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A NOTE ON QUASI AND BI-IDEALS IN TERNARY SEMIGROUPS

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ABSTRACT. In this paper we have studied the properties of Quasi-ideals and Bi-ideals in ternary semi groups. We prove that every quasi-ideal is a bi-ideal in T but the converse is not true in general by giving several example in different context .

KEY WORDS AND PHRASES. Quasi-ideal, Bi-ideal, Ternary Semi group.
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1. INTRODUCTION.

D.H. Lehmer [4] gave the definition of a ternary semi group as follows:

DEFINITION 1.1. A non-empty set T is called a ternary semigroup if a ternary operation [ ] on T is defined and satisfies the associative law

$$[[x_1\ x_2\ x_3]x_4\ x_5] = [x_1[x_2\ x_3\ x_4]x_5] = [x_1\ x_2[x_3\ x_4\ x_5]]$$

for all $x_i \in T$, $1 \leq i \leq 5$.

Banach showed by an example that a ternary semi group does not necessarily reduce to an ordinary semi group. This has been shown by the following example.

EXAMPLE 1.2. Let T = $\{-i,o,i\}$ be a ternary semi group under the multiplication over complex number while T is not a binary semi group under the multiplication over complex number.

Los [5] showed that any ternary semi group however may be embedded in an ordinary semi group in such a way that the operation in the ternary semi group is an (ternary) extension of the (binary) operation of the containing semi group.

Dudek [1], Feizullaer [2], Kim and Roush [3], Lyapin [6] and Sioson [7] has also studied the properties of the ternary semi groups.

We give the following definitions of ideals [7] as follows:

DEFINITION 1.3. A left (right, lateral) ideal of a ternary semi group T is a non-empty subset L(R,M) of T such that

$$[TTL] \subseteq L([RTT] \subseteq R,\ [TMT] \subseteq M)$$

DEFINITION 1.4. If a non-empty subset of T is a left, right and lateral ideal of T, then it is called an ideal of T.

DEFINITION 1.5. For each element t in T, the left, right and lateral ideal generated by 't' are respectively given by:

$$(t)_L = \{t\} \cup [TTt]$$

$$(t)_R = \{t\} \cup [tTT]$$

$$(t)_M = \{t\} \cup [TtT] \cup [TTtTT]$$

Due to associative law in T, one may write Sioson [7]

$$[x_1\ x_2 \ldots x_{2n+1}] = [x_1 \ldots x_m x_{m+1} \ldots x_{m+4} \ldots x_{2n+1}],\ m \leq n$$

$$= [x_1 \ldots [[x_m x_{m+1} x_{m+2}] x_{m+3}\ x_{m+4}] \ldots x_{2n+1}],\ m \leq n$$

DEFINITION 1.6. Quasi-ideal in a ternary semi group [7] is also a subset Q of T (possibly empty) satisfying following two conditions:

(1) $[QTT] \cap [TQT] \cap [TTQ] \subseteq Q$

(2) $[QTT] \cap [TTQTT] \cap [TTQ] \subseteq Q$

REMARK 1.7. Every right, left and lateral ideal is a quasi-ideal. But every quasi-ideal is not a right, a left and a lateral ideal of T. This follows from the following example

EXAMPLE 1.8. Let T = $\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ be the ternary semi group under matrix multiplication. Then Q = $\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}$ be the quasi-ideal of T, which is neither a left, nor a right nor a lateral ideal of T.

DEFINITION 1.9. A ternary sub semi group is a subset S of a ternary semi group T such that

$$[SSS] \subseteq S$$

DEFINITION 1.10. A ternary semi group T is said to be a ternary group if it satisfies the following property that for all x,y and z in T, there exists unique a,b,c in T such that

$$[xab] = c,\ [ayb] = c,\ [abz] = c$$

DEFINITION 1.11. A ternary group T is said to be a ternary group with 0 if for all a,b,c in T

$$[oab] = 0 = [aob] = [abo] = [aoo] = [obo] = [ooc].$$

DEFINITION 1.12. A ternary semi group T is with identity if there exists an idempotent e in T such that

$$[aae] = [eaa] = [aea] = a, \quad \forall a \in T.$$

2. SOME RESULTS ON QUASI-IDEAL IN T WHICH ARE TRIVIALLY TRUE

PROPOSITION 2.1. A ternary group T with 0 and $[TTT] \neq 0$ has no proper quasi-ideal.

