A NOTE ON REDUCED JORDAN ALGEBRAS

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In this paper we give short proofs of two of the main theorems concerning reduced exceptional Jordan algebras $A = H(C_3, \gamma)$: the Albert-Jacobson Theorem that the Cayley coordinate algebra C is determined by A up to isomorphism, and the Springer Theorem that two such algebras A, A' are isomorphic if and only if they have isomorphic coordinate algebras and equivalent trace forms. We avoid using the generic norm by working directly with the reduced idempotents. Our proofs do not require that the algebras be exceptional, and are valid for arbitrary reduced simple algebras.

1. **Reduced idempotents.** Throughout this paper, A will denote a Jordan algebra with identity element 1 over a field Φ of characteristic $\neq 2$. In this section we make no assumptions about the simplicity or finite-dimensionality of A. Recall that an idempotent e is *reduced* if $A_1(e) = U_eA = \Phi e$. We assume the reader is familiar with the operators $U_x = 2L_x^2 - L_{x^2}$ and the basic identities involving them (e.g. [**3**, pp. 1072–1075]).

LEMMA 1. *Let e be a reduced idempotent in A, $x = x_1 + x_{1/2} + x_0$ (x_i in the Peirce space $A_i(e)$) an element with $U_ex = x_1 = \alpha e$, $U_ex^2 = \beta e$, $U_ex_{1/2}^2 = \gamma e$ for β, $\gamma \neq 0$. Then $U_xe = \beta g$, $U_{x_{1/2}}e = \gamma f$ where g, f are reduced idempotents with f orthogonal to e and g of the form $g = \rho e + y + (1-\rho)f$ with $\rho = \alpha^2\beta^{-1}$, $y = \alpha\beta^{-1}x_{1/2}$ an element in $A_{1/2}(e, f)$ satisfying $y^2 = \rho(1-\rho)(e+f)$ since $x_{1/2} \in A_{1/2}(e, f)$ has $x_{1/2}^2 = \gamma(e+f)$.*

PROOF. Since $U_eU_x(\beta g) = U_eU_{x^2}e = (U_ex^2)^2 = \beta^2e^2 \neq 0$, we see $g \neq 0$, and $\beta^2g^2 = (U_xe)^2 = U_xU_ex^2 = \beta U_xe = \beta^2 g$ shows g is an idempotent. It is reduced since $\beta^2U_gA = U_{U(x)e}A = U_xU_eU_xA \subset U_x(\Phi e) = \Phi g$. Repeating this argument for $x_{1/2}$, we get $U_{x_{1/2}}e = \gamma f$ for f a reduced idempotent, which is orthogonal to e since $U_{x_{1/2}}A_1(e) \subset A_0(e)$. Then $U_xe = \alpha^2U_ee + 2\alpha\{eex_{1/2}\} + U_{x_{1/2}}e = \alpha^2e + \alpha x_{1/2} + \gamma f = \beta\{\rho e + y + (1-\rho)f\}$ since $\gamma = \beta - \alpha^2$ follows from $\beta e = U_ex^2 = \alpha^2 e + U_ex_{1/2}^2 = (\alpha^2 + \gamma)e$. Since $x_{1/2}^2 = U_ex_{1/2}^2 + U_{1-e}x_{1/2}^2 = \gamma e + U_{x_{1/2}}e = \gamma(e+f)$ and $\rho(1-\rho) = \alpha^2\gamma\beta^{-2}$, we have $y^2 = \rho(1-\rho)(e+f)$. Since $2\gamma f \cdot x_{1/2} = 2x_{1/2} \cdot U_{x_{1/2}}e = 2\{x_{1/2}ex_{1/2}^2\} = 2\{x_{1/2}e\gamma e\} = \gamma x_{1/2}$, we have $x_{1/2} \in A_{1/2}(e) \cap A_{1/2}(f) = A_{1/2}(e, f)$.

The next lemma is a variant of a result of N. Jacobson [**2**, p. 82].

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If e is an idempotent, then $c=2e-1$ satisfies $c^2=1$, so U_c is an automorphism of A of period two. This automorphism will be called the *Peirce reflection* S_e determined by e; S_e acts as the identity I on the Peirce spaces $A_1(e)$ and $A_0(e)$, and as $-I$ on $A_{1/2}(e)$.

