Tohoku Math. J.
60 (2008), 349–356

A NOTE ON RELATIVE DUALITY FOR VOEVODSKY MOTIVES

LUCA BARBIERI-VIALE AND BRUNO KAHN

(Received April 27, 2007, revised January 15, 2008)

Abstract. Let k be a perfect field which admits resolution of singularities in the sense of Friedlander and Voevodsky (for example, k of characteristic 0). Let X be a smooth proper k-variety of pure dimension n and Y, Z two disjoint closed subsets of X. We prove an isomorphism

$$M(X - Z, Y) \simeq M(X - Y, Z)^*(n)[2n],$$

where $M(X - Z, Y)$ and $M(X - Y, Z)$ are relative Voevodsky motives, defined in his triangulated category $\mathrm{DM}_{\mathrm{gm}}(k)$.

Introduction. Relative duality is a useful tool in algebraic geometry and has been used several times. Here we prove a version of it in Voevodsky's triangulated category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(k)$ [10], where k is a (perfect) field which admits resolution of singularities. (Recall that, according to [6, Def. 3.4], this means that every k-scheme of finite type may be dominated by a smooth k-scheme via a proper surjective morphism, and that moreover any modification with base a smooth k-scheme may be dominated by a composition of blow-ups with smooth centres: this is the case if k is of characteristic 0, by Hironaka's main theorems.)

Namely, let X be a smooth proper k-variety of pure dimension n and Y, Z two disjoint closed subsets of X. We prove in Theorem 3.1 an isomorphism

$$M(X - Z, Y) \simeq M(X - Y, Z)^*(n)[2n],$$

where $M(X - Z, Y)$ and $M(X - Y, Z)$ are relative Voevodsky motives, see Definition 1.1.

This isomorphism remains true after application of any $\otimes$-functor from $\mathrm{DM}_{\mathrm{gm}}(k)$, for example one of the realisation functors appearing in [9, I.VI.2.5.5 and I.V.2], [7], [8] or [2]. In particular, taking the Hodge realisation, this makes the recourse to M. Saito's theory of mixed Hodge modules unnecessary in [1, Proof of 2.4.2].

The main tools in the proof of Theorem 3.1 are a good theory of extended Gysin morphisms, readily deduced from Déglise's work (Section 2), Voevodsky's localisation theorem for motives with compact supports [10, 4.1.5], and his theorem that, for any scheme of finite type $X \in Sch/k$, the object $M(X) := \underline{C}_*(L(X))$ of $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ actually belongs to $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ (*ibid.*, 4.1.4). This may be used for an alternative presentation of some of the duality results of [10, §4.3]. The arguments seem axiomatic enough to be transposable to other contexts.

We assume familiarity with Voevodsky's paper [10], and use its notation throughout.

2000 *Mathematics Subject Classification.* Primary 14C25.
Key words and phrases. Duality, motives.

1. Relative motives and motives with supports.

DEFINITION 1.1. Let $X \in Sch/k$ and $Y \subseteq X$, closed. We set

$$M(X, Y) = \underline{C}_*(L(X)/L(Y)),$$
$$M^Y(X) = \underline{C}_*(L(X)/L(X - Y)).$$

REMARK 1.2. This convention is different from the one of Déglise in [3, 4, 5] where what we denote by $M^Y(X)$ is written $M(X, Y)$ (and occasionally $M_Y(X)$ as well). Like Déglise, we shall only consider these motives for X smooth (but Y may be singular).

Note that $L(Y) \to L(X)$ and $L(X - Y) \to L(X)$ are monomorphisms, so that we have functorial exact triangles

$$M(Y) \to M(X) \to M(X, Y) \xrightarrow{+1},$$
$$M(X - Y) \to M(X) \to M^Y(X) \xrightarrow{+1}. \tag{1}$$

We can mix the two ideas: for $Y, Z \subseteq X$ closed, define

$$M^Z(X, Y) = \underline{C}_*(L(X)/L(Y) + L(X - Z)).$$

LEMMA 1.3. *If $Y \cap Z = \emptyset$, the obvious map $M^Z(X) \to M^Z(X, Y)$ is an isomorphism, and we have an exact triangle*

$$M(X - Z, Y) \to M(X, Y) \xrightarrow{\delta} M^Z(X) \xrightarrow{+1}. \qquad \square$$