PROPOSITION 2.2. The intersection of a quasi-ideal Q and a ternary sub semi-group A of a ternary semi group T is either empty or a quasi-ideal of A.

PROPOSITION 2.3. Let Q be any non-empty subset of aternary semi group T, then the following are true:

(1) $Q \cup [TTQ]$ is the smallest left ideal of T containing Q.

(2) $Q \cup [QTT]$ is the smallest right ideal of T containing Q.

(3) $Q \cup [TQT] \cup [TTQTT]$ is the smallest lateral ideal of T containing Q.

(4) If Q is a quasi-ideal of T. Then

$$Q = (Q \cup [TTQ]) \cap (Q \cup [TQT] \cup [TTQTT]) \cap (Q \cup [QTT]).$$

PROPOSITION 2.4. The intersection of arbitrary set of quasi-ideals in a ternary semi group is either empty or a quasi-ideal of T.

DEFINITION 2.5. Let X be a non-empty subset of a ternary semi group T. The quasi-ideal of T generated by X is intersection of all quasi-ideals $(X)_q$ of T containing X.

If the subset X consists of a single element X, then $(X)_q$ is the cyclic quasi-ideal of T.

PROPOSITION 2.6. Let X be a non-empty subset of ternary semi group T, then

$$(X)_q = (X \cup [TTX]) \cap (X \cup [TXT] \cup [TTXTT]) \cap (X \cup [XTT])$$

is the smallest quasi-ideal containing X.

PROOF. Sioson [7] shows that the intersection of a right, a left and a lateral ideal of a ternary semi group T is a quasi-ideal. Therefore the proof easily follows by using 2.3.

From 2.6 it follows that

$$(X)_q = (\{X\} \cup [TTX]) \cap (\{X\} \cup [TXT] \cup [TTXTT]) \cap (\{X\} \cup [XTT])$$

is the smallest quasi-ideal of T containing X.

3. BI-IDEALS IN TERNARY SEMI GROUP

DEFINITION 3.1. A ternary sub semi group B of a ternary semi group T is a bi-ideal of T if $[BTBTB] \subseteq B$.

PROPOSITION 3.2. Every quasi-ideal of a ternary semi group T is a bi-ideal.

PROOF. Let Q be a quasi-ideal of T. Then Q is a ternary semi group of T.
Now $[QTQTQ] \subseteq [Q[TTT]T] \subseteq [QTT]$.
Similarly $[QTQTQ] \subseteq [TTQ] \cap [TTQTT]$.
Therefore $[QTQTQ] \subseteq [TTQ] \cap [TTQTT] \cap [QTT] \subseteq Q$.

PROPOSITION 3.3. Let A be an ideal and Q be a quasi-ideal of T. Then $A \cap Q$ is a bi-ideal and a quasi-ideal of T.

PROOF. $[A \cap Q \quad A \cap Q \quad A \cap Q] \subseteq [AAA] \cap [QQQ] \subseteq A \cap Q$ implies that $A \cap Q$ is a ternary sub semi group of T. Also

$$[A \cap Q\ T\ A \cap Q\ T\ A \cap Q] \subseteq [QTQTQ] \cap [A[TAT]A] \subseteq Q \cap [AAA]$$

by (3.2) and the given hypothesis implies that L.H.S. $\subseteq Q \cap A$. Thus $A \cap Q$ is a bi-ideal of T. Since A is an ideal of T and it is also a quasi-ideal of T. Hence $A \cap Q$ is a quasi-ideal of T.

PROPOSITION 3.4. Let X, Y be non-empty subsets of ternary semi group T, then $N = [XTY]$ is a bi-ideal of T.

PROOF. Clearly N is a ternary sub semi group of T. Also

$$[NTNTN] \subseteq [X[TTT][TTT][TTT]Y] \subseteq [X[TTT]Y]$$

$$\subseteq [XTY] = N.$$

Then N is a bi-ideal of T.

PROPOSITION 3.5. The intersection of arbitrary set of bi-ideals of T is either empty or a bi-ideal of T.

We omit the trivial proof.

PROPOSITION 3.6. Every left, right or lateral ideal of T is a bi-ideal of T.

PROOF. Trivial.

PROPOSITION 3.7. Let Q be a subset of a ternary semi group T and Y be a non-empty proper subset of T such that

(1) $[TTQ] \cup [TQT] \cup [QTT] \cup [TTQTT] \subseteq Y$.

(2) $Y \subseteq Q$.

Then Y is an ideal of T. Moreover Y is a bi-ideal of T.

