text{for} \quad \langle mj; \mathbf{p}\lambda | mj; \mathbf{p}'\lambda' \rangle = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{\lambda\lambda'}, \quad (4.1)$$

where j and λ, λ' are the spin and polarisations of the particle, \mathbf{p}, \mathbf{p}' are its three-momenta and $E_{\mathbf{p}}$ is its energy. The decomposition eq. (3.2) of the EMT into quark and gluon EMTs may be used to define the respective quark and gluon contributions to the mass. Unfortunately, this decomposition is not unique. If the occurring operators $\mathcal{O}_{1,\dots,3}$ are defined in the $\overline{\text{MS}}$ scheme, then the quark and gluon contributions to the mass depend on the unphysical renormalisation scale μ . With the renormalisation scheme specified in section 3, on the other hand, these contributions do not depend on μ anymore, but there still remains a dependence on the parameter k , see eqs. (3.14). This remaining ambiguity is finally removed with our proposal, eq. (3.20). In this section, we compare the quark and gluon contributions to the mass of the proton and pion defined in the $\overline{\text{MS}}$ scheme to those obtained with our proposal.

We begin with the $\overline{\text{MS}}$ -renormalised quark and gluon contributions to the proton mass originally presented in ref. [37] at three-loop order. Since the anomalous dimensions, γ_{11} and γ_{33} , of the $\mathcal{O}_{1,3}$ operators, eq. (2.32), have recently been determined to four-loop order [45, 46], and the QCD β -function as well as the quark-mass anomalous dimension are available at the same order from refs. [47, 48] and [49, 50], it is now possible to extend the analysis of ref. [37] to one order higher.¹ To this end, we use four-loop accurate generalisations of eqs. (2.34) substituted in eqs. (3.7). Due to the length of the resulting expressions, we refrain from reproducing them explicitly in the text, but rather include them in the supplementary material attached to this paper.

Let $|P\rangle$ be a proton state with some (irrelevant) polarisation and three-momentum, and consider QCD with $n_f = 3$ light quarks. The expectation values of the traces of the quark and gluon EMTs read:

$$\begin{aligned} \langle \text{Tr}([\Theta_g]_R^{\overline{\text{MS}}}) \rangle_P &= \langle [O_F]_R \rangle_P \left(-0.437676 \alpha_s - 0.261512 \alpha_s^2 - 0.183827 \alpha_s^3 - 0.256096 \alpha_s^4 \right) \\ &\quad + \langle [O_m]_R \rangle_P \left(0.495149 \alpha_s + 0.776587 \alpha_s^2 + 0.865492 \alpha_s^3 + 0.974674 \alpha_s^4 \right), \\ \langle \text{Tr}([\Theta_q]_R^{\overline{\text{MS}}}) \rangle_P &= \langle [O_F]_R \rangle_P \left(0.079578 \alpha_s + 0.058870 \alpha_s^2 + 0.021604 \alpha_s^3 + 0.013675 \alpha_s^4 \right) \\ &\quad + \langle [O_m]_R \rangle_P \left(1 + 0.141471 \alpha_s - 0.008235 \alpha_s^2 - 0.064351 \alpha_s^3 - 0.065869 \alpha_s^4 \right), \end{aligned} \quad (4.2)$$

where $\langle [O_F]_R \rangle_P, \langle [O_m]_R \rangle_P$ denote, respectively, the non-perturbative expectation values of the $\overline{\text{MS}}$ -renormalised operators O_F and O_m . Both α_s and $\langle [O_F]_R \rangle_P$ depend on the renormalisation scale μ , while $\langle [O_m]_R \rangle_P$ is independent of μ since $[O_m]_R$ is RG invariant. Eqs. (4.2) agree with ref. [37] up to $\mathcal{O}(\alpha_s^3)$, while the $\mathcal{O}(\alpha_s^4)$ terms are new. These results

¹The state-of-the-art five-loop results for β [51–53] and γ_m [54–56] cannot be used, since the anomalous dimensions of the $\mathcal{O}_{1,3}$ operators are not known at this order yet.