2. Extended Gysin. In the situation of Lemma 1.3, assume that Z is smooth of pure codimension c. F. Déglise has then constructed a purity isomorphism

$$p_{Z \subset X} : M^Z(X) \xrightarrow{\sim} M(Z)(c)[2c] \tag{2}$$

with the following properties:

(1) $p_{Z \subset X}$ coincides with Voevodsky's purity isomorphism of [10, 3.5.4] (see [5, 1.11]).

(2) If $f : X' \to X$ is transverse to Z in the sense that $Z' = Z \times_X X'$ is smooth of pure codimension c in X', then the diagram

$$\begin{array}{ccc} M^{Z'}(X') & \xrightarrow{p_{Z' \subset X'}} & M(Z')(c)[2c] \\ {\scriptstyle (f,g)_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ M^Z(X) & \xrightarrow{p_{Z \subset X}} & M(Z)(c)[2c] \end{array}$$

commutes, where $g = f_{|Z'}$ ([3, Rem. 4] or [4, 2.4.5]).

(3) If $i : T \subset Z$ is a closed subset, smooth of codimension d in X, the diagram

$$
\begin{array}{ccccc}
M^Z(X) & \xrightarrow{p_{Z\subset X}} & M(Z)(c)[2c] & & \\
 & & & \searrow{\scriptstyle \alpha} & \\
\downarrow{\scriptstyle i^*} & & & & M^T(Z)(c)[2c] \\
 & & & \swarrow{\scriptstyle p_{T\subset Z}} & \\
M^T(X) & \xrightarrow{p_{T\subset X}} & M(T)(d)[2d] & &
\end{array}
$$

commutes, where α is the twist/shift of the second map in the triangle corresponding to (1) [5, proof of 2.3].

DEFINITION 2.1. We set:

$$g^Y_{Z\subset X} = p_{Z\subset X} \circ \delta$$

where $p_{Z\subset X}$ is as in (2) and δ is the morphism appearing in Lemma 1.3.

In view of the properties of $p_{Z\subset X}$, these extended Gysin morphisms have the following properties:

PROPOSITION 2.2. (a) *Let $f : X' \to X$ be a morphism of smooth schemes. Let $Z' = f^{-1}(Z)$ and $Y' = f^{-1}(Y)$. If f is transverse to Z, the diagram*

$$
\begin{array}{ccc}
M(X', Y') & \xrightarrow{g^{Y'}_{Z'\subset X'}} & M(Z')(c)[2c] \\
{\scriptstyle f_*}\downarrow & & {\scriptstyle g_*}\downarrow \\
M(X, Y) & \xrightarrow{g^{Y}_{Z\subset X}} & M(Z)(c)[2c]
\end{array}
$$

commutes, with $g = f_{|Z}$.

(b) *Let $X \supset Z \supset Z'$ be a chain of smooth k-schemes of pure codimensions, and let $d = \operatorname{codim}_Z Z'$. Let $Y \subset X$ be closed, with $Y \cap Z = \emptyset$. Then*

$$g^Y_{Z'\subset X} = g_{Z'\subset Z}(d)[2d] \circ g^Y_{Z\subset X}\,.$$

3. Relative duality. In this section, X is a smooth proper variety purely of dimension n and Y, Z are two disjoint closed subsets of X. Consider the diagonal embedding of X into $X \times X$: its intersection with $(X - Y) \times (X - Z)$ is closed and isomorphic to $X - Y - Z$. The closed subset $(X - Y) \times Y \cup Z \times (X - Z)$ is disjoint from $X - Y - Z$; from Definition 2.1 we get a extended Gysin map

$$M((X - Y) \times (X - Z), (X - Y) \times Y \cup Z \times (X - Z)) \to M(X - Y - Z)(n)[2n]\,.$$

Note that the left hand side is isomorphic to $M(X - Y, Z) \otimes M(X - Z, Y)$ by an explicit computation from the definition of relative motives. Composing with the projection

$M(X - Y - Z)(n)[2n] \to \mathbf{Z}(n)[2n]$, we get a map

$$M(X - Y, Z) \otimes M(X - Z, Y) \to \mathbf{Z}(n)[2n]$$

and hence a map

$$M(X - Z, Y) \xrightarrow{\alpha_X^{Y,Z}} M(X - Y, Z)^*(n)[2n]\,. \tag{3}$$

THEOREM 3.1. *The map* (3) *is an isomorphism.*

The proof is given in the next section.