PROOF. It is obvious that $[TTY]$, $[TYT]$. $[YTT]$ and $[TTYTT]$ are contained in Y under the condition (2) therefore Y is an ideal of T. And hence a quasi-ideal of T which by 3.2 is a bi-ideal of T.

In the following example we show that if both or either of the conditions (1) and (2) of above proposition are not satisfied then Y is neither a left, a right, a lateral, a quasi nor a bi-ideal of T.

EXAMPLE 3.8. Let $T = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$.
Then T is a ternary semi group under matrix multiplication.

(1) Take $Y = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \}$ and the quasi-ideal

$$Q = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \} \text{ of } T.$$

We see that $Y \nsubseteq Q$, and

$$[TTQ] \cup [TQT] \cup [TTQTT] \cup [QTT]$$
$$= \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$$
$$\nsubseteq Y$$

Also each of $[TTY]$, $[TYT]$, $[YTT]$ and $[TTYTT]$ is not in Y.
Therefore Y is neither a left, nor a right nor a lateral ideal of T.
Moreover $[TTY] \cap [TYT] \cap [YTT] \nsubseteq Y$.
So, Y is not a quasi-ideal of T.

(2) Take $Y = \{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \}$ and $Q = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \}$. Then $Y \subseteq Q$.
Again $[TTQ] \cup [TQT] \cup [QTT] \cup [TTQTT] \nsubseteq Y$.
Since each of $[TTY]$, $[TYT]$, $[YTT]$ contains $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, they are not contained in Y.
Hence Y is neither a left, a lateral nor a right ideal of T.
Also $[TTY] \cap [TYT] \cap [YTT] \nsubseteq Y$.
So Y is not a quasi-ideal of T.
Further $[YTYTY] \nsubseteq Y$ implies Y is not a bi-ideal of T.

(3) Now we take $Y = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \}$ and $Q = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \}$ of T. Then $Y \nsubseteq Q$.

$$[TTQ] \cup [TQT] \cup [TTQTT] \cup [QTT] = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \} \subseteq Y.$$

We find that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in [TTY]$, $[TYT]$ and $[YTT]$.

But $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \notin Y$. So Y is either a left, a lateral nor a right ideal of T. Similarly Y is neither a quasi nor a bi-ideal of T.

THEOREM 3.9. Let X, Y and Z be three non-empty subsets of a ternary semigroup T and $N = [XYZ]$. Then N is a bi-ideal of T if one of the following conditions holds:

(1) $X, Y \subseteq Z$ and Z is a bi-ideal of T.
(2) $Y, Z \subseteq X$ and X is a bi-ideal of T.
(3) $X, Z \subseteq Y$ and Y is a bi-ideal of T.
(4) At least one of X, Y, Z is a right, or a left or a lateral ideal of T.

PROOF. (1) $[NNN] \subseteq [XYZ[ZZZ][ZZZ]]$

$$\subset [XY[ZZZ]] \subseteq N$$

and $[NTNTN] \subseteq [XY[ZTZTZ]] \subseteq N$.
Similar proofs establish (2) and (3).

(4) Asssume X is a right ideal of T. Then

$$[NNN] \subseteq [X[TTT][TTT]YZ] \subseteq [XTTYZ] \subseteq N$$

$$[NTNTN] \subseteq [X[TTT][TTT]TTYZ] \subseteq [XTTYZ] \subseteq N.$$

Similar proofs can be given when either X or Y or Z is a left, or a lateral or a right ideal of T.

DEFINITION 3.10 [7]. An element 't' in a ternary semi group T is said to be regular if there exists x,y in T such that

$$[txtyt] = t.$$

If all the elements of T are regular then it is said to be regular ternary semi group.

EXAMPLE 3.11. This example shows that there exists a ternary semi group while T is not a regular ternary semi group such that T has a minimal right, a minimal lateral and a minimal left ideal of T.

Let T = {0,e,a,b} be the ternary semi group under the operation ( ), (given below in the table)

( )	0	e	a	b
0	0	0	0	0
e	0	e	a	b
a	0	a	0	0
b	0	b	0	0

$\forall a,b,c \in T$, $[abc] = a(bc) = (ab)c$.

Hence {0} is a minimal right, a minimal left and a minimal lateral ideal of T. Since a and b are not the regular elements of T. Therefore T is not a regular ternary semi group.

Now we use theorem 3.9 to give an example of a ternary semi group in which a bi-ideal is not a quasi-ideal.

