A note on recursively enumerable predicates in groups

by

Paul E. Schupp (Urbana, Illinois)

This note is prompted by a paper of A. Włodzimierz Mostowski [3]. It has benefited from discussions with C. F. Miller. Among the interesting results of Mostowski's paper it is shown that the conjugacy problem and the extended word problem are solvable for finitely generated nilpotent groups. The purpose of this note is to point out that the arguments of Mostowski can be stated in a unified and general manner. The possibility of applying this method in the general setting is suggestive.

Let $P(x_1, \ldots, x_n)$ be a predicate which is meaningful in groups, and which is recursively enumerable in a finitely presented group G; that is, the set of n-tuples of words representing elements of G which satisfy P is recursively enumerable. We do not assume that P is a first order predicate. The relevant P for the extended word problem is "x_n is in the subgroup generated by $x_1, \ldots, x_{n-1}$".

We also assume that P is preserved under homomorphism. If φ is a homomorphism of G into any group H, then $P(g_1, \ldots, g_n)$ implies $P(\varphi(g_1), \ldots, \varphi(g_n))$.

We use the symbol $\neg$ for negation.

Let Σ be a recursively enumerable sequence, $H_0, H_1, \ldots,$ of finitely presented groups. (Precisely, Σ is a recursively enumerable sequence of Gödel numbers of finite presentations of groups.) We say that G is *P-separable in Σ* if for any n-tuple $(g_1, \ldots, g_n)$ of elements of G such that $\neg P(g_1, \ldots, g_n)$, there exists a homomorphism φ of G into some H_i in Σ such that $\neg P(\varphi(g_1), \ldots, \varphi) g_n))$ holds in H_i.

THEOREM. *Let G be P-separable in Σ. If there are uniform algorithms for deciding the predicate P in the H_i and the word problem in the H_i, then the predicate P is decidable in G.*

Proof. P is recursively enumerable in G by assumption. The idea is to prove that $\neg P$ is also recursively enumerable in G. The decidability of P in G then follows.

Start enumerating the groups in Σ. We claim that we can effectively enumerate all homomorphisms of G into the groups H_i. Suppose G has presentation $\langle a_1, \ldots, a_k;\ r_1, \ldots, r_s\rangle$.

A homomorphism is completely determined by its effect on $a_1, \ldots, a_k$. A candidate for a homomorphism into H_i is a k-tuple of elements of H_i. Since we have a uniform algorithm for solving the word problem of the H_i, we can, given an index i and a k-tuple $(h_1, \ldots, h_k)$ of elements of H_i, check if the mapping $a_j \to h_j$, $j = 1, \ldots, k$, defines a homomorphism of G into H_i. To do this, simply check if each relator of G is sent to the identity under the proposed mapping. Label the homomorphisms enumerated $\varphi_1, \varphi_2, \ldots$

We also begin enumerating n-tuples of elements of G. Label these $(g_{11}, \ldots, g_{1n})$, etc. Let $\varphi_j: G \to H_{i_j}$. We can, by hypothesis, decide whether or not $P(\varphi_j(g_{l1}), \ldots, \varphi_j(g_{ln}))$ holds in H_{i_j}.

If $\neg P(\varphi_j(g_{l1}), \ldots, \varphi_j(g_{ln}))$ holds in H_{i_j}, we know that $\neg P(g_{l1}, \ldots, g_{ln})$ holds in G since P is preserved by homomorphisms. We then enumerate $(g_{l1}, \ldots, g_{ln})$ as an n-tuple of G for which $\neg P$ holds. The hypothesis that G is P-separable in Σ ensures that we can enumerate *all* n-tuples of G for which $\neg P$ holds. Hence $\neg P$ is recursively enumerable in G.

If $P(x)$ is "$x = 1$" and G is P-separable in Σ, we say that G is *residually imbeddable* in Σ. If Σ is a recursively enumerable sequence of presentations of all finite groups, we have the case where G is residually finite. It is interesting to note that we may take for Σ a recursive enumeration of all finitely presented one-relator groups. By the theorem of Magnus, there is a uniform algorithm for solving the word problem for the groups in Σ. Hence, if G is a finitely presented group that is residually imbeddable in a one relator group, then G itself has solvable word problem. We could also take Σ to be a recursive enumeration of presentations of groups which have "small cancellation among the relators" for which Lyndon has solved the word problem. (Cf. Lyndon [2]) (Admittedly, the hope of verifying one of these conditions for a given G is quite dim.)

New classes of groups for which various problem are decidable are being found. If $P(x, y)$ is "x is conjugate to y" we say, following Mostowski, that G is conjugacy separable in Σ. Garside [1] has solved the conjugacy problem for braid groups. It might be that some groups arising in a topological context would be conjugacy separable in the braid groups and thus have solvable conjugacy problem.

It is perhaps worthwhile to make two comments on the theorem. In most cases of interest, a decision procedure for P solves the word problem. C. F. Miller has pointed out that the separate assumption of a uniform algorithm for deciding the word problem in the H_i is unnecessary. It is not dificult to see that the homomorphisms of G into the H_i are recursively enumerable without assuming a solution to the word problem in the H_i.

We have stated the theorem in terms of absolute computability. Everything may be taken to be computable relative to some oracle.

References

[1] F. A. Garside, *The theory of knots and associated problems*, Thesis, Oxford University, 1965.

[2] R. C. Lyndon, *On Dehn's Algorithm*, Math. Ann. 166, (1966), pp. 208-228.

[3] A. W. Mostowski, *On the decidability of some problems in special classes of groups*, Fund. Math. 59 (1966), pp. 123-135.

UNIVERSITY OF ILLINOIS
Urbana, Illinois

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