

A note on Ricci solitons

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Abstract. In this paper we consider a complete connected Ricci soliton (M, g, ξ, λ) of positive Ricci curvature and assign the Ricci tensor $Ric = \bar{g}$, a role of another Riemannian metric on M . It is shown that the identity map $i : (M, g) \rightarrow (M, \bar{g})$ is a harmonic map. In addition, we also study compact shrinking gradient Ricci soliton $(M, g, \nabla f, \lambda)$ of positive Ricci curvature and obtain a lower bound for the average value of the potential function f and show that if the lower bound is attended then the gradient Ricci soliton is an Einstein manifold.

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Key words: Ricci soliton; gradient Ricci soliton; harmonic map; second fundamental form of a map; Einstein manifold; scalar curvature.

1 Introduction

A Riemannian manifold (M, g) is said to be a Ricci soliton if there exists a smooth vector field ξ on M satisfying

$$(1.1) \quad Ric + \frac{1}{2}\mathcal{L}_\xi = \lambda g$$

where Ric denotes the Ricci tensor of M , \mathcal{L}_ξ denotes the Lie derivative in the direction of ξ and λ is a constant. A Ricci soliton (M, g, ξ, λ) is shrinking soliton, steady soliton or expanding soliton according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. Compact Ricci solitons are the fixed points of the Ricci flow

$$(1.2) \quad \frac{\partial g}{\partial t} = -2Ric$$

projected from the space of metrics onto its quotient modulo diffeomorphism and scaling (cf. [4], [5]) and the complete Ricci solitons arise as blow-up limits for the Ricci flow on compact manifolds. Topology of Ricci solitons has been studied by Derdzinski, Lopez and Garcia-Rio, Wylie (cf. [7], [11], [15], [17]). If the vector field ξ is gradient ∇f of a smooth function f , the Ricci soliton $(M, g, \nabla f, \lambda)$ is called a gradient Ricci soliton and the function f is called the potential function. Gradient

Ricci solitons have been studied quite extensively in last decade (cf. [6], [9], [12], [13], [17]). Hamilton [8], conjectured that a compact gradient shrinking Ricci soliton with positive curvature operator must be Einstein, which is settled in [2] and since then the question of obtaining conditions under which a Ricci soliton is an Einstein manifold has been taken up with interest.

A harmonic map from a Riemannian manifold to other Riemannian manifold has played an important role in linking the geometry to global analysis on Riemannian manifolds as well as its importance in physics is also well established (cf. [2], [14]). Therefore it is an interesting question to find harmonic maps on Ricci soliton. In this paper we are interested in Ricci solitons (M, g, ξ, λ) of positive Ricci curvature and in assigning the Ricci tensor Ric of M , the role of a Riemannian metric $\bar{g} = Ric$ on M . It turns out that the identity map $i : (M, g)$ and (M, \bar{g}) is a harmonic map. In fact we prove the following:

Theorem 1.1. *Let (M, g, ξ, λ) be an n -dimensional complete connected Ricci soliton of positive Ricci curvature. Then the identity map $i : (M, g) \rightarrow (M, \bar{g})$ is a harmonic map, where $\bar{g} = Ric$ is the Ricci tensor of (M, g) .*

It is interesting to note that, this is the first attempt to assign the role of Riemannian metric to the Ricci tensor on a Ricci soliton (M, g, ξ, λ) of positive Ricci curvature. We found the relation between the Levi-Civita connections of these two metrics on the manifold M (cf. Lemma 2.1), which then ultimately relates the curvature tensor fields of these two metrics. We hope it will be interesting to analyze the geometry of Ricci soliton vis-a-vis the geometry of the Riemannian manifold (M, \bar{g}) , where $\bar{g} = Ric$.

In case of a gradient Ricci soliton $(M, g, \nabla f, \lambda)$, where ∇f is the gradient of a smooth function f on M , the average value of the potential function f , f_{av} is defined as

$$f_{av} = \frac{\int f}{V(M)},$$

where $V(M)$ is the volume of M . We find a lower bound for f_{av} , in the case of non-Einstein gradient Ricci soliton and as a consequence, we show that the lower bound is attained only by Einstein manifolds. In fact we prove the following:

Theorem 1.2. *Let $(M, g, \nabla f, \lambda)$ be an n -dimensional compact non-Einstein gradient shrinking Ricci soliton of positive Ricci curvature. Then*

$$f_{av} \geq \frac{n}{2}$$

We also show that on an n -dimensional compact gradient shrinking Ricci soliton $(M, g, \nabla f, \lambda)$ of positive Ricci curvature the inequality $2f_{av} \leq n$ implies that M is an Einstein manifold with Einstein constant λ (cf. Corollary 4.1 in Section 4).