EXAMPLE 3.12. Let T be a ternary semi group such that T is not regular, X,Y,Z be respectively a minimal right, a minimal lateral and a minimal left ideal of T satisfying the condition of 3.9. Thus N = [XYZ] is a bi-ideal of T. We will show that N is not a quasi-ideal of T.

PROOF. $[XYZ] \subseteq [XTT] \subseteq X$, $[XYZ] \subseteq Y$, $[XYZ] \subseteq Z$. So, $[XYZ] \subset X \cap Y \cap Z$ which is a minimal quasi-ideal of T [7].

If we assume that [XYZ] is a quasi-ideal then $[XYZ] = X \cap Y \cap Z$ which (by Sioson [7]) thus implies that T is a regular ternary semi group. Hence it contradicts the hypothesis. So [XYZ] is not a quasi-ideal but bi-ideal by Theorem 3.9.

PROPOSITION 3.13. In a regular ternary semi group every bi-ideal is a quasi-ideal.

PROOF. Sioson [7] shows that a subset Q of a regular ternary semi group T is a quasi-ideal if and only if

$$[QTQTQ] \cap [QTTQTTQ] \subseteq Q.$$

Since a bi-ideal of T, clearly satisfies the above condition, so we get the proof.

PROPOSITION 3.14. Let C be a non-empty subset of a ternary semi group T without identity. Then $C \cup [CCC] \cup [CTCTC]$ is the smallest bi-ideal of T containing C.

PROOF. Let x be any elemnt of $C \cup [CCC] \cup [CTCTC]$. Then either $x = x_1$ for x_1 in C or $x = [c_1\ c_2\ c_3] \in [CCC]$ for all c_i in C. $i = 1,2,3$ or $x = [c_1t_1c_2t_2c_3]$ $[CTCTC]$ for all c_i in C, $i = 1,2,3$, t_i in T, $i = 1,2$.

We will consider the elements of $[CTCTC]$. The other two cases will be done in similar manner. Let $x,y,z \in [CTCTC]$.
i.e., $x = [c_1t_1\ c_2t_2\ c_3]$, $y = [c_4t_3\ c_5t_4\ c_6]$. $z = [c_7t_5\ c_8t_6\ c_9]$, $c_i \in C$, $\forall i = 1,2,\ldots,9$, $t_i \in T$, $\forall i = 1,2,\ldots,6$.
Then

$$\begin{aligned}
[xyz] &= [[c_1t_1\ c_2t_2\ c_3][c_4t_3\ c_5t_4\ c_6][c_7t_5\ c_8t_6\ c_9]] \\
&= [c_1[[t_1c_2t_2][c_3c_4t_3][c_5t_4c_6]]c_7[t_5c_8t_6]c_9] \\
&= [c_1t_7c_7t_8c_9] \text{ where} \\
t_7 &= [[t_1c_2t_2][c_3c_4t_3][c_5t_4c_6]] \\
t_8 &= [t_5c_8t_6]
\end{aligned}$$

so $[xyz] \in C \cup [CCC] \cup [CTCTC]$.
Further,
$$\begin{aligned}
[xt_9yt_{10}z] &= [[c_1t_1c_2t_2c_3]t_9[c_4t_3c_5t_4c_6]t_{10}[c_7t_5c_8t_6c_0]] \\
&= [c_1[[t_1c_2t_2][c_3t_9c_4][t_3c_5t_4]]c_6[t_{10}c_7[t_5c_8t_6]]c_9] \\
&= [c_1\ t_{11}\ c_6\ t_{12}\ c_9]
\end{aligned}$$

where
$$\begin{aligned}
t_{11} &= [[t_1c_2t_2][c_3t_9c_4][t_3c_5t_4]] \\
t_{12} &= [t_{10}c_7[t_5t_8t_6]],\ t_9,\ t_{10} \in T.
\end{aligned}$$

Thus

$$[xt_9\ yt_{10}\ z] \in C \cup [CCC] \cup [CTCTC].$$

Hence $C \cup [CCC] \cup [CTCTC]$ is a bi-ideal of T containing C.
Suppose there exists a bi-ideal R of T containing C such that

$$R \subseteq C \cup [CCC] \cup [CTCTC].$$

Then R being a bi-ideal implies that

$$R \subseteq C \cup [CCC] \cup [CTCTC] \subseteq R \cup [RRR] \cup [RTRTR] \subseteq R.$$

Thus $R = C \cup [CCC] \cup [CTCTC]$ is the smallest bi-ideal of T contain ing C.

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