LEMMA 2. *Let e_1, e_2 be reduced idempotents in A. Then*

(a) *There is a $\rho\in\Phi$, called the projection coefficient of e_1 and e_2, such that $U_{e_1}e_2=\rho e_1$ and $U_{e_2}e_1=\rho e_2$.*

(b) *If $\rho=1$, then $e=\frac{1}{4}(e_1+e_2)^2$ is idempotent and the Peirce reflection S_e maps e_1 onto e_2.*

(c) *If $\rho=0$, then $g=e_1\cdot(e_1-e_2)$ is idempotent and S_g maps e_1 onto the reduced idempotent $f_1=e_1\cdot(e_1-2e_2)$, which is orthogonal to e_2.*

(d) *If $\rho\neq 0$, 1, there are reduced idempotents f_i orthogonal to e_i $(i=1,\ 2)$ with $h=e_1+f_1=e_2+f_2$ an idempotent such that for some $y\in A_1(h)$ the element $s=(1-h)+y$ is invertible and $U_se_1=\rho e_2$, $U_sf_1=\rho f_2$.*

PROOF. We have $U_{e_i}e_j=\rho_ie_i$ for some ρ_i $(i\neq j)$. Since $\rho_1^3e_1=U_{\rho_1e_1}e_2$ $=U_{U(e_1)e_2}e_2=U_{e_1}U_{e_2}U_{e_1}e_2=\rho_1\rho_2\rho_1e_1$ we have $\rho_1^2(\rho_1-\rho_2)=0$; similarly $\rho_2^2(\rho_2-\rho_1)=0$, so either $\rho_1-\rho_2=0$ or $\rho_1^2=\rho_2^2=0$, and in either case $\rho_1=\rho_2=\rho$. We claim the subalgebra generated by e_1, e_2 is $\Phi[e_1,\ e_2]$ $=\Phi e_1+\Phi x+\Phi e_2$, where

$$x = 2e_1\cdot e_2 \qquad e_i\cdot x = \rho e_i + \tfrac{1}{2}x \qquad x^2 = \rho(e_1 + x + e_2).$$

Indeed, such an x has $e_i\cdot x=2e_i\cdot(e_i\cdot e_j)=U_{e_i}e_j+e_i^2\cdot e_j=\rho e_i+e_i\cdot e_j$, by [3, p. 1073, (10)] and $x^2=U_{e_1}e_2^2+U_{e_2}e_1^2+2e_1\cdot U_{e_2}e_1=\rho e_1+\rho e_2$ $+2\rho e_1\cdot e_2$.

If $\rho=1$, set $e=\frac{1}{4}(e_1+e_2+x)$, $z=\frac{1}{2}(e_1-e_2)$, $z^2=\frac{1}{4}(e_1+e_2-x)$. Then we see $e^2=e$, $e\cdot z=\frac{1}{2}z$, $e\cdot z^2=0$. Thus e is an idempotent with $z\in A_{1/2}(e)$, $z^2\in A_0(e)$. Since $e_1=e+z+z^2$, $e_2=e-z+z^2$, this shows $S_ee_1=e_2$.

If $\rho=0$, set $g=e_1\cdot(e_1-e_2)=e_1-\frac{1}{2}x$, $f_1=e_1\cdot(e_1-2e_2)=e_1-x$. Then $x^2=0$, $g^2=g$, $g\cdot x=\frac{1}{2}x$ imply g is an idempotent with $x\in A_{1/2}(g)$. Since $e_1=g+\frac{1}{2}x$, $f_1=g-\frac{1}{2}x$, this shows $S_ge_1=f_1$. Because S_g is an automorphism, we know f_1 is a reduced idempotent, and $e_2\cdot f_1=e_2\cdot e_1-e_2\cdot x=0$.