may also be rewritten in terms of the expectation values of RG-invariant operators using $\langle \text{Tr}(\Theta) \rangle_{\text{P}} = 2m_{\text{P}}^2$ in eqs. (3.7):

$$\begin{aligned} \frac{\langle \text{Tr}([\Theta_{g|R}^{\overline{\text{MS}}}] \rangle_{\text{P}}}{2m_{\text{P}}^2} &= 1.22222 + 0.0386426 \alpha_s - 0.0622081 \alpha_s^2 - 0.0945524 \alpha_s^3 \\ &+ \frac{\langle [O_m]_R \rangle_{\text{P}}}{2m_{\text{P}}^2} \left(-1.22222 - 0.321585 \alpha_s - 0.124903 \alpha_s^2 - 0.00921612 \alpha_s^3 \right), \\ \frac{\langle \text{Tr}([\Theta_{q|R}^{\overline{\text{MS}}}] \rangle_{\text{P}}}{2m_{\text{P}}^2} &= -0.222222 - 0.0386426 \alpha_s + 0.0622081 \alpha_s^2 + 0.0945524 \alpha_s^3 \\ &+ \frac{\langle [O_m]_R \rangle_{\text{P}}}{2m_{\text{P}}^2} \left(1.22222 + 0.321585 \alpha_s + 0.124903 \alpha_s^2 + 0.00921612 \alpha_s^3 \right). \end{aligned} \quad (4.3)$$

The scale dependence of the expectation values is now entirely due to the strong coupling constant. In order to illustrate the size of this dependence, we need a numerical estimate for $\langle [O_m]_R \rangle_{\text{P}} / 2m_{\text{P}}^2$. One possibility is to set it to null by taking the chiral limit as has been done in ref. [37]. On the other hand, we may also obtain it from lattice simulations, for example from table 1 of ref. [27] (see also refs. [57, 58]) where it is denoted $f_{q\text{total}}^N$ at $n_f = 2 + 1$:

$$\frac{\langle [O_m]_R \rangle_{\text{P}}}{2m_{\text{P}}^2} \approx 0.092. \quad (4.4)$$

As for the strong coupling constant, we make use of the program RUNDEC [59] with four-loop running and default parameter settings which yields:²

$$\alpha_s(\mu) \approx \begin{cases} 0.48 & \text{for } \mu = 1 \text{ GeV} \sim m_{\text{P}}, \\ 0.30 & \text{for } \mu = 2 \text{ GeV}, \\ 0.80 & \text{for } \mu = 1/\sqrt{2} \text{ GeV}. \end{cases} \quad (4.5)$$

As can be seen from these numbers, a typically-used scale choice for the lower scale amounting to half of the central scale would certainly be non-perturbative. This is the reason for our unusual choice of $1/\sqrt{2} \text{ GeV}$. Substitution in eqs. (4.3) results in:

$$\begin{aligned} \frac{\langle \text{Tr}([\Theta_{g|R}^{\overline{\text{MS}}}] \rangle_{\text{P}}}{2m_{\text{P}}^2} &= \begin{cases} 1.087 & \text{for } \mu = 1 \text{ GeV} \sim m_{\text{P}}, \\ 1.103 & \text{for } \mu = 2 \text{ GeV}, \\ 1.021 & \text{for } \mu = 1/\sqrt{2} \text{ GeV}, \end{cases} \\ \frac{\langle \text{Tr}([\Theta_{q|R}^{\overline{\text{MS}}}] \rangle_{\text{P}}}{2m_{\text{P}}^2} &= 1 - \frac{\langle \text{Tr}([\Theta_{g|R}^{\overline{\text{MS}}}] \rangle_{\text{P}}}{2m_{\text{P}}^2} = \begin{cases} -0.087 & \text{for } \mu = 1 \text{ GeV} \sim m_{\text{P}}, \\ -0.103 & \text{for } \mu = 2 \text{ GeV}, \\ -0.021 & \text{for } \mu = 1/\sqrt{2} \text{ GeV}. \end{cases} \end{aligned} \quad (4.6)$$

²The values in eq. (4.5) were determined using RUNDEC [59] via solving the RGE equation for α_s at 4-loop accuracy numerically with initial value $\alpha_s = 0.118$ at the Z-boson pole mass.

The gluon content of the proton is rather stable with respect to the scale choice according to the above results. Indeed, the deviations from the value for the central scale reach only +1.5% and -5.9% for the upper and lower scales respectively. The quark content, on the other hand, is rather uncertain with deviations ranging from +19% to -74%.