2 Preliminaries

Let (M, g, ξ, λ) be an n -dimensional Ricci soliton of positive Ricci curvature. We treat the Ricci tensor $Ric = \bar{g}$ as another Riemannian metric on M . Let ∇ and $\bar{\nabla}$ be the Riemannian connections with respect to the metrics g and \bar{g} respectively. The Ricci

operator Q of the Riemannian manifold (M, g) is defined by $Ric(X, Y) = g(QX, Y)$, $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie-algebra of smooth vector fields on M . Using Kozul's formula together with equation (1.1) of the Ricci soliton, after a straight forward calculation we arrive at the following expression for the covariant derivative with respect to the connection $\bar{\nabla}$

$$2\bar{g}(\bar{\nabla}_X Y, Z) = 2\bar{g}(\nabla_X Y, Z) + R(X, \xi; Y, Z) - g(\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi, Z),$$

$X, Y, Z \in \mathfrak{X}(M)$, where R is the curvature tensor of the Riemannian manifold (M, g) .

Since $Ric = \bar{g}$, the above equation gives

$$(2.1) \quad 2Q(\bar{\nabla}_X Y - \nabla_X Y) = R(X, \xi)Y - \nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y} \xi.$$

Now, since the Ricci curvature is positive, at each point $p \in M$, the Ricci operator $Q : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ gives an isomorphism $Q_p : T_p M \rightarrow T_p M$ of the tangent space $T_p M$ of M at p , and consequently for the tangent vector $\frac{1}{2}(R(X, \xi)Y - \nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y} \xi)_p \in T_p M$, there exists a vector $(T(X, Y))_p \in T_p M$ such that

$$Q_p(T(X, Y))_p = \frac{1}{2}(R(X, \xi)Y - \nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y} \xi)_p$$

Consequently we get

$$(2.2) \quad Q(T(X, Y)) = \frac{1}{2}(R(X, \xi)Y - \nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y} \xi), \quad X, Y \in \mathfrak{X}(M),$$

where $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is a tensor field of type $(1, 2)$ on M . Thus equation (2.1) can be expressed as

$$2Q(\bar{\nabla}_X Y - \nabla_X Y) = 2Q(T(X, Y))$$

and as Q is non-singular (as the Ricci tensor is positive definite), we have the following:

Lemma 2.1. *Let (M, g, ξ, λ) be an n -dimensional Ricci soliton of positive Ricci curvature. If $\bar{g} = Ric$ and $\bar{\nabla}, \nabla$ are Riemannian connections on the Riemannian manifolds $(M, \bar{g}), (M, g)$ respectively, then*

$$\bar{\nabla}_X Y = \nabla_X Y + T(X, Y), \quad X, Y \in \mathfrak{X}(M)$$

where

$$2Q(T(X, Y)) = R(X, \xi)Y - \nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y} \xi.$$

Note that as both connections $\bar{\nabla}, \nabla$ being Riemannian connections, are torsion free, and consequently

$$(2.3) \quad T(X, Y) = T(Y, X), \quad X, Y \in \mathfrak{X}(M)$$

that is the tensor field T is symmetric.

Let $(M, g), (N, g')$ be two Riemannian manifolds of dimensions m and n respectively. Consider a smooth map $f : (M, g) \rightarrow (N, g')$, and define the Lagrangian of f by

$$\mathcal{L}_f = e(f) = \frac{1}{2} \|df\|^2.$$

The map f is said to be harmonic if it is stationary point of the variational principle for \mathcal{L}_f on any compact subset $U \subset M$, that is, if f is solution of the Euler-Lagrange equation

$$\delta \mathcal{L}_f = 0,$$

where δ denotes the functional derivative (cf. [1], [6], [14]). If we denote the covariant derivative operators on (M, g) and (N, g') by ∇ and $\bar{\nabla}$ respectively, then the second fundamental form α_f of the map f is defined by (cf. [12], [14])

$$(2.4) \quad \alpha_f(X, Y) = \bar{\nabla}_{df(X)} df(Y) - df(\nabla_X Y), \quad X, Y \in \Gamma(TM).$$

It is known that a smooth map $f : (M, g) \rightarrow (N, g')$ is harmonic if and only if the **Trace** of the second fundamental form α_f is zero (cf. [6]).