If $\rho\neq 0$, 1, set $x_i=x-2\rho e_i$ $(i=1,2)$. Then $e_i\cdot x_i=\frac{1}{2}x_i$, so $x_i\in A_{1/2}(e_i)$, and $x_i^2=\rho(e_i+e_j-x)$, so $U_{e_i}x_i^2=\rho U_{e_i}(e_i+e_j-x)=\rho(1-\rho)e_i\neq 0$. Applying Lemma 1, $U_{x_i}e_i=\rho(1-\rho)f_i$ where f_i is a reduced idempotent orthogonal to e_i such that $x_i\in A_{1/2}(e_i,f_i)$ has $x_i^2=\rho(1-\rho)(e_i+f_i)$. But $x_1^2=\rho(e_1+e_2-x)=x_2^2$, so $e_1+f_1=e_2+f_2=h$. Also, $(1-\rho)(e_i+f_i)=\rho^{-1}x_i^2$ $= e_i + e_j - x = e_i - x_i - 2\rho e_i + e_j$, so $e_j = \rho e_i + x_i + (1 - \rho)f_i$. Set $y=\rho(e_1-f_1)+x_1$; then $y^2=\rho^2(e_1+f_1)+\rho(1-\rho)(e_1+f_1)=\rho h$, so y is invertible in $A_1(h)$, and $U_ye_1=\rho^2e_1+\rho(1-\rho)f_1+\rho x_1=\rho e_2$, U_yf_1 $=U_y(h-e_1)=y^2-U_ye_1=\rho(h-e_2)=\rho f_2$. If $s=(1-h)+y$ then s is invertible in A and $U_s=U_y$ on $A_1(h)$, so $U_se_1=\rho e_2$, $U_sf_1=\rho f_2$.

2. **Peirce quadratic forms.** If e_1, e_2 are reduced orthogonal idempotents then the *Peirce quadratic form* Q_{e_1,e_2} determined by e_1 and e_2 is the generic norm of the Peirce subalgebra $A_1(e_1+e_2)=\Phi e_1+A_{12}+\Phi e_2$. Thus $Q(\alpha_1e_1+a_{12}+\alpha_2e_2)=\alpha_1\alpha_2-Q_0(a_{12})$ where $a_{12}^2=Q_0(a_{12})(e_1+e_2)$. A *similarity* of Peirce quadratic forms $Q=Q_{e_1,e_2}$ in A and $Q'=Qe_1'e_2'$ in A' is a bijection S of $A_1(e_1+e_2)$ onto $A_1'(e_1+e_2)$ satisfying $Q'(Sx)=\sigma Q(x)$ for all $x\in A_1(e_1+e_2)$ and such that $Se_i=\sigma_ie_i'$ for some nonzero scalars σ_1, σ_2 $(\sigma=\sigma_1\sigma_2)$. The main source of similarities is the following

LEMMA 3. *Suppose e_1, e_2 are reduced orthogonal idempotents in A, and similarly for f_1, f_2. If s is invertible and $U_se_i=\sigma_if_i$ $(i=1, 2)$ then $S=U_s$ is a similarity of Q_{e_1,e_2} with Q_{f_1,f_2} with ratio $\sigma=\sigma_1\sigma_2$.*

PROOF. Clearly U_s is a similarity of $\Phi e_1+\Phi e_2$ onto $\Phi f_1+\Phi f_2$, with ratio σ, so it suffices to check it from $A_{1/2}(e_1, e_2)$ to $A_{1/2}(f_1, f_2)$. The formula $U_sU_{e_1,e_2}A=U_sU_{U(s^{-1})f_1,U(s^{-1})f_2}A=U_sU_s^{-1}U_{f_1,f_2}U_s^{-1}A=U_{f_1,f_2}A$ show that U_s is a bijection of $A_{1/2}(e_1, e_2)$ onto $A_{1/2}(f_1, f_2)$. The component of s^2 in $A_1(e_1+e_2)$ is $U_{e_1+e_2}s^2=U_{U(s^{-1})(\sigma_1f_1+\sigma_2f_2)}U_s1=U_s^{-1}U_{\sigma_1f_1+\sigma_2f_2}U_s^{-1}U_s1=U_s^{-1}(\sigma_1^2f_1+\sigma_2^2f_2)=\sigma_1e_1+\sigma_2e_2$, and hence for $a\in A_{1/2}(e_1, e_2)\subset A_1(e_1+e_2)$ we have $(U_sa)^2=U_sU_as^2=U_sU_aU_{e_1+e_2}s^2=U_sU_a(\sigma_1e_1+\sigma_2e_2)=U_s\{\sigma_1U_{e_2}a^2+\sigma_2U_{e_1}a^2\}=Q_0(a)U_s\{\sigma_1e_2+\sigma_2e_1\}=Q_0(a)\{\sigma_1\sigma_2f_2+\sigma_2\sigma_1f_1\}=\sigma Q_0(a)(f_1+f_2)$. Thus $Q_0(U_sa)=\sigma Q_0(a)$, and U_s is a similarity with ratio σ.