Let us now compare the above results to those obtained from eqs. (3.14) and (3.17). Taking into account the construction presented in section 3.1, it should be clear that the desired results can be obtained by simply setting $\alpha_s = 0$ in eqs. (4.3):

$$\begin{aligned} \frac{\langle \text{Tr}([\Theta_g]_R^{\text{RGltr}}) \rangle_{\text{P}}}{2m_{\text{P}}^2} &= 1.22222 - 1.22222 \frac{\langle [O_m]_R \rangle_{\text{P}}}{2m_{\text{P}}^2} = 1.11, \\ \frac{\langle \text{Tr}([\Theta_q]_R^{\text{RGltr}}) \rangle_{\text{P}}}{2m_{\text{P}}^2} &= -0.222222 + 1.22222 \frac{\langle [O_m]_R \rangle_{\text{P}}}{2m_{\text{P}}^2} = -0.11. \end{aligned} \tag{4.7}$$

These numbers are, expectedly, very close to those listed in eqs. (4.3) for the central scale choice. Their main advantage is that they are scale independent and have the same value even in the non-perturbative regime of the strong coupling constant.

Finally, let us note that if the quark and gluon EMTs were not defined through renormalised operators in eqs. (3.2), their traces would nevertheless be finite and, of course, RG invariant. For reference, we also list the corresponding values:

$$\begin{aligned} \frac{\langle \text{Tr}([\Theta_g]_B) \rangle_{\text{P}}}{2m_{\text{P}}^2} &= 1 - \frac{\langle [O_m]_R \rangle_{\text{P}}}{2m_{\text{P}}^2} = 0.908, \\ \frac{\langle \text{Tr}([\Theta_q]_B) \rangle_{\text{P}}}{2m_{\text{P}}^2} &= \frac{\langle [O_m]_R \rangle_{\text{P}}}{2m_{\text{P}}^2} = 0.092. \end{aligned} \tag{4.8}$$

These numbers correspond to the classical EMT trace assigned entirely to the quark EMT, as in Ji's decomposition scheme [14], see also ref. [27]. Furthermore, they are identical to those obtained with $k = 1$ in eq. (3.14), i.e. with the D2 scheme of ref. [22]. As far as the gluon contribution to the proton mass is concerned, the difference between (4.7) and (4.8) is about 20%. However, there is a qualitative difference as far as the quark contribution is concerned, because of the sign difference between (4.7) and (4.8). (4.7) implies that quarks reduce the mass of the proton, while (4.8) implies that quarks increase the mass of the proton.

Let us now turn to a pion state $|\pi\rangle$ with mass m_π . We base our discussion on ref. [37], where chiral perturbation theory has been used to derive the following approximation:

$$\langle [O_m]_R \rangle_\pi = m_\pi^2. \tag{4.9}$$

In this case eq. (3.7) turns into:

$$\begin{aligned} \frac{\langle \text{Tr}([\Theta_g]_R^{\overline{\text{MS}}}) \rangle_\pi}{2m_\pi^2} &= (1 + 2\gamma_m) \frac{x_1^{\overline{\text{MS}}}}{\beta} + \frac{y_1^{\overline{\text{MS}}}}{2} \\ &= 0.611111 - 0.122150 \alpha_s - 0.124659 \alpha_s^2 - 0.099160 \alpha_s^3, \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \text{Tr}([\Theta_q]_R^{\overline{\text{MS}}}) \rangle_\pi}{2m_\pi^2} &= 1 - \frac{\langle \text{Tr}([\Theta_g]_R^{\overline{\text{MS}}}) \rangle_\pi}{2m_\pi^2} \\
 &= 0.388889 + 0.122150 \alpha_s + 0.124659 \alpha_s^2 + 0.099160 \alpha_s^3. \quad (4.10)
 \end{aligned}$$

Eqs. (4.10) agree with the results presented in ref. [37] only up to $\mathcal{O}(\alpha_s^2)$. The discrepancy at $\mathcal{O}(\alpha_s^3)$ is due to the presence of $1/\beta$ in the coefficient of x_1 . Indeed, x_1 is therefore needed to $\mathcal{O}(\alpha_s^4)$. At the time of writing of ref. [37] this coefficient could not be determined because of the missing terms in the expansions of the anomalous dimensions of the operators $\mathcal{O}_{1,3}$.

