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LETTERS TO THE EDITOR

A NOTE ON REPEATED SEQUENCES IN MARKOV CHAINS

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Abstract

If (non-overlapping) repeats of specified sequences of states in a Markov chain are considered, the result is a Markov renewal process. Formulae somewhat simpler than those given in Biggins and Cannings (1987) are derived which can be used to obtain the transition matrix and conditional mean sojourn times in this process.

SEQUENCE PATTERNS; MARKOV RENEWAL PROCESS

This note forms a continuation of Biggins and Cannings (1987), which will be referred to as BC, and uses the same notation.

A realization of a finite Markov chain (whose state space forms the alphabet of available letters) is cut in the following way. A set of words, none of which is contained in any other, is given. The realization of the underlying chain is now viewed as a sequence of letters to be read, from its start, until the end of one of the words is reached, at which point the realization is cut to produce a fragment. This procedure is then repeated on the remainder of the realization, and so on. Each time a particular word occurs in the realization its end forms the site of a potential cut; only some of these words actually result in cuts, the remainder being inhibited by some other cut occurring within the word in question. Both the sequences of potential cuts and that of the actual cuts form Markov renewal processes, with the type of a cut being identified with the word causing it. The mechanism for obtaining the second of these processes from the first is essentially that of a Type-I counter. The generating function of the semi-Markov matrix for the process of potential cuts is denoted by $\boldsymbol{a}(z)$ and that for actual cuts by $\boldsymbol{a}^0(z)$. The types of the actual cuts develop as a Markov chain with transition matrix $\boldsymbol{a}^0(1)$, whilst the conditional mean fragment lengths (conditional on the types at the end) are simply related to $\boldsymbol{a}^{0\prime}(1)$. In BC formulae for obtaining $\boldsymbol{a}^0(1)$ and $\boldsymbol{a}^{0\prime}(1)$ from the transition matrix of the underlying chain were developed. These formulae involved first finding $\boldsymbol{a}(1)$ and $\boldsymbol{a}'(1)$ and then using BC(5·2) which relates $\boldsymbol{a}^0(z)$ and $\boldsymbol{a}(z)$. Here simpler , more direct formulae are obtained for $\boldsymbol{a}^0(1)$ and $\boldsymbol{a}^{0\prime}(1)$, and the same idea also yields slightly simpler formulae for $\boldsymbol{a}(1)$ and $\boldsymbol{a}'(1)$ too. Besides their computational advantage, these formulae also afford some structural insight, in particular into the validity of certain approximations for $\boldsymbol{a}^0(1)$ and $\boldsymbol{a}^{0\prime}(1)$ used by Bishop et al. (1983). In obtaining the formulae we draw heavily on the ideas in Section 4 of BC, but here the structure of the transform of the renewal function, $\boldsymbol{r}$, will be exploited rather more.

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We have, from BC(3·2) and BC(6·7), $(\boldsymbol{I}-\boldsymbol{a})^{-1}=\boldsymbol{r}=(\boldsymbol{q}+\boldsymbol{s})$ so that

$$\boldsymbol{a}(\boldsymbol{q}+\boldsymbol{s})=\boldsymbol{q}+\boldsymbol{s}-\boldsymbol{I}, \tag{1}$$

and, from BC(6·12),

$$\boldsymbol{a}^0(\boldsymbol{q}+\boldsymbol{s})=\boldsymbol{s}. \tag{2}$$

Here $\boldsymbol{q}$ is given by BC(6·4) and $\boldsymbol{s}$ by BC(6·11), and the important point is that the denominator of s_{ij}, which causes a singularity in Equations (1) and (2) at $z=1$, does not depend on i. Denote this denominator by $e_j(z)\pi_j z^{d_j-1}$ so that

$$e_j(z)=\frac{\nu_{A(j)}(1-f_{A(j)A(j)}(z))}{\pi_j z^{d_j-1}}.$$

We will rewrite (1) and (2) with the denominators cleared. To facilitate this let

$$\begin{aligned}\zeta^0_{ij}(z)&=f_{F(i)A(j)}(z),\\ \eta_{ij}(z)&=e_j(z)(q_{ij}(z)+s_{ij}(z))=e_j(z)q_{ij}(z)+\zeta^0_{ij}(z),\\ \zeta_{ij}(z)&=\eta_{ij}(z)+I_{ij}e_j(z)=(q_{ij}(z)-I_{ij})e_j(z)+\zeta^0_{ij}(z).\end{aligned} \tag{3}$$