Recall that two orthogonal idempotents e_1, e_2 are *connected* if there is an element $y\in A_{1/2}(e_1, e_2)$ which is invertible in $A_1(e_1+e_2)$.

THEOREM 1. *If A is a Jordan algebra in which any two reduced orthogonal idempotents are connected, then any two Peirce quadratic forms in A are similar, and the similarity may be taken in the group $\mathfrak{U}(A)$ generated by the U_s for s invertible.*

PROOF. We write $Q\sim Q'$ if such a similarity exists; this defines an equivalence relation. The connectivity assumption guarantees

$$Q_{e_1,e_2}\sim Q_{e_2,e_1} \quad \text{if } e_1\perp e_2 \text{ are orthogonal,} \tag{1}$$

since there is an element $y\in A_{1/2}(e_1, e_2)$ with $y^2=\sigma(e_1+e_2)$ for $\sigma\neq 0$, and $s=(1-e_1-e_2)+y$ is invertible in A with $U_se_i=U_ye_i=U_{e_j}y^2=\sigma e_j$ $(i\neq j)$, so we can apply the lemma. Using this same U_s we see

$$Q_{e_0,e_1}\sim Q_{e_0,e_2} \quad \text{if } e_0\perp e_1\perp e_2 \tag{2}$$

since $U_se_0=e_0$, $U_se_1=\sigma e_2$.

We use this to establish the more general case

$$Q_{e_0,e\cdot}\sim Q_{e_0,e_2} \quad \text{if } e_0\perp e_1,\ e_0\perp e_2. \tag{3}$$

Let ρ be the projection coefficient of e_1 and e_2. If $\rho=1$, there is a Peirce reflection, viz. the S_e of Lemma 2 ($e\in\Phi[e_1, e_2]\subset A_0(e_0)$), with $S_ee_1=e_2$, $S_ee_0=e_0$, so $Q_{e_0,e_1}\sim Q_{e_0,e_2}$ by Lemma 3 (or directly, since S_e is an automorphism). If $\rho=0$ then $S_ge_1=f_1\perp e_2$, $S_ge_0=e_0$ so $Q_{e_0,e_1}\sim Q_{e_0,f_1}\sim Q_{e_0,e_2}$ (where the last step follows from (2) since $e_0\perp f_1\perp e_2$). If $\rho\neq 0$, 1, there is $U_s\in\mathfrak{U}(A)$ with $U_se_0=e_0$, $U_se_1=\rho e_2$ and again $Q_{e_0,e_1}\sim Q_{e_0,e_2}$ by Lemma 3.

Finally, we turn to the general case:

$$Q_{e_1,e_2}\sim Q_{f_1,f_2}\quad\text{if } e_1\perp e_2, f_1\perp f_2. \tag{4}$$

Let ρ be the projection coefficient of e_1 and f_1. If $\rho=1$, then $S_ee_1=f_1$, so $Q_{e_1,e_2}\sim Q_{Se_1,Se_2}=Q_{f_1,Se_2}\sim Q_{f_1,f_2}$ (using (3) for the last step). If $\rho=0$ we have $S_ge_1=g_1\perp f_1$ so $Q_{e_1,e_2}\sim Q_{Se_1,Se_2}=Q_{g_1,Se_2}\sim Q_{g_1,f_1}\sim Q_{f_1,g_1}\sim Q_{f_1,f_2}$ (using (3), (1), (3) for the last steps). If $\rho\neq 0$, 1 we have $e_1\perp g_1$, $f_1\perp h_1$ and $U_s\in\mathfrak{U}(A)$ with $U_se_1=\rho f_1$, $U_sg_1=\rho h_1$, and hence $Q_{e_1,e_2}\sim Q_{e_1,g_1}$ $\sim Q_{f_1,h_1}\sim Q_{f_1,f_2}$ by (3), Lemma 3, and (3) again.