We now finally note that at the scale of the pion mass, the strong coupling constant is non-perturbative. In consequence, eqs. (4.10) are not particularly useful. On the other hand, in the case of RG-invariant EMT traces, there is:

$$\begin{aligned}
 \frac{\langle \text{Tr}([\Theta_g]_R^{\text{RGltr}}) \rangle_\pi}{2m_\pi^2} &= \left. \frac{x_1^{\overline{\text{MS}}}}{\beta} \right|_{\alpha_s=0} = \frac{k}{2} = \frac{\beta_0|_{n_f=0}}{2\beta_0} = 0.611111, \\
 \frac{\langle \text{Tr}([\Theta_q]_R^{\text{RGltr}}) \rangle_\pi}{2m_\pi^2} &= 1 - \frac{\langle \text{Tr}([\Theta_g]_R^{\text{RGltr}}) \rangle_\pi}{2m_\pi^2} = 0.388889, \quad (4.11)
 \end{aligned}$$

which provides a decomposition of the mass of the pion into quark and gluon components even in the non-perturbative regime. Of course, the exact physical meaning of these numbers without guidance from perturbation theory may still be disputed due to all the ambiguities that we have discussed.

5 Conclusions

In this publication, we have provided a general construction of quark and gluon EMTs with RG-invariant traces. The only two constraints that we have assumed are: 1) $\overline{\text{MS}}$ -renormalisation of the occurring irreducible rank-two tensor operators, 2) equality of the renormalised and the bare total EMT. An alternative construction to the one presented here consists of renormalising all physical operators occurring in the definition of the quark and gluon EMTs, i.e. not only the irreducible rank-two tensor operators but also scalar operators, in the $\overline{\text{MS}}$ scheme, followed by the addition of finite contributions proportional to the two renormalised scalar operators. The two approaches are equivalent by eq. (3.3).

A possible generalisation of our study would consist in allowing for finite contributions to the quark and gluon EMTs proportional to the two renormalised irreducible rank-two tensor operators. This would not change the analysis of the traces of the EMTs, since irreducible tensor operators are traceless. On the other hand, the additional freedom would allow constructing quark and gluon EMTs that are RG invariant themselves rather than only after taking the trace. While such a construction is certainly interesting, it is irrelevant to the physical motivation behind the study of separate quark and gluon EMTs, namely the understanding of the origin of mass of strongly interacting particles.

Our analysis has shown that the finite coefficients entering the renormalisation of the scalar operators are strongly constrained. Expressing the traces of the EMTs in terms of RG-invariant operators, we have obtained RG equations that these coefficients must satisfy.

These equations still allow for rather general solutions. Hence, we have discussed the case where they do not explicitly depend on the renormalisation scale. This case covers the majority of definitions of quark and gluon EMTs presented in the literature and leaves the freedom to choose a single numeric parameter. We have proposed a value for this parameter that corresponds to a minimal modification of quark and gluon EMTs defined in the $\overline{\text{MS}}$ scheme. The corresponding operator-renormalisation scheme removes the perturbative correction terms in eq. (4.3), which are consequences of the $\overline{\text{MS}}$ renormalisation of the quark and gluon trace operators. It nevertheless coincides with $\overline{\text{MS}}$ renormalisation at leading order. This feature is desirable in applications to mesons whose mass scales are in the non-perturbative regime of the strong coupling constant. As one possible example where the physical analysis may benefit from our operator-renormalisation scheme, one may consider the so-called nucleon's twist-four gravitational form factors in the forward limit, see e.g. refs. [18, 20, 27, 37, 60–62]. It is straightforward to prove that the anomalous dimensions in the homogeneous part of the RG equation for these renormalised objects defined in our scheme will be simply those in eq. (2.32). The latter are known to high orders in perturbation theory from the literature. We hope to return to this point in the future.

Finally, we have updated previous theoretical results for the $\overline{\text{MS}}$ -renormalised quark and gluon contributions to the proton and pion mass induced by the respective EMTs to achieve four-loop accuracy.

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References

- [1] R.J. Crewther, *Nonperturbative evaluation of the anomalies in low-energy theorems*, *Phys. Rev. Lett.* **28** (1972) 1421 [[INSPIRE](#)].
- [2] M.S. Chanowitz and J.R. Ellis, *Canonical Anomalies and Broken Scale Invariance*, *Phys. Lett. B* **40** (1972) 397 [[INSPIRE](#)].
- [3] M.S. Chanowitz and J.R. Ellis, *Canonical Trace Anomalies*, *Phys. Rev. D* **7** (1973) 2490 [[INSPIRE](#)].