4. Proof of Theorem 3.1.

LEMMA 4.1. *If* $Y = Z = \emptyset$ *and* X *is projective, then* (3) *is an isomorphism.*

PROOF. As pointed out in [10, p. 221], $\alpha_X^{\emptyset,\emptyset}$ corresponds to the class of the diagonal; then Lemma 4.1 follows from the functor of [10, 2.1.4] from Chow motives to $\mathrm{DM}_{\mathrm{gm}}(k)$. (This avoids a recourse to [10, 4.3.2 and 4.3.6].) □

The next step is when Z is empty. For any $U \in Sch/k$, write $M^c(U) := \underline{C}_*(L^c(U))$ [10, p. 224]. Since X is proper, by [10, 4.1.5] there is a canonical isomorphism

$$M(X, Y) \xrightarrow{\sim} M^c(X - Y)$$

induced by the map of Nisenvich sheaves

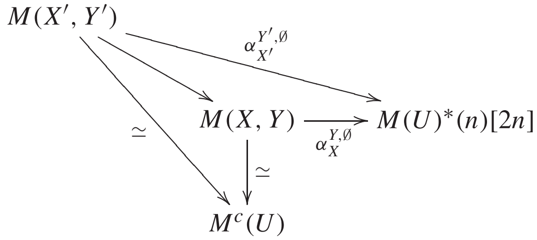
$$L(X)/L(Y) \to L^c(X - Y)\,.$$

Therefore, from $\alpha_X^{Y,\emptyset}$, we get a map

$$\beta_X^Y : M^c(X - Y) \to M(X - Y)^*(n)[2n]\,.$$

LEMMA 4.2. *The map* β_X^Y *only depends on* $X - Y$.

PROOF. Let $U = X - Y$. If X' is another smooth compactification of U, with $Y' = X' - U$, we need to show that $\beta_X^Y = \beta_{X'}^{Y'}$. By resolution of singularities, X and X' may be dominated by a third smooth compactification; therefore, without loss of generality, we may assume that the rational map $q : X' \to X$ is a morphism. The point is that, in the diagram



both triangles commute. For the left one it is obvious, and for the upper one this follows from the naturality of the pairing (3). Indeed, the square

$$\begin{array}{ccc} X' - Y' & \xrightarrow{\Delta'} & (X' - Y') \times X' \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q' \times q} \\ X - Y & \xrightarrow{\Delta} & (X - Y) \times X \end{array}$$

is clearly transverse, where $q' = q_{|X'-Y'}$ (an isomorphism) and Δ, Δ' are the diagonal embeddings; therefore we may apply Proposition 2.2 (a). □

From now on, we write β_{X-Y} for the map β_X^Y.

LEMMA 4.3. (a) *Let $U \in Sm/k$ of pure dimension n, $T \xrightarrow{i} U$ closed, smooth of pure dimension m and $V = U - T \xrightarrow{j} U$. Then the diagram*

$$\begin{array}{ccc} M^c(T) & \xrightarrow{\beta_T} & M(T)^*(m)[2m] \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle g^*_{T\subset U}(n)[2n]} \\ M^c(U) & \xrightarrow{\beta_U} & M(U)^*(n)[2n] \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ M^c(V) & \xrightarrow{\beta_V} & M(V)^*(n)[2n] \end{array}$$

commutes.

(b) *Suppose that β_T is an isomorphism. Then β_U is an isomorphism if and only if β_V is.*

PROOF. (a) The bottom square commutes by a trivial case of Proposition 2.2 (a). For the top square, the statement is equivalent to the commutation of the diagram

$$\begin{array}{ccccc} & & M^c(T) \otimes M(T)(c)[2c] & & \\ & \nearrow^{1\otimes g_{T\subset U}} & & \searrow & \\ M^c(T) \otimes M(U) & & & & \mathbf{Z}(n)[2n] \\ & \searrow_{i_*\otimes 1} & & \nearrow & \\ & & M^c(U) \otimes M(U) & & \end{array}$$

with $c = n - m$.