As an immediate corollary we have

THEOREM 2 (ALBERT-JACOBSON [1]). *If two reduced simple Jordan matrix algebras $A=H(D_n, \gamma)$ and $A'=H(D_n', \gamma')$ are isomorphic ($n\geqq 2$, D and D' composition algebras) then D and D' are isomorphic.*

PROOF. By simplicity, any two reduced orthogonal idempotents of A (or A') are connected. Hence, by Theorem 1, any two Peirce quadratic forms of A (or A') are similar. The isomorphism of A and A' implies that for any Peirce quadratic form of A there is a similar one of A'. Consequently, the Peirce quadratic forms of A are similar to those of A'. Taking in A the diagonal idempotents e_{11}, e_{22}, the corresponding Peirce quadratic form is the sum of a hyperbolic 2-dimensional quadratic form and of a multiple of the norm form N of D. Similarly in A'. Consequently, Witt's theorem implies that the norm forms N and N' of D and D' are similar. Then D and D' are known to be isomorphic (see [1]).

3. **Reduced algebras of degree three.** Throughout this section, we will be concerned with reduced simple algebras $H(D_3, \gamma)$ of degree three, D a composition algebra. Given a reduced idempotent e_1, we can write $1-e_1=e_2+e_3$, where e_2, e_3 are reduced orthogonal idempotents. The set of values $Q_0(A_{23})=-Q_{e_2,e_3}(A_{23})$ of Q_0 on A_{23} is called the *norm class* of e_1, and is denoted by $K(e_1)$. By Witt's Theorem, it is independent of the decomposition of $1-e_1$ into reduced idempotents, since Q is nondegenerate on $A_1(1-e_1)$ by simplicity. The nomenclature is justified by

(5) $$K(e_i) = \gamma_k^{-1}\gamma_j N(D) \quad \text{if } e_l = e_{ll} \text{ in } H(D_3, \gamma),$$

since $A_{jk} = D[jk]$, $x[jk]^2 = \gamma_k^{-1}\gamma_j N(x)$. From this we immediately see

(6) $$K(e_i) = K(e_j)K(e_k) \quad \text{if } 1 = e_i + e_j + e_k.$$

THEOREM 3. *If A, A' are reduced simple Jordan algebras of degree three with equivalent trace forms and isomorphic coordinate algebras, then any two reduced idempotents e, e' with the same norm class $K(e) = K(e')$ may be mapped into each other under an isomorphism of A onto A'.*

PROOF. Let $1 = e_1 + e_2 + e_3$, $1' = e_1' + e_2' + e_3'$ for e_i, e_i' reduced and $e = e_1$, $e' = e_1'$. By the Coordinatization Theorem [**3**, p. 1077] there are isomorphisms $A \cong H(D_3, \gamma)$, $A' \cong H(D_3', \gamma')$ mapping e_i, e_i' into the diagonal idempotents e_{ii}, e_{ii}'; we may assume $D = D'$. Here $\gamma = (1, \gamma_2, \gamma_3)$, $\gamma' = (1', \gamma_2', \gamma_3')$, where $K(e) = \gamma_3^{-1}\gamma_2 N(D) = \gamma_3'^{-1}\gamma_2' N(D) = K(e')$. Since $\tau(x[23], x[23]) = 2\gamma_3^{-1}\gamma_2 N(x)$ and $\tau'(x[23]', x[23]') = 2\gamma_3'^{-1}\gamma_2' N(x)$, the generic trace forms on $D[23]$ and $D[23]'$ are equivalent, and this extends to an equivalence of $V' = \Phi e_1' + \Phi e_2' + \Phi e_3' + D[23]'$ onto $V = \Phi e_1 + \Phi e_2 + \Phi e_3 + D[23]$. Since τ and τ' are equivalent by hypothesis, Witt's Theorem furnishes us with an equivalence of $V'^{\perp} = D[12]' + D[13]'$ onto $V^{\perp} = D[12] + D[13]$. Since $\tau'(1[31]', 1[31]') = 2\gamma_3'$, there must be an element $u = x[21] + y[31]$ $(x, y \in D)$ with $\tau(u, u) = 2\{\gamma_2 N(x) + \gamma_3 N(y)\} = 2\gamma_3'$. Then $U_{e_1}u^2 = \gamma_3' e_1 \neq 0$, and by Lemma 1, $U_u e_1 = \gamma_3' f_1$ where f_1 is a reduced idempotent orthogonal to e_1 (because $u \in A_{1/2}(e_1)$) with $u \in A_{1/2}(e_1, f_1)$ and $u^2 = \gamma_3'(e_1 + f_1)$. Then $1 = e_1 + f_1 + g_1$ for g_1 reduced and $K(g_1) = \gamma_3' N(D)$. We already had $K(e_1) = K(e_1') = \gamma_3'^{-1}\gamma_2' N(D)$, hence $K(f_1) = K(e_1)K(g_1) = \gamma_2' N(D)$ by (6). But this means that we can choose $u_{12} \in A_{1/2}(e_1, g_1)$, $u_{13} \in A_{1/2}(e_1, f_1)$ with $u_{12}^2 = \gamma_2'^{-1}(e_1 + g_1)$, $u_{13}^2 = \gamma_3'^{-1}(e_1 + f_1)$, and by the Coordinatization Theorem there is an isomorphism $A \cong H(D_3, \gamma')$ for $\gamma' = (1, \gamma_2', \gamma_3')$ sending $e_1 \to e_{11}'$, $g_1 \to e_{22}'$, $f_1 \to e_{33}'$. Thus $A \cong H(D_3, \gamma') \cong A'$ under isomorphisms mapping $e = e_1 \leftrightarrow e_{11}' \leftrightarrow e_1' = e'$.