3 Proof of Theorem 1.1

Let (M, g, ξ, λ) be an n -dimensional complete connected Ricci soliton of positive Ricci curvature. Then by Lemma 2.1, it follows that T is the second fundamental form of the identity map $i : (M, g) \rightarrow (M, \bar{g})$. Choose a local orthonormal frame $\{e_1, \dots, e_n\}$ on (M, g) , which together with equation (2.2) gives

$$(3.1) \quad \begin{aligned} 2Q \left(\sum_{i=1}^n T(e_i, e_i) \right) &= \sum_{i=1}^n (R(e_i, \xi)e_i - \nabla e_i \nabla e_i \xi + \nabla_{\nabla_{e_i} e_i} \xi) \\ &= -Q(\xi) - \Delta \xi \end{aligned}$$

where $\Delta \xi$ is the rough Laplacian of the vector field ξ . Let η be the 1-form dual to ξ and define a skew-symmetric operator $\phi : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by

$$(3.2) \quad d\eta(X, Y) = 2g(\phi(X), Y), \quad X, Y \in \mathfrak{X}(M).$$

Then using Kozul's formula, it is straight forward to verify that

$$2g(\nabla_X \xi, Y) = (\mathcal{L}_\xi g)(X, Y) + d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Using equations (1.1) and (3.2) in above equation we arrive at

$$(3.3) \quad \nabla_X \xi = \lambda X - Q(X) + \phi(X), \quad X \in \mathfrak{X}(M).$$

The covariant derivative (∇Q) of the operator Q is defined as $(\nabla Q)(X, Y) = \nabla_X QY - Q(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$ and it is well known that

$$(3.4) \quad \sum_{i=1}^n (\nabla Q)(e_i, e_i) = \frac{1}{2} \nabla S,$$

where S is the scalar curvature of the Riemannian manifold (M, g) . Using above equation together with equation (3.3) to compute $\Delta \xi$, we get

$$(3.5) \quad \Delta \xi = -\frac{1}{2} \nabla S + \sum_{i=1}^n (\nabla \phi)(e_i, e_i).$$

Note that ϕ being skew symmetric we have

$$(3.6) \quad g((\nabla\phi)(X, Y), Z) = -g(Y, (\nabla\phi)(X, Z)), \quad X, Y, Z \in \mathfrak{X}(M).$$

Choosing a point wise constant local orthonormal frame $\{e_1, \dots, e_n\}$ on the Riemannian manifold (M, g) , we use equation (3.3) to compute

$$R(e_i, X)\xi = -(\nabla Q)(e_i, X) + (\nabla\phi)(e_i, X) + \nabla_X Q(e_i) - \nabla_X \phi(e_i), \quad X \in \mathfrak{X}(M).$$

Taking inner product with e_i in above equation with respect to metric g and summing the equations we arrive at

$$Ric(X, \xi) = -\frac{1}{2}g(\nabla S, X) - g\left(X, \sum_{i=1}^n (\nabla\phi)(e_i, e_i)\right) + g(X, \nabla S),$$

where we have used equations (3.4), (3.6), the facts that Q is symmetric, ϕ is skew-symmetric and $\mathbf{Trace}Q = S$, $\mathbf{Trace}\phi = 0$. Thus the above equation gives

$$Q(\xi) = \frac{1}{2}\nabla S - \sum_{i=1}^n (\nabla\phi)(e_i, e_i),$$

which together with equation (3.5) gives

$$\Delta\xi + Q(\xi) = 0.$$

Thus Q being non-singular, the above equation together with equation (3.1) implies that

$$(3.7) \quad \sum_{i=1}^n T(e_i, e_i) = 0.$$

This proves that the tension $\tau(i) = \mathbf{Trace}T = 0$ that is the identity map is harmonic.

Remark 3.1. *Suppose that the Ricci soliton (M, g, ξ, λ) has positive Ricci curvature. We denote by R and \bar{R} the curvature tensor fields of the Riemannian manifolds (M, g) and (M, \bar{g}) respectively. Then using Lemma 2.1, it is a straight forward calculation to show that*

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + (\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z) \\ &\quad + T(X, T(Y, Z)) - T(Y, T(X, Z)). \end{aligned}$$

If we assume that the Ricci tensor of the Ricci soliton (M, g, ξ, λ) is parallel, then the uniqueness of the Levi-Civita connection on the Riemannian manifold (M, \bar{g}) , where $\bar{g} = Ric$ and Lemma 2.1 will imply that $T = 0$ and consequently in this case the above relation between curvature tensor fields reduces to

$$(3.8) \quad \bar{R}(X, Y)Z = R(X, Y)Z.$$

Choosing a local orthonormal frame $\{e_1, \dots, e_n\}$ on an open subset U of the Ricci soliton (M, g, ξ, λ) that diagonalizes Q with $Qe_i = \mu_i e_i$. Then as $\mu_i > 0$ on U , we define

$$E_i = \frac{1}{\sqrt{\mu_i}} e_i,$$

which gives a local orthonormal frame $\{E_1, \dots, E_n\}$ for the Riemannian manifold (M, \bar{g}) . Consequently the equation (3.8) gives the Ricci tensor \bar{Ric} of the Riemannian manifold (M, \bar{g}) as $\bar{Ric} = \bar{g}$, that is the Riemannian manifold (M, \bar{g}) is an Einstein manifold. Thus we have the following corollary.