- [4] C.G. Callan Jr., *Broken scale invariance in scalar field theory*, *Phys. Rev. D* **2** (1970) 1541 [[INSPIRE](#)].
- [5] K. Symanzik, *Small distance behavior in field theory and power counting*, *Commun. Math. Phys.* **18** (1970) 227 [[INSPIRE](#)].
- [6] S.R. Coleman and R. Jackiw, *Why dilatation generators do not generate dilatations?*, *Annals Phys.* **67** (1971) 552 [[INSPIRE](#)].
- [7] D.Z. Freedman, I.J. Muzinich and E.J. Weinberg, *On the Energy-Momentum Tensor in Gauge Field Theories*, *Annals Phys.* **87** (1974) 95 [[INSPIRE](#)].
- [8] D.Z. Freedman and E.J. Weinberg, *The Energy-Momentum Tensor in Scalar and Gauge Field Theories*, *Annals Phys.* **87** (1974) 354 [[INSPIRE](#)].
- [9] S.L. Adler, J.C. Collins and A. Duncan, *Energy-Momentum-Tensor Trace Anomaly in Spin 1/2 Quantum Electrodynamics*, *Phys. Rev. D* **15** (1977) 1712 [[INSPIRE](#)].
- [10] J.C. Collins, A. Duncan and S.D. Joglekar, *Trace and Dilatation Anomalies in Gauge Theories*, *Phys. Rev. D* **16** (1977) 438 [[INSPIRE](#)].
- [11] N.K. Nielsen, *The Energy Momentum Tensor in a Nonabelian Quark Gluon Theory*, *Nucl. Phys. B* **120** (1977) 212 [[INSPIRE](#)].
- [12] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Remarks on Higgs Boson Interactions with Nucleons*, *Phys. Lett. B* **78** (1978) 443 [[INSPIRE](#)].
- [13] V.P. Spiridonov, *Anomalous Dimension of $G_{\mu\nu}^2$ and β Function*, IYaI-P-0378 (1984) [[INSPIRE](#)].
- [14] X.-D. Ji, *A QCD analysis of the mass structure of the nucleon*, *Phys. Rev. Lett.* **74** (1995) 1071 [[hep-ph/9410274](#)] [[INSPIRE](#)].
- [15] X.-D. Ji, *Breakup of hadron masses and energy-momentum tensor of QCD*, *Phys. Rev. D* **52** (1995) 271 [[hep-ph/9502213](#)] [[INSPIRE](#)].
- [16] C. Lorcé, *The light-front gauge-invariant energy-momentum tensor*, *JHEP* **08** (2015) 045 [[arXiv:1502.06656](#)] [[INSPIRE](#)].
- [17] C.D. Roberts, *Perspective on the origin of hadron masses*, *Few Body Syst.* **58** (2017) 5 [[arXiv:1606.03909](#)] [[INSPIRE](#)].
- [18] C. Lorcé, *On the hadron mass decomposition*, *Eur. Phys. J. C* **78** (2018) 120 [[arXiv:1706.05853](#)] [[INSPIRE](#)].
- [19] Y. Hatta and D.-L. Yang, *Holographic J/ψ production near threshold and the proton mass problem*, *Phys. Rev. D* **98** (2018) 074003 [[arXiv:1808.02163](#)] [[INSPIRE](#)].
- [20] Y. Hatta, A. Rajan and K. Tanaka, *Quark and gluon contributions to the QCD trace anomaly*, *JHEP* **12** (2018) 008 [[arXiv:1810.05116](#)] [[INSPIRE](#)].
- [21] S. Rodini, A. Metz and B. Pasquini, *Mass sum rules of the electron in quantum electrodynamics*, *JHEP* **09** (2020) 067 [[arXiv:2004.03704](#)] [[INSPIRE](#)].
- [22] A. Metz, B. Pasquini and S. Rodini, *Revisiting the proton mass decomposition*, *Phys. Rev. D* **102** (2020) 114042 [[arXiv:2006.11171](#)] [[INSPIRE](#)].
- [23] C. Lorcé, A. Metz, B. Pasquini and S. Rodini, *Energy-momentum tensor in QCD: nucleon mass decomposition and mechanical equilibrium*, *JHEP* **11** (2021) 121 [[arXiv:2109.11785](#)] [[INSPIRE](#)].