COROLLARY. *Two reduced idempotents e, e' in a reduced simple Jordan algebra of degree three are conjugate under the group of automorphisms if and only if $K(e) = K(e')$.*

THEOREM 4 (SPRINGER [**4**]). *Two reduced simple Jordan algebras of degree three are isomorphic if and only if they have isomorphic coordinate algebras and equivalent trace forms.*

PROOF. The conditions are necessary by the Albert-Jacobson Theorem. The previous theorem shows they will be sufficient if they allow us to find reduced idempotents in A and A' with the same norm class.

Let $1' = e_1' + e_2' + e_3'$ where $e' = e_1'$ has $K(e') = \gamma_3'^{-1}\gamma_2' N(D')$ and $A' = H(D_3', \gamma')$ for $\gamma' = (1, \gamma_2', \gamma_3')$. We must find $e \in A$ with $K(e) = K(e')$.

Choose a (temporary) coordinatization $A = H(D_3, \gamma)$ with $1 = e_1 + e_2 + e_3$, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (to have complete symmetry among the indices we do not assume $\gamma_1 = 1$). Trivially the trace forms τ on $\Phi e_1 + \Phi e_2 + \Phi e_3$ and τ' on $\Phi e_1' + \Phi e_2' + \Phi e_3'$ are equivalent, so if τ and τ' are equivalent we can use Witt's Theorem again to obtain an equivalence of $D[12] + D[13] + D[23]$ with $D'[12]' + D'[13]' + D'[23]'$. Since $\tau'(1[23]', 1[23]') = 2\gamma_3'^{-1}\gamma_2'$, there is an element $u = x[12] + y[23] + z[31]$ with $\tau(u, u) = 2\{\gamma_2^{-1}\gamma_1 N(x) + \gamma_3^{-1}\gamma_2 N(y) + \gamma_1^{-1}\gamma_3 N(z)\} = 2\gamma_3'^{-1}\gamma_2'$.

We cannot have $N(x) = N(y) = N(z) = 0$. First suppose that two of the norms are zero; by symmetry we may assume $N(x) = N(z) = 0$. Then $\gamma_3^{-1}\gamma_2 N(y) = \gamma_3'^{-1}\gamma_2'$ implies $K(e_1) = K(e')$, and we may take $e = e_1$.

Suppose only one of the norms is zero, say $N(y) = 0$. Then $v = x[12] + z[31]$ has $\tau(v, v) = \tau(u, u) = 2\gamma_3'^{-1}\gamma_2'$. Since $N(x), N(z) \neq 0$ we can re-coordinatize A (keeping the same idempotents e_1, e_2, e_3) in such a way that v has the form $v = 1[21] + 1[31]$ in $H(D_3, \gamma)$ for some new $\gamma = (1, \gamma_2, \gamma_3)$. Then $U_{e_1} v^2 = \frac{1}{2}\tau(v, v)e_1 = \gamma_3'^{-1}\gamma_2' e_1 \neq 0$. By Lemma 1 we have $U_v e_1 = \gamma_3'^{-1}\gamma_2' f_1$ where f_1 is a reduced idempotent orthogonal to e_1 with $v \in A_{1/2}(e_1, f_1)$ and $v^2 = \gamma_3'^{-1}\gamma_2'(e_1 + f_1)$. This implies $g_1 = 1 - e_1 - f_1$ has $K(g_1) = \gamma_3'^{-1}\gamma_2' N(D) = \gamma_3'^{-1}\gamma_2' N(D') = K(e')$, and we may take $e = g_1$.

