3. Flora Dinkines, *Semi-automorphisms of symmetric and alternating groups*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 478–486.

UNIVERSITY OF KANSAS AND
INSTITUTE FOR ADVANCED STUDY

A NOTE ON REPRESENTATIONS OF INVERSE SEMIGROUPS

G. B. PRESTON

It is known [1; 2] that every inverse semigroup S has a faithful representation as a semigroup of (1, 1)-mappings of subsets of a set A into A. The set A may be taken as the set of elements of S and the (1, 1)-mappings as mappings of principal left ideals of S onto principal left ideals of S. If E is the set of idempotents of S then there is also a representation of S, not necessarily faithful, as a semigroup of (1, 1)-mappings of subsets of E into E [2]. If $e \in E$ denote by S_e the subsemigroup eSe of S. In this note we give a representation of any inverse semigroup S as a semigroup of isomorphisms between the semigroups S_e. The representation is faithful if (a more general condition is given below) the center of each maximal subgroup of S is trivial.

We recall that an inverse semigroup [3] is a semigroup S in which for any $a \in S$ the equations $xax = x$ and $axa = a$ have a unique common solution $x \in S$ called the inverse of a and denoted by a^{-1} [5; 6]. This implies that the idempotents of S commute and that to each $a \in S$ there corresponds a pair of idempotents e, f such that $aa^{-1} = e$, $a^{-1}a = f$, $ea = a$, $af = a$. The idempotents e, f are called respectively the left and right units of a. For any two elements $a, b \in S$, $(ab)^{-1} = b^{-1}a^{-1}$ (see [3]). Throughout what follows S will denote an inverse semigroup and E will denote its set of idempotents. If $e \in E$ then S_e will denote the subsemigroup eSe of S.

LEMMA 1. *If $e, f \in E$ then $S_e \cap S_f = S_{ef}$.*

PROOF. By Lemma 1 of [4] and its left-right dual $Se \cap Sf = Sef$ and $eS \cap fS = efS$. Hence since $S_e = eS \cap Se$ and $S_f = fS \cap Sf$, it follows that $S_e \cap S_f = efS \cap Sef = S_{ef}$.

Received by the editors March 9, 1957.

LEMMA 2. *If a is an element of S with left unit e and right unit f then S_e is isomorphic to S_f.*

PROOF. We show that the mapping $\alpha_a: s \rightarrow s\alpha_a = a^{-1}sa$, where $s \in S_e$, is an isomorphism of S_e onto S_f.

Let $a^{-1}sa = t$; then since $af = a$ and $fa^{-1} = a^{-1}$, $ft = t = tf$ and so $t \in S_f$. Hence α_a maps S_e into S_f. Now let t be any element of S_f. Then a repetition of the above argument with a^{-1} replacing a shows that $ata^{-1} = s \in S_e$. Thus, since $s\alpha_a = a^{-1}sa = a^{-1}ata^{-1}a = ftf = t$, the mapping α_a is onto S_f.

If $a^{-1}s_1a = a^{-1}s_2a$ for $s_1, s_2 \in S_e$ then $aa^{-1}s_1aa^{-1} = aa^{-1}s_2aa^{-1}$, so that since $aa^{-1} = e$ and $es_i = s_i = s_ie$, $s_1 = s_2$. Hence α_a is a (1, 1)-mapping.

Finally that α_a is a homomorphism follows because if $s_1, s_2 \in S_e$, then $s_1\alpha_a s_2\alpha_a = a^{-1}s_1aa^{-1}s_2a = a^{-1}s_1s_2a = (s_1s_2)\alpha_a$ because s_1 (or s_2) $\in S_e$ implies that $s_1es_2 = s_1s_2$.

The set of all elements of S with e as both left unit and right unit forms a group denoted by G_e [**3**]. The groups G_e are clearly the maximal subgroups of S. We now have as a corollary to Lemma 2 the result

COROLLARY. *If a is an element of S with left unit e and right unit f then G_e is isomorphic to G_f.*

PROOF. It is easily seen that the restriction of α_a to G_e maps G_e onto G_f.

Denote by $A(S)$ the set of isomorphisms $\{\alpha_a: a \in S\}$, defined in the proof of Lemma 2. Since an isomorphism is a (1, 1)-mapping, the set $A(S)$ generates a semigroup $MA(S)$, say, of (1, 1)-mappings formed by taking all finite products of the elements of $A(S)$. If α and β are (1, 1)-mappings the product $\alpha\beta$ is the mapping α followed by the mapping β applied to those elements for which this sequence of mappings can be carried out [**2**]. If there are no such elements we write $\alpha\beta = 0$, and can regard 0 as the unique (1, 1)-mapping of the elements of the empty set into the empty set. Then we have

LEMMA 3. $A(S) = MA(S)$.

PROOF. It is sufficient to show that for any $a, b \in S$, $\alpha_a\alpha_b \in A(S)$.

Let $aa^{-1} = e$, $a^{-1}a = f$, $bb^{-1} = g$, $b^{-1}b = h$, so that α_a maps S_e onto S_f and α_b maps S_g onto S_h. Then, by Lemma 1, $S_f \cap S_g = S_{fg}$ and so $\alpha_a\alpha_b$ maps $S_{fg}\alpha_a^{-1}$ onto $S_{fg}\alpha_b$. We show that $\alpha_a\alpha_b = \alpha_{ab}$.