Then (1) and (2) become

$$\boldsymbol{a\eta}=\boldsymbol{\zeta} \quad\text{and}\quad \boldsymbol{a}^0\boldsymbol{\eta}=\boldsymbol{\zeta}^0 \tag{4}$$

respectively. These equations are similar to BC(4·3), and differentiating the first one once and twice, setting $z=1$, and using BC(4·6) and BC(4·11) gives

$$\boldsymbol{CM}=\boldsymbol{L} \quad\text{and}\quad \tilde{\boldsymbol{C}}\boldsymbol{M}=\tilde{\boldsymbol{L}} \tag{5}$$

where, using $\mathbf{1}$ for a vector of ones,

$$\begin{aligned}&\boldsymbol{C}=(k\,\vdots\,\boldsymbol{a}(1)) \qquad \boldsymbol{M}=\begin{pmatrix}0 & \mathbf{1}^{\mathrm{T}}\\ \mathbf{1} & \boldsymbol{\eta}'(1)\end{pmatrix} \qquad \boldsymbol{L}=(\mathbf{1}\,\vdots\,\boldsymbol{\zeta}'(1))\\ &\tilde{\boldsymbol{C}}=(v\,\vdots\,\boldsymbol{a}'(1)) \quad\text{and}\quad \tilde{\boldsymbol{L}}=(k\,\vdots\,\tfrac{1}{2}(\boldsymbol{\zeta}''(1)-\boldsymbol{a}(1)\boldsymbol{\eta}''(1))),\end{aligned} \tag{6}$$

which are of course similar to BC(4·7) and BC(4·12). Similarly, differentiating $\boldsymbol{a}^0\boldsymbol{\eta}=\boldsymbol{\zeta}^0$ gives

$$\boldsymbol{C}^0\boldsymbol{M}=\boldsymbol{L}^0 \quad\text{and}\quad \tilde{\boldsymbol{C}}^0\boldsymbol{M}=\tilde{\boldsymbol{L}}^0 \tag{7}$$

with definitions analogous to (6) (henceforth $(6)^0$).

It can be shown that $\boldsymbol{M}$ is invertible when its entries are finite (which they must be here as the underlying chain is finite and irreducible) and, excluding its diagonal terms, each row of $\boldsymbol{\eta}'(1)$ has a non-zero entry. The proof of this is similar to that used in Section 4 of BC which is in turn similar to that given by Gerber and Li (1981). The Equations (7) constitute the main result here, providing a more direct route to the quantities of interest, $\boldsymbol{a}^0(1)$ and $\boldsymbol{a}^{0\prime}(1)$, than that used in BC. Of course similar equations hold for the case, discussed in Section 8 of BC, where partial overlaps are permitted.

It is routine to establish, using (3) and BC(6·15), that

$$\eta'_{ij}(1)=\mu_{ij}-\frac{1}{\pi_j}q_{ij}(1) \quad\text{and}\quad \zeta^{0\prime}_{ij}(1)=\mu_{ij}, \tag{8}$$

and, using BC(6·16) and BC(6·17), that

$$\eta''_{ij}(1)=\rho_{ij}-\frac{2}{\pi_j}q'_{ij}(1)+\frac{1}{\pi_j}\{2(d_j-1)-\xi_j\}q_{ij}(1) \quad\text{and}\quad \zeta^{0\prime\prime}_{ij}(1)=\rho_{ij}, \tag{9}$$

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which are needed for (7). To use (5) one also needs

$$\zeta'_{ij}(1) = \eta'_{ij}(1) + I_{ij}\frac{1}{\pi_j} \quad \text{and} \quad \zeta''_{ij}(1) = \eta''_{ij}(1) - \frac{1}{\pi_j}\{2(d_j - 1) - \xi_j\}I_{ij}.$$

These formulae can be used on the example discussed in Section 9 of BC. Substitution into (8) and then into $(6)^0$ gives

$$\boldsymbol{M} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & -10 \end{pmatrix} \qquad \boldsymbol{L}^0 = \begin{pmatrix} 1 & 4 & \frac{3}{2} \\ 1 & 3 & 2 \end{pmatrix}$$

so that

$$\boldsymbol{M}^{-1} = \frac{1}{14}\begin{pmatrix} 40 & 10 & 4 \\ 10 & -1 & 1 \\ 4 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{C}^0 = \boldsymbol{L}^0\boldsymbol{M}^{-1} = \frac{1}{28}\begin{pmatrix} 172 & 15 & 13 \\ 156 & 18 & 10 \end{pmatrix},$$