- [24] X. Ji, *Proton mass decomposition: naturalness and interpretations*, *Front. Phys. (Beijing)* **16** (2021) 64601 [[arXiv:2102.07830](#)] [[INSPIRE](#)].
- [25] D.E. Kharzeev, *Mass radius of the proton*, *Phys. Rev. D* **104** (2021) 054015 [[arXiv:2102.00110](#)] [[INSPIRE](#)].
- [26] Y.-B. Yang et al., *Proton Mass Decomposition from the QCD Energy Momentum Tensor*, *Phys. Rev. Lett.* **121** (2018) 212001 [[arXiv:1808.08677](#)] [[INSPIRE](#)].
- [27] K.-F. Liu, *Proton mass decomposition and hadron cosmological constant*, *Phys. Rev. D* **104** (2021) 076010 [[arXiv:2103.15768](#)] [[INSPIRE](#)].
- [28] X. Ji and Y. Liu, *Quantum anomalous energy effects on the nucleon mass*, *Sci. China Phys. Mech. Astron.* **64** (2021) 281012 [[arXiv:2101.04483](#)] [[INSPIRE](#)].
- [29] B.-d. Sun, Z.-h. Sun and J. Zhou, *Trace anomaly contribution to hydrogen atom mass*, *Phys. Rev. D* **104** (2021) 056008 [[arXiv:2012.09443](#)] [[INSPIRE](#)].
- [30] χ QCD collaboration, *Demonstration of the hadron mass origin from the QCD trace anomaly*, *Phys. Rev. D* **104** (2021) 074507 [[arXiv:2101.04942](#)] [[INSPIRE](#)].
- [31] J. Dudek et al., *Physics Opportunities with the 12 GeV Upgrade at Jefferson Lab*, *Eur. Phys. J. A* **48** (2012) 187 [[arXiv:1208.1244](#)] [[INSPIRE](#)].
- [32] A. Accardi et al., *Electron Ion Collider: The Next QCD Frontier: Understanding the glue that binds us all*, *Eur. Phys. J. A* **52** (2016) 268 [[arXiv:1212.1701](#)] [[INSPIRE](#)].
- [33] D. Kharzeev, *Quarkonium interactions in QCD*, *Proc. Int. Sch. Phys. Fermi* **130** (1996) 105 [[nucl-th/9601029](#)] [[INSPIRE](#)].
- [34] D. Kharzeev, H. Satz, A. Syamtomov and G. Zinovjev, *J/ψ photoproduction and the gluon structure of the nucleon*, *Eur. Phys. J. C* **9** (1999) 459 [[hep-ph/9901375](#)] [[INSPIRE](#)].
- [35] D.P. Anderle et al., *Electron-ion collider in China*, *Front. Phys. (Beijing)* **16** (2021) 64701 [[arXiv:2102.09222](#)] [[INSPIRE](#)].
- [36] H. Makino and H. Suzuki, *Lattice energy-momentum tensor from the Yang-Mills gradient flow — inclusion of fermion fields*, *PTEP* **2014** (2014) 063B02 [[arXiv:1403.4772](#)] [[INSPIRE](#)].
- [37] K. Tanaka, *Three-loop formula for quark and gluon contributions to the QCD trace anomaly*, *JHEP* **01** (2019) 120 [[arXiv:1811.07879](#)] [[INSPIRE](#)].
- [38] F.J. Belinfante, *On the spin angular momentum of mesons*, *Physica* **6** (1939) 887.
- [39] F.J. Belinfante, *On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields*, *Physica* **7** (1940) 449.
- [40] L. Rosenfeld, *On the energy-momentum tensor*, *Mém. Acad. Roy. Belg.* **18** (1940) 1.
- [41] M.F. Zoller and K.G. Chetyrkin, *OPE of the energy-momentum tensor correlator in massless QCD*, *JHEP* **12** (2012) 119 [[arXiv:1209.1516](#)] [[INSPIRE](#)].
- [42] N.K. Nielsen, *Gauge Invariance and Broken Conformal Symmetry*, *Nucl. Phys. B* **97** (1975) 527 [[INSPIRE](#)].
- [43] R. Tarrach, *The renormalization of FF*, *Nucl. Phys. B* **196** (1982) 45 [[INSPIRE](#)].
- [44] J.C. Collins and R.J. Scalise, *The Renormalization of composite operators in Yang-Mills theories using general covariant gauge*, *Phys. Rev. D* **50** (1994) 4117 [[hep-ph/9403231](#)] [[INSPIRE](#)].