Let $(ab)(ab)^{-1} = k$ and $(ab)^{-1}(ab) = l$. Let $x \in S_{fg}\alpha_a^{-1}$, so that $x \in S_e$ and hence $aa^{-1}xaa^{-1} = x$, and also $x\alpha_a \in S_{fg}$ so that $fga^{-1}xafg = a^{-1}xa$, from which it follows that $afga^{-1}xafga^{-1} = aa^{-1}xaa^{-1} = x$. But $afga^{-1} = aa^{-1}abb^{-1}a^{-1} = ab(ab)^{-1} = k$, and so $kxk = x$ and $x \in S_k$. Conversely, if $x \in S_k$, then $fg(x\alpha_a)fg = fga^{-1}xafg = fga^{-1}xagf = a^{-1}(ab)(ab)^{-1} \cdot x(ab)(ab)^{-1}a = a^{-1}kxka = a^{-1}xa = x\alpha_a$, and so $x\alpha_a \in S_{fg}$. We similarly prove that $S_{fg}\alpha_b = S_l$.

Thus $\alpha_a\alpha_b$ maps S_k onto S_l and since $x\alpha_a\alpha_b = (a^{-1}xa)\alpha_b = b^{-1}a^{-1}xab = (ab)^{-1}x(ab) = x\alpha_{ab}$ for $x \in S_k$, $\alpha_a\alpha_b = \alpha_{ab}$. This completes the proof of the lemma.

In [3] it was shown that the homomorphic image of an inverse semigroup is an inverse semigroup. If $\mu: S \to T$ is a homomorphic mapping of S onto T, then the kernel N of μ is the inverse image under μ of the set of idempotents of T, and N is the union of its components, each component being the inverse image of a single idempotent of T. It was shown in [3] that, given S and T, the homomorphism μ is determined by the components of the kernel of μ.

The center of a semigroup T is the set $Z(T) = \{z: z \in T, zt = tz$ for all $t \in T\}$. $Z(T)$ is clearly a subsemigroup of T. Denote by Z_e the center of the maximal subgroup G_e of S. Then we have the

THEOREM. *The mapping $\mu: S \to A(S)$ of S onto $A(S)$ defined by $a\mu = \alpha_a$ for $a \in S$ is a homomorphism. The kernel of μ is $N = \bigcup N_e$, where N_e is the normal subgroup $Z(S_e) \cap Z_e$ of G_e and the union is taken over all $e \in E$.*

PROOF. We have already seen in the course of the proof of Lemma 3 that μ is a homomorphism of S onto $A(S)$.

It remains to determine the kernel of μ. Let α_a be an idempotent of $A(S)$. Then α_a must be the identical mapping of some set S_e, and so for $s \in S_e$, $s\alpha_a = a^{-1}sa = s$. Hence $aa^{-1}sa = as$, and since $aa^{-1} = e$, $aa^{-1}s = s$, therefore $sa = as$, and so $a \in Z(S_e)$. Since also $a \in G_e$, and $G_e \subseteq S_e$, therefore $a \in Z_e$. Hence $a \in Z(S_e) \cap Z_e$.

Conversely, let $b \in Z(S_e) \cap Z_e$. Then $b \in G_e$ so that $bb^{-1} = e = b^{-1}b$, and $bs = sb$ for all $s \in S_e$. Hence $s\alpha_b = b^{-1}sb = b^{-1}bs = es = s = s\alpha_a$, so that $\alpha_a = \alpha_b$.

Thus $N_e = \mu^{-1}(\alpha_a) = Z(S_e) \cap Z_e$; which completes the proof of the theorem.

COROLLARY. *If for each $e \in E$, $Z(S_e) \cap Z_e = e$, then the mapping μ is an isomorphism.*

It follows, as we remarked earlier, that if each $Z_e = e$, that is if the center of each maximal subgroup of S is trivial, then S has a faithful representation as a semigroup of isomorphisms between the semigroups S_e.

Finally we remark that in this latter case when each $Z_e = e$, the isomorphisms α_a are determined by their restrictions to the groups G_e.

Thus we can regard $A(S)$ as a semigroup of isomorphisms between the groups G_e. When the elements of $A(S)$ are so regarded the product of two elements of $A(S)$ cannot be defined in a natural way independently of S.

References

1. V. V. Vagner, *Obobščennye gruppy*, C. R. (Doklady) Acad. Nauk URSS. vol. 84 (1952) pp. 1119–1122.

2. G. B. Preston, *Representations of inverse semigroups*, J. London Math. Soc. vol. 29 (1954) pp. 411–419.

3. G. B. Preston, *Inverse semigroups*, J. London Math. Soc. vol. 29 (1954), pp. 396–403.

4. G. B. Preston, *Inverse semigroups with minimal right ideals*, J. London Math. Soc. vol. 29 (1954) pp. 404–411.

5. A. E. Liber, *K teorii obobščennyh grupp*, C. R. (Doklady) Acad. Nauk URSS. vol. 97 (1954) pp. 25–28.

6. W. D. Munn and R. Penrose, *A note on inverse semigroups*, Proc. Cambridge Philos. Soc. vol. 51 (1955) pp. 396–399.

The Tulane University and
The Royal Military College of Science, Shrivenham, Eng.