Corollary 3.2. *Let (M, g, ξ, λ) be an n -dimensional Ricci soliton of positive Ricci curvature. If $\bar{g} = Ric$ is parallel on the Riemannian manifold (M, g) , then the Riemannian manifold (M, \bar{g}) is an Einstein manifold.*

Note that in Corollary 3.2 we did not assume compactness of the Ricci soliton nor we have assumed any condition on the sectional curvature of the Ricci soliton and as such even though the Ricci tensor being parallel, the Ricci soliton need not be an Einstein manifold. For, if (M, g, ξ, λ) is Einstein manifold of positive scalar curvature, then (M, \bar{g}) will be homothetic to (M, g, ξ, λ) and therefore trivially will be an Einstein manifold.

4 Proof of Theorem 1.2

Suppose that $(M, g, \nabla f, \lambda)$ is an n -dimensional compact shrinking gradient soliton with potential function f . Then equation (1.1) takes the form

$$(4.1) \quad Q + A = \lambda I,$$

where A is the Hessian operator of the function f defined by $A(X) = \nabla_X \nabla f$. The Hessian operator A satisfies

$$(4.2) \quad (\nabla A)(X, Y) - (\nabla A)(Y, X) = R(X, Y)\nabla f, \quad tr A = \Delta f = n\lambda - S,$$

where Δf is the Laplacian of the function f and S is the scalar curvature of the Riemannian manifold (M, g) . Using symmetry of the operator A and local orthonormal frame $\{e_1, \dots, e_n\}$ in equation (4.2) we immediately get

$$\sum_{i=1}^n (\nabla A)(e_i, e_i) = Q(\nabla f) - \nabla S.$$

Using equations (3.5) and (4.1) in above equation we conclude

$$(4.3) \quad Q(\nabla f) = \frac{1}{2}\nabla S.$$

Also, using equations (4.1) and (4.3), we have that

$$\frac{1}{2}X \left(\|\nabla f\|^2 \right) = \lambda X(f) - g(Q(\nabla f), X) = \lambda X(f) - \frac{1}{2}X(S), \quad X \in \mathfrak{X}(M),$$

which proves that $\frac{1}{2}(\|\nabla f\|^2 + S) - \lambda f = c$, where c is a constant. We can replace f by $f - \frac{c}{\lambda}$ to conclude that the potential function f of the gradient soliton $(M, g, \nabla f, \lambda)$ satisfies

$$(4.4) \quad 2\lambda f = \|\nabla f\|^2 + S,$$

which together with the second equation in (4.2) gives

$$(4.5) \quad \Delta f + 2\lambda f = n\lambda + \|\nabla f\|^2.$$

Let $k_0(n-1)$ be the infimum and $K_0(n-1)$ be the supremum of the Ricci curvatures of the compact Ricci soliton $(M, g, \nabla f, \lambda)$. As the Ricci curvature is positive, both numbers k_0 and K_0 are positive, multiplying equation (4.5) by $(n-1)k_0$ and $(n-1)K_0$ respectively and integrating the resulting equations, we arrive at

$$\begin{aligned} 2\lambda(n-1)k_0 \int_M f &\leq n\lambda(n-1)k_0 V(M) + \int_M Ric(\nabla f, \nabla f), \\ n\lambda(n-1)K_0 V(M) + \int_M Ric(\nabla f, \nabla f) &\leq 2\lambda(n-1)K_0 \int_M f, \end{aligned}$$

where $V(M)$ is the volume of M . Adding these two inequalities, we conclude that

$$n\lambda(n-1)(K_0 - k_0)V(M) \leq 2\lambda(n-1)(K_0 - k_0) \int_M f.$$

As the Ricci soliton $(M, g, \nabla f, \lambda)$ is shrinking and non-Einstein, we get that

$$f_{av} \geq \frac{n}{2}.$$

This proves the result. □

Finally we have the following:

Corollary 4.1. *Let $(M, g, \nabla f, \lambda)$ be an n -dimensional compact shrinking gradient Ricci soliton of positive Ricci curvature. If the average value of the potential function f satisfies $2f_{av} \leq n$, then M is an Einstein manifold.*

Proof. Integrating equation (4.5) and using

$$f_{av}V(M) = \int_M f,$$

we get

$$\int_M \|\nabla f\|^2 \leq 0,$$

which proves that $A = 0$ and consequently M is an Einstein manifold with Einstein constant λ . □

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