

A note on regular modules

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Kaplansky's observation, namely, a commutative ring R is (von Neumann) regular if and only if each simple R -module is injective, is generalized to projective modules over a commutative ring.

It is a well-known observation due to Kaplansky that a commutative ring R is (von Neumann) regular if and only if every simple epi-image of R is injective. One may ask whether this can be extended to projective modules over arbitrary commutative rings. To be precise, call a module P over a ring R

- (i) a *regular module* if, for each $p \in P$, there is $f \in \text{hom}_R(P, R)$ such that $p = p(fp)$ (Zelmanowitz [10]) and
- (ii) a *V-module* if each simple epi-image of P , when it exists, is injective.

Then the question is: is a projective module P over a commutative ring R a regular module if and only if it is a V -module? In [9] Ware proved the implication 'only if' for all P and the implication 'if' for finitely generated P , leaving unsettled the validity of 'if' for arbitrary P . This problem was solved by S. Alamelu (private communication). In this note we formulate this problem in a more general setting and prove the validity of 'if' for all P . We call a module P over a ring R

- (i) a *weakly regular module*, if for each $p \in P$, there are $f_1, \dots, f_n \in \text{hom}_R(P, R)$ and $r_1, \dots, r_n \in R$ such that

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$$P = P \sum_1^n f_i (pr_i), \text{ and}$$

- (ii) an *HV-module* if every simple image of every cyclic submodule of P is injective.

We show that any projective *HV-module* is weakly regular. For commutative R , this yields the desired result; namely, any projective V -module is regular.

Throughout this paper R will denote an associative ring with identity and all modules considered will be unitary right R -modules. Homomorphisms will be written opposite to scalars. For any module M , M^* will denote $\text{hom}_R(M, R)$, and for any $m \in M$,

$$T_m = \left\{ \sum_1^n f_i (mr_i); f_i \in M^*, r_i \in R \right\}.$$

Note that T_m is a two-sided ideal of R .

DEFINITION 1. An R -module M is said to be

- (i) *regular*, if, for every $m \in M$, there is $f \in M^*$ with $m = m(fm)$ (cf. [10]);
- (ii) *weakly regular* if, for every $m \in M$, $m \in mT_m$;
- (iii) a *V-module* if each simple epi-image of M (when it exists) is injective;
- (iv) an *HV-module* if each cyclic submodule of M is a V -module.

EXAMPLES. (i) If R is a simple ring with identity and I is a right ideal of R , then I is a weakly regular module. For, if $a \in I$, then $a = a \cdot 1 = a \sum r_i s_i$ ($r_i, s_i \in R$) and clearly the r_i can be considered as elements of I^* . More generally, any right ideal of a weakly regular ring is a weakly regular module. (R is called a *weakly regular ring* if $a \in aRa$ for every $a \in R$. For information on these rings, see [6].)

- (ii) Any regular module is weakly regular. The converse is not true.

For instance, if R is a simple integral domain which is not a division ring (for an example of such a ring see [5], p. 211), then any right ideal of R is weakly regular but not regular.

(iii) A ring R has been called a V -ring, if each simple right R -module is injective (see [3], [4] for properties and examples of such rings). It is clear that any module over a V -ring is an HV -module.

(iv) Any HV -module is a V -module. The converse is not true (see Remark 3 (ii) below).

PROPOSITION 2. *Any projective HV -module is weakly regular.*

Proof. Let P be a projective HV -module and $p \in P$. If $p \notin pT_p$, choose a submodule X of P maximal with respect to $p \notin X$, $pT_p \subset X$. Then $Y = (pR+X)/X \simeq pR/pR \cap X$ is a simple R -module and as it is an epimorphic image of $pR \subset P$, Y is injective, by hypothesis. But Y is an essential submodule of P/X so that $Y = P/X$. Now consider the diagram

$$\begin{array}{ccc} & P & \\ & \downarrow g & \\ R & \xrightarrow{f} & P/X = (pR+X)/X, \end{array}$$

where g is the projection map and f is the canonical map obtained by using the fact that P/X is simple. As P is projective, this diagram yields a map $h : P \rightarrow R$ with $fh = g$. In particular,

$$p + X = gp = (fh)p = (fl)(hp),$$

and if $fl = pr + X$, then this gives $p - (pr)(hp) \in X$. But $(pr)(hp) \in pT_p$. Thus $p \in X$, a contradiction.