Take a smooth compactification X of U, and let $\bar{T}$ be a desingularisation of the closure of T in X. Let $q : \bar{T} \to X$ be the corresponding morphism, $Y = X - U$ and $W = \bar{T} - T$:

we have to show that the diagram

$$\begin{array}{ccccc} & & M(\bar{T}, W) \otimes M(T)(c)[2c] & & \\ & \overset{1\otimes g_{T\subset U}}{\nearrow} & & \searrow & \\ M(\bar{T}, W) \otimes M(U) & & & & \mathbf{Z}(n)[2n]\,, \\ & \underset{q_*\otimes 1}{\searrow} & & \nearrow & \\ & & M(X, Y) \otimes M(U) & & \end{array}$$

or equivalently

$$\begin{array}{ccccc} & & M(\bar{T} \times T, W \times T)(c)[2c] & & \\ & \overset{f\circ g^{W\times U}_{\bar{T}\times T\subset \bar{T}\times U}}{\nearrow} & & \searrow & \\ M(\bar{T} \times U, W \times U) & & & & \mathbf{Z}(n)[2n] \\ & \underset{(q\times 1)_*}{\searrow} & & \nearrow & \\ & & M(X \times U, Y \times U) & & \end{array}$$

commutes, where f is the map $M(\bar{T} \times T)(c)[2c] \to M(\bar{T} \times T, W \times T)(c)[2c]$. For this, it is enough to show that the diagram

$$\begin{array}{ccccc} & & M(\bar{T} \times T, W \times T)(c)[2c] & \xrightarrow[g^{W\times T}_{T\subset \bar{T}\times T}(c)[2c]]{} & M(T)(n)[2n] \\ & \overset{f\circ g^{W\times U}_{\bar{T}\times T\subset \bar{T}\times U}}{\nearrow} & & & \\ M(\bar{T} \times U, W \times U) & & & & \downarrow i_* \\ & \underset{(q\times 1)_*}{\searrow} & & & \\ & & M(X \times U, Y \times U) & \xrightarrow{g^{Y\times U}_{U\subset X\times U}} & M(U)(n)[2n] \end{array}$$

commutes. Since extended Gysin extends Gysin, Proposition 2.2 (a) shows that this amounts to the commutatvity of

$$\begin{array}{ccc} M(\bar{T} \times U, W \times U) & \xrightarrow{g^{W\times U}_{T\subset \bar{T}\times U}} & M(T)(n)[2n] \\ {\scriptstyle (q\times 1)_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ M(X \times U, Y \times U) & \xrightarrow{g^{Y\times U}_{U\subset X\times U}} & M(U)(n)[2n]\,, \end{array}$$

which follows from the functoriality of the extended Gysin maps (Proposition 2.2 (b)).

(b) This follows immediately from (a). □

PROPOSITION 4.4. *β_U is an isomorphism for all smooth U.*

PROOF. We argue by induction on $n = \dim U$, the case $n = 0$ being known by Lemma 4.1. In general, let V be an open affine subset of U and pick a smooth projective compactification X of V, with $Z = X - V$. Let $Z \supset Z_1 \supset \cdots \supset Z_r = \emptyset$, where Z_{i+1} is the singular locus of Z_i. Let also $T = U - V$ and define similarly $T \supset T_1 \supset \cdots \supset T_s = \emptyset$ (all Z_i and T_j are taken with their reduced structure). Let $V_i = X - Z_i$ and $U_j = U - T_j$. Then $V_i - V_{i-1}$ and $U_j - U_{j-1}$ are smooth for all i, j. Thus β_U is an isomorphism by Lemma 4.1 (case of β_X) and a repeated application of Lemma 4.3 (b). □

REMARK 4.5. We have not tried to check whether β_U is the inverse of the isomorphism appearing in the proof of [10, 4.3.7]: we leave this interesting question to the interested reader.