Finally, suppose none of the norms is zero. Then again we can re-coordinatize A in such a way that u has the form $u = 1[21] + 1[31] + x[23]$ in $H(D_3, \gamma)$ for some new $\gamma = (1, \gamma_2, \gamma_3)$ and some new $x \in D$. Here $\tau(u, u) = 2\{\gamma_2 + \gamma_3 + \gamma_2\gamma_3 N(y)\} = 2\gamma_3'^{-1}\gamma_2'$ for $y = \gamma_3^{-1}x$ (and $\gamma_2 + \gamma_3 \neq 0$). Since y is invertible in D, set $v = e_1 + \gamma_2^{-1}y^{-1}[21] + \gamma_3^{-1}y^{-1}[31] = e_1 + v_{1/2}$. Then $U_{e_1} v^2 = \beta e_1$, $U_{e_1} v_{1/2}^2 = \delta e_1$ for $\beta = 1 + \gamma_2^{-2}\gamma_2 N(y^{-1}) + \gamma_3^{-2}\gamma_3 N(y^{-1}) = \gamma_3'^{-1}\gamma_2'\{\gamma_2\gamma_3 N(y)\}^{-1} \neq 0$ and $\delta = (\gamma_2^{-1} + \gamma_3^{-1})N(y^{-1}) = (\gamma_2 + \gamma_3)\{\gamma_2\gamma_3 N(y)\}^{-1} \neq 0$. Applying Lemma 1, $U_v e_1 = \beta g_1$ and $U_{v_{1/2}} e_1 = \delta f_1$ for e_1, f_1, g_1 reduced, e_1 and f_1 orthogonal, and $g_1 \in A_1(e_1 + f_1)$. Thus there is another reduced idempotent h_1 orthogonal to g_1 with $g_1 + h_1 = e_1 + f_1$. If $1 = e_1 + f_1 + k_1 = g_1 + h_1 + k_1$ then k_1 is reduced. Now v is invertible in $A_1(e_1 + f_1)$ because $v^2 - v = \delta(e_1 + f_1)$, so $s = k_1 + v$ is invertible in A, and $U_s k_1 = k_1$, $U_s e_1 = U_v e_1 = \beta g_1$. From Lemma 3 we see U_s is an equivalence of Q_{k_1, g_1} with $\beta Q_{k_1, e_1}$. Thus $K(h_1) = \beta K(f_1)$. Applying (6) twice, $K(g_1) = K(h_1)K(k_1) = \beta K(f_1)K(k_1) = \beta K(e_1)$. But $\beta K(e_1) = \gamma_3'^{-1}\gamma_2'\{\gamma_2\gamma_3 N(y)\}^{-1}\gamma_3^{-1}\gamma_2 N(D) = \gamma_3'^{-1}\gamma_2' N(D) = K(e')$, and again we can take $e = g_1$.

Remark. As the referee has kindly pointed out, most of the above results can be extended to the case of algebras A, A' over different ground fields Φ, Φ'. In Theorem 2, an isomorphism of A and A' as simple rings induces an isomorphism of their centers Φ and Φ', and the argument shows that the coordinate algebras D and D' are semilinearly isomorphic. In Theorem 3, if A and A' have trace forms which are semilinearly equivalent and coordinate algebras which are semilinearly isomorphic relative to some given isomorphism of the ground fields Φ and Φ', and e, e' are reduced idempotents whose norm classes correspond under the field isomorphism, then e and e' may be mapped into each other under a semilinear isomorphism of A onto A'. Then Theorem 4 says that two reduced simple Jordan algebras of degree 3 are (ring) isomorphic if and only if they have trace forms which are semilinearly equivalent and coordinate algebras which are semilinearly isomorphic relative to a fixed isomorphism of the ground fields.

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