- [45] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*, *JHEP* **10** (2017) 041 [[arXiv:1707.08315](#)] [[INSPIRE](#)].
- [46] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Low moments of the four-loop splitting functions in QCD*, *Phys. Lett. B* **825** (2022) 136853 [[arXiv:2111.15561](#)] [[INSPIRE](#)].
- [47] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, *The Four loop beta function in quantum chromodynamics*, *Phys. Lett. B* **400** (1997) 379 [[hep-ph/9701390](#)] [[INSPIRE](#)].
- [48] M. Czakon, *The Four-loop QCD beta-function and anomalous dimensions*, *Nucl. Phys. B* **710** (2005) 485 [[hep-ph/0411261](#)] [[INSPIRE](#)].
- [49] K.G. Chetyrkin, *Quark mass anomalous dimension to $O(\alpha_s^4)$* , *Phys. Lett. B* **404** (1997) 161 [[hep-ph/9703278](#)] [[INSPIRE](#)].
- [50] J.A.M. Vermaseren, S.A. Larin and T. van Ritbergen, *The four loop quark mass anomalous dimension and the invariant quark mass*, *Phys. Lett. B* **405** (1997) 327 [[hep-ph/9703284](#)] [[INSPIRE](#)].
- [51] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Five-Loop Running of the QCD coupling constant*, *Phys. Rev. Lett.* **118** (2017) 082002 [[arXiv:1606.08659](#)] [[INSPIRE](#)].
- [52] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *The five-loop beta function of Yang-Mills theory with fermions*, *JHEP* **02** (2017) 090 [[arXiv:1701.01404](#)] [[INSPIRE](#)].
- [53] T. Luthe, A. Maier, P. Marquard and Y. Schröder, *The five-loop Beta function for a general gauge group and anomalous dimensions beyond Feynman gauge*, *JHEP* **10** (2017) 166 [[arXiv:1709.07718](#)] [[INSPIRE](#)].
- [54] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Quark Mass and Field Anomalous Dimensions to $O(\alpha_s^5)$* , *JHEP* **10** (2014) 076 [[arXiv:1402.6611](#)] [[INSPIRE](#)].
- [55] T. Luthe, A. Maier, P. Marquard and Y. Schröder, *Five-loop quark mass and field anomalous dimensions for a general gauge group*, *JHEP* **01** (2017) 081 [[arXiv:1612.05512](#)] [[INSPIRE](#)].
- [56] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Five-loop fermion anomalous dimension for a general gauge group from four-loop massless propagators*, *JHEP* **04** (2017) 119 [[arXiv:1702.01458](#)] [[INSPIRE](#)].
- [57] xQCD collaboration, *πN and strangeness sigma terms at the physical point with chiral fermions*, *Phys. Rev. D* **94** (2016) 054503 [[arXiv:1511.09089](#)] [[INSPIRE](#)].
- [58] xQCD collaboration, *Strangeness and charmness content of the nucleon from overlap fermions on 2 + 1-flavor domain-wall fermion configurations*, *Phys. Rev. D* **88** (2013) 014503 [[arXiv:1304.1194](#)] [[INSPIRE](#)].
- [59] K.G. Chetyrkin, J.H. Kühn and M. Steinhauser, *RunDec: A Mathematica package for running and decoupling of the strong coupling and quark masses*, *Comput. Phys. Commun.* **133** (2000) 43 [[hep-ph/0004189](#)] [[INSPIRE](#)].
- [60] X.-D. Ji, *Gauge-Invariant Decomposition of Nucleon Spin*, *Phys. Rev. Lett.* **78** (1997) 610 [[hep-ph/9603249](#)] [[INSPIRE](#)].
- [61] M.V. Polyakov and P. Schweitzer, *Forces inside hadrons: pressure, surface tension, mechanical radius, and all that*, *Int. J. Mod. Phys. A* **33** (2018) 1830025 [[arXiv:1805.06596](#)] [[INSPIRE](#)].
- [62] K. Tanaka, *Operator relations for gravitational form factors of a spin-0 hadron*, *Phys. Rev. D* **98** (2018) 034009 [[arXiv:1806.10591](#)] [[INSPIRE](#)].