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A NOTE ON RIEMANN INTEGRABILITY

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ABSTRACT. In this note we define Riemann integrability for real valued functions defined on a compact metric space accompanied by a finite Borel measure. If the measure of each open ball equals the measure of its corresponding closed ball, then a bounded function is Riemann integrable if and only if its set of points of discontinuity has measure zero.

Let $\mathscr{A}$ denote the algebra of sets generated by the open and closed subintervals of an interval $[a,b]$. A bounded real valued function f defined on $[a,b]$ is Riemann integrable if for each positive ε, there exist two functions ϕ and ψ that are linear combinations of characteristic functions of sets in $\mathscr{A}$ satisfying $\phi \le f \le \psi$ and

$$\int_a^b \psi \, dm - \int_a^b \phi \, dm < \varepsilon$$

where m denotes ordinary Lebesgue measure. Riemann integrability may be defined in an analagous way for real valued functions defined on a compact metric space K accompanied by a finite Borel measure. If we make a simple assumption about the balls of K, then the following famous theorem of Lebesgue extends: a bounded real valued function f defined on $[a,b]$ is Riemann integrable if and only if the set of points at which f is not continuous has Lebesgue measure zero.

Suppose that K is a compact metric space and μ is a finite Borel measure on K. Let $B_r(x) = \{y: d(x,y) < r\}$ and $\overline{B}_r(x) = \{y: d(x,y) \le r\}$ denote the open and closed balls of radius r about a point x in K. Let $\mathscr{A}$ denote the algebra generated by all such balls. Any element of $\mathscr{A}$ is of the form

$$\bigcup_{1 \le i \le m} \ \bigcap_{1 \le k \le n_i} A_{ik} \qquad (1)$$

where A_{ik} is a ball or its complement and $\{m,n_1,\ldots,n_m\}$ are positive integers. A <u>step function</u> is a linear combination of characteristic functions determined by elements of $\mathscr{A}$. Hence a step function ϕ has the form $\Sigma\, d_i\chi_{A_i}$ where each d_i is real and $A_i \in \mathscr{A}$. Since $\mathscr{A}$ is an algebra, the $\{A_i\}$ can be taken to be pairwise disjoint. It is easy to see that if ϕ and ψ are step functions, then so are $\phi + \psi$, $\phi - \psi$, $\inf\{\phi,\psi\}$, and $\sup\{\phi,\psi\}$.

DEFINITION. A bounded real valued function f defined on K is <u>Riemann integrable</u> if for each positive ε there exist step functions ϕ and ψ such that $\phi \le f \le \psi$ and $\int \psi\, d\mu - \int \phi\, d\mu < \varepsilon$.

Given a bounded real valued function f defined on K, the <u>upper envelope</u> h of f is the function defined by

$$h(x) = \inf_{\delta>0} \sup_{y \in B_\delta(x)} f(y) \qquad x \in K$$

and the <u>lower envelope</u> g of f is defined by

$$g(x) = \sup_{\delta>0} \inf_{y \in B_\delta(x)} f(y) \qquad x \in K$$

It is well known that h is upper semicontinuous, g is lower semicontinuous, $g(x) \leq f(x) \leq h(x)$ for each x, and $g(x) = h(x)$ if and only if f is continuous at x (see Royden [1, p.49]).

THEOREM. Suppose $\mu(B_r(x)) = \mu(\overline{B}_r(x))$ for each x in K and for each positive r. A bounded real valued function f defined on K is Riemann integrable if and only if the set of points at which f is discontinuous has μ-measure zero.

<u>Proof</u>. Let h be the upper envelope of f and g its lower envelope. Let ψ be any step function that exceeds f. Since each member of $\mathscr{A}$ can be expressed in the form depicted in (1), the condition on the balls of K implies that each member of $\mathscr{A}$ is the union of an open set and a set of μ-measure zero. It follows that ψ can be represented as

$$\sum_{j=1}^{n} a_j \chi_{A_j}$$

where (i) A_j is an open set for $1 \leq j \leq m$ (ii) $\mu(A_j) = 0$ for $m < j \leq n$ (iii) $\{A_1, A_2, \ldots, A_n\}$ partition K.

Let $x \in \bigcup_{j=1}^{m} A_j$. Since ψ is constant near x, there exists $\delta > 0$ such that $\psi(x) \geq \sup_{y \in B_\delta(x)} f(y)$ so that $\psi(x) \geq h(x)$. Hence, $\mu\{x\colon \psi(x) < h(x)\} = 0$, and we have $\int \psi \, d\mu \geq \int h \, d\mu$. We now construct a decreasing sequence of step functions converging pointwise to h so that

$\inf \{\int \psi \, d\mu : \psi \geq f$ and ψ is a step function$\} = \int h \, d\mu$.

Let N be a fixed positive integer. Let $\{B_{r_1}(x_1), \ldots, B_{r_m}(x_m)\}$ be a cover of K by balls of radius at most $1/N$ such that if $y \in B_{r_i}(x_i)$, then $h(y) < h(x_i) + 1/N$. Now let $\theta_N : K \to R$ be the step function described by $\theta_N(x) = \inf \{h(x_i) + 1/N : x \in B_{r_i}(x_i)\}$. Define ψ_N to be θ_N. Given any positive integer p, define θ_{N+p} as above, and let ψ_{N+p} be $\inf \{\theta_{N+p}, \psi_{N+p-1}\}$. Clearly, for each p ψ_{N+p} is a step function, and $\psi_{N+p} \geq \psi_{N+p+1} \geq h$. To establish the pointwise convergence, suppose to the contrary that for some x_0 in K and $\varepsilon > 0$ we have for each p

$$\psi_{N+p}(x_0) > h(x_0) + 2\varepsilon$$

Pick n so large that $1/n < \varepsilon$. There exists a point x_n such that $d(x_0, x_n) < 1/n$ and $\psi_n(x_0) \leq h(x_n) + 1/n$. Clearly, $h(x_n) > h(x_0) + \varepsilon$ which violates the upper semicontinuity of h. Hence, $\{\psi_n\}$ is the desired sequence.

Using the above technique we can show in the same manner that $\int g \, d\mu = \sup \{\int \phi \, d\mu : \phi \leq f$ and ϕ is a step function$\}$. The proof is now completed by observing the equivalence of the following statements: (i) f is Riemann integrable (ii) $\int g \, d\mu = \int h \, d\mu$ (iii) f is continuous except at a set of points of μ-measure zero.

A simple example shows that the theorem need not hold if our condition on the balls of the metric space is omitted. Let K be the closed unit disc in the plane with the usual metric. If B is a Borel subset of K, define $\mu(B)$ to be $\mu_1(B \cap \{(x,y) : x^2 + y^2 = 1\}) + \mu_2(B \cap \{(x,y) : x^2 + y^2$

< 1} where μ_2 is two dimensional Lebesgue measure and μ_1 is one dimensional Lebesgue measure, considering the circle as having measure 2π. Then the characteristic function of the unit circle is Riemann integrable (being a step function), but its set of discontinuities has measure 2π.

REFERENCES

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