and deleting the first column of $\boldsymbol{C}^0$ gives $\boldsymbol{a}^0(1)$. The deleted column forms the first column of $\tilde{\boldsymbol{L}}^0$, the rest of which is obtained by substituting into (9) and then into $(6)^0$ to give

$$\tilde{\boldsymbol{L}}^0 = \frac{1}{14}\begin{pmatrix} 86 & 118 & -103 \\ 78 & 52 & -62 \end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{C}}^0 = \tilde{\boldsymbol{L}}^0\boldsymbol{M}^{-1} = \frac{1}{196}\begin{pmatrix} 4208 & 639 & 565 \\ 3392 & 666 & 426 \end{pmatrix},$$

and deleting the first column of $\tilde{\boldsymbol{C}}^0$ gives $\boldsymbol{a}^{0\prime}(1)$.

To illustrate, without going into details, the usefulness of having the comparatively explicit formula (7) available let $\boldsymbol{\Pi} = \operatorname{diag}\{\pi_i\}$ and $\boldsymbol{Z} = (\boldsymbol{I} - \boldsymbol{\mu}\boldsymbol{\Pi}\boldsymbol{q}(1)^{-1})^{-1}$ then straightforward algebra using $(6)^0$, (7) and (8) gives

$$\text{(10)} \qquad \boldsymbol{a}^0(1) = \boldsymbol{Z}\left(\mathbf{1}\frac{\mathbf{1}^{\mathrm{T}}\boldsymbol{\Pi}\boldsymbol{q}(1)^{-1}\boldsymbol{Z}}{\mathbf{1}^{\mathrm{T}}\boldsymbol{\Pi}\boldsymbol{q}(1)^{-1}\boldsymbol{Z}\mathbf{1}} - \boldsymbol{\mu}\boldsymbol{\Pi}\boldsymbol{q}(1)^{-1}\right).$$

Now suppose the underlying Markov chain, a realization of which is to be cut, is such that the expected number of transitions to reach any state from any other is not large in comparison with the distance between cuts, so that the terms of $\boldsymbol{\mu}$ are not large in comparison with the inverse of the elements of π. It is then plausible that the transition probabilities $a^0_{ij}(1)$ are in fact nearly independent of i, for the underlying Markov chain will have 'forgotten' that the previous cut was of type i long before the next cut occurs, and so they should be given approximately by the stationary probability α^0_j.

This argument can now be made more precise for if $\boldsymbol{\mu}\boldsymbol{\Pi}\boldsymbol{q}(1)^{-1}$ is small, so that $\boldsymbol{Z} \simeq \boldsymbol{I}$, then, using BC(6·6) and (10).

$$\boldsymbol{a}^0(1) \simeq \mathbf{1}\frac{\mathbf{1}^{\mathrm{T}}\boldsymbol{\Pi}\boldsymbol{q}(1)^{-1}}{\mathbf{1}^{\mathrm{T}}\boldsymbol{\Pi}\boldsymbol{q}(1)^{-1}\mathbf{1}} = \mathbf{1}\frac{\pi^{0\mathrm{T}}}{\pi^{0\mathrm{T}}\mathbf{1}} = \mathbf{1}\alpha^{0\mathrm{T}}.$$

This is one of the aproximations proposed by Bishop et al. (1983). (In the cases they consider the terms of $\boldsymbol{\mu}\boldsymbol{\Pi}$ will be small and $\boldsymbol{q}(1) \simeq \boldsymbol{I}$.) Notice that when this approximation is acceptable only BC(6·6) is needed to obtain $\boldsymbol{a}^0(1)$. Similar, but more complicated, discussion of the approximation of $(\pi^{0\mathrm{T}}\mathbf{1})\boldsymbol{a}^{0\prime}(1)$ by $\mathbf{1}\alpha^{0\mathrm{T}}$ is possible, and bounds on both approximations using a matrix norm can also be obtained; details are given in Biggins (1986).

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References

BIGGINS, J. D. (1986) On repeated sequences in Markov chains. Department of Probability and Statistics, University of Sheffield, Research Report no. 279/86.

BIGGINS, J. D. AND CANNINGS, C. (1987) Markov renewal processes, counters and repeated sequences in Markov chains. *Adv. Appl. Prob.* **19,** 521–545.

BISHOP, D. T., WILLIAMSON, J. A. AND SKOLNICK, M. H. (1983) A model for restriction fragment length distributions. *Am. J. Hum. Genet.* **35,** 795–815.

GERBER, H. U. AND LI, S.-Y.P. (1981) The occurrence of sequence patterns in repeated experiments and hitting times in a Markov chain. *Stoch. Proc. Appl.* **11,** 101–108.