REMARKS 3. (i) A projective weakly regular module need not be an HV -module. This follows from the fact that there are (von Neumann) regular rings which are not V -rings (see [4]).

(ii) A projective V -module need not be weakly regular. To see this, let R be a non-semi-simple artinian, right hereditary ring such that the injective hull E of the right R -module R is projective. Then R is a semi-primary QF-3 ring and E is projective (cf. Colby and Rutter [2]). As R is hereditary, E is a projective V -module, and if it is a weakly regular module, then R (being a submodule of E) will be a weakly

regular module over itself. In other words, R will be a weakly regular ring and consequently its Jacobson radical will be equal to zero [6]. Since R is already semi-primary, this will imply that R is semi-simple artinian, contradicting our assumption that R is not so. Thus E is not a weakly regular module. By Proposition 2, it also follows that E is not an HV -module.

(iii) The above proposition generalizes to projective modules the fact that any V -ring is a weakly regular ring; proved in [7].

(iv) A more detailed study of weakly regular modules will be presented in a subsequent paper.

THEOREM 4. *If R is a commutative ring, then the following are equivalent on any projective R -module P :*

- (i) P is weakly regular;
- (ii) P is regular;
- (iii) P is a V -module;
- (iv) P is an HV -module.

Proof. We give a cyclic proof.

(i) \Rightarrow (ii). Let $p \in P$. Then $p = p \sum f_i(pr_i)$ where $f_i \in P^*$ and $r_i \in R$. But

$$\sum f_i(pr_i) = \sum (f_i p)r_i = \sum r_i(f_i p) = \sum (r_i f_i)p = \left(\sum r_i f_i \right) p.$$

Thus $p = p(fp)$ where $f = \sum r_i f_i \in P^*$ and hence P is regular.

(ii) \Rightarrow (iii) has been proved by Ware ([9], Proposition 2.5).

(iii) \Rightarrow (iv). Let $p \in P$. As P is projective, $p = \sum p_i(f_i p)$ for $p_i \in P$, $f_i \in P^*$. Let $m_i = f_i p$ and

$$T = \left\{ \sum f(\bar{p}), f \in P^* \text{ and } \bar{p} \in P \right\} = \text{trace ideal of } P \text{ in } R.$$

It is well known (and easily proved by considering localizations of P and T at maximal ideals) that R/T is a flat module (see, for instance, [8]). Hence, by Proposition 2.2 of [1], there is a morphism f from R to T

such that $f(m_i) = m_i$ for each i . Writing $f(1) = t$, we have $m_i = m_i t$

for each i and hence $p = pt$. Hence, if $t = \sum_1^n g_i x_i$ for some

$g_i \in P^*$, $x_i \in P$, then $(q_i) \mapsto p \left(\sum g_i q_i \right)$ defines an epimorphism from $P \oplus \dots \oplus P$ (n copies) to pR . It follows from this, that any simple epi-image of pR is also an epi-image of P and hence is injective. Thus P is an *HV*-module.

(iv) \Rightarrow (i) follows from Proposition 2 above.

This completes the proof.

Note added in proof (11 November 1974). It seems worthwhile noting explicitly the following result, which is a consequence of the proofs given in this paper.

Let R be a (not necessarily commutative) ring with identity, and let P be a projective V -module with trace ideal T . Then the following are equivalent:

- (i)' R/T is left R -flat;
- (ii)' P is an *HV*-module;
- (iii)' P is a weakly regular module.

The proof of (iii) \Rightarrow (iv) of Theorem 4 actually establishes (i)' \Rightarrow (ii)' here. (ii)' \Rightarrow (iii)' is Proposition 2. (iii)' \Rightarrow (i)' follows by using the dual basis property of projective modules. Note that if R is commutative, then (i)' holds for all projectives P .

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