END OF PROOF OF THEOREM 3.1. By Lemma 1.3, the triangle

$$M(Z) \to M(X - Y) \to M(X - Y, Z) \xrightarrow{+1}$$

and the duality pairings induce a map of triangles

$$\begin{array}{ccccc} M(X-Y,Z)^*(n)[2n] & \longrightarrow & M(X-Y)^*(n)[2n] & \longrightarrow & M(Z)^*(n)[2n] \\ {\scriptstyle \alpha_X^{Y,Z}}\uparrow & & {\scriptstyle \alpha_X^{Y,\emptyset}}\uparrow & & {\scriptstyle \Phi}\uparrow \\ M(X-Z,Y) & \longrightarrow & M(X,Y) & \longrightarrow & M^Z(X)\,. \end{array}$$

(The left square commutes by a trivial application of Proposition 2.2 (a), and Φ is some chosen completion of the commutative diagram by the appropriate axiom of triangulated categories.)

Consider the following diagram (which is the previous diagram with $Y = \emptyset$):

$$\begin{array}{ccccc} M(X,Z)^*(n)[2n] & \longrightarrow & M(X)^*(n)[2n] & \longrightarrow & M(Z)^*(n)[2n] \\ {\scriptstyle \alpha_X^{\emptyset,Z}}\uparrow & & {\scriptstyle \alpha_X^{\emptyset,\emptyset}}\uparrow & & {\scriptstyle \Phi}\uparrow \\ M(X-Z) & \longrightarrow & M(X) & \longrightarrow & M^Z(X)\,. \end{array}$$

Note that $\alpha_X^{\emptyset,Z}$ is dual to $\alpha_X^{Z,\emptyset}$; therefore it is an isomorphism by Lemma 4.2 and Proposition 4.4. It follows that Φ is an isomorphism. Coming back to the first diagram and using Lemma 4.2 and Proposition 4.4 a second time, we get the theorem. □

REMARK 4.6. It would be interesting to produce a canonical pairing

$$\cap_{(X,Z)} : M^Z(X) \otimes M(Z) \to \mathbf{Z}(n)[2n]$$

playing the rôle of Φ in the above proof, i.e., compatible with $\alpha_X^{Y,Z}$.

REFERENCES

[1] L. BARBIERI-VIALE AND V. SRINIVAS, Albanese and Picard 1-motives, Mémoire Soc. Math. France (N.S.) 87, Paris, 2001.

[2] D. C. CISINSKI AND F. DÉGLISE, Mixed Weil cohomologies, preprint, 2007.

[3] F. DÉGLISE, Interprétation motivique de la formule d'excès d'intersection, C. R. Math. Acad. Sci. Paris 338 (2004), 41–46.
[4] F. DÉGLISE, Motifs génériques, preprint, 2005.
[5] F. DÉGLISE, Around the Gysin triangle, preprint, 2005.
[6] E. FRIEDLANDER AND V. VOEVODSKY, Bivariant cycle cohomology, Ann. of Math. Stud. 143 (2000), 138–187.
[7] A. HUBER, Realization of Voevodsky's motives, J. Algebraic. Geom. 9 (2000), 755–799./Corrigendum, ibid. 13 (2004), 195–207.
[8] F. IVORRA, Réalisation ℓ-adique des motifs triangulés géométriques, I, Doc. Math. 12 (2007), 607–671.
[9] M. LEVINE, Mixed motives, Math. Surveys Monogr. 57, American Mathematical Society, Providence, RI, 1998.
[10] V. VOEVODSKY, Triangulated categories of motives over a field, Ann. of Math. Stud. 143 (2000), 188–238.

DIPARTIMENTO DI MATEMATICA PURA E APPLICATA
UNIVERSITÀ DEGLI STUDI DI PADOVA
VIA TRIESTE, 63, I-35121–PADOVA
ITALY

E-mail address: barbieri@math.unipd.it

INSTITUT DE MATHÉMATIQUES DE JUSSIEU
175–179 RUE DU CHEVALERET
75013 PARIS
FRANCE

E-mail address: kahn@math.jussieu.fr