

## A NOTE ON RIEMANN INTEGRABILITY

**G. A. BEER**

Department of Mathematics  
California State University  
Los Angeles, California 90032

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ABSTRACT. In this note we define Riemann integrability for real valued functions defined on a compact metric space accompanied by a finite Borel measure. If the measure of each open ball equals the measure of its corresponding closed ball, then a bounded function is Riemann integrable if and only if its set of points of discontinuity has measure zero.

Let  $\mathcal{A}$  denote the algebra of sets generated by the open and closed subintervals of an interval  $[a,b]$ . A bounded real valued function  $f$  defined on  $[a,b]$  is Riemann integrable if for each positive  $\epsilon$ , there exist two functions  $\phi$  and  $\psi$  that are linear combinations of characteristic functions of sets in  $\mathcal{A}$  satisfying  $\phi \leq f \leq \psi$  and

$$\int_a^b \psi \, dm - \int_a^b \phi \, dm < \epsilon$$

where  $m$  denotes ordinary Lebesgue measure. Riemann integrability may be defined in an analagous way for real valued functions defined on a compact metric space  $K$  accompanied by a finite Borel measure. If we make a simple

assumption about the balls of  $K$ , then the following famous theorem of Lebesgue extends: a bounded real valued function  $f$  defined on  $[a,b]$  is Riemann integrable if and only if the set of points at which  $f$  is not continuous has Lebesgue measure zero.

Suppose that  $K$  is a compact metric space and  $\mu$  is a finite Borel measure on  $K$ . Let  $B_r(x) = \{y: d(x,y) < r\}$  and  $\bar{B}_r(x) = \{y: d(x,y) \leq r\}$  denote the open and closed balls of radius  $r$  about a point  $x$  in  $K$ . Let  $\mathcal{A}$  denote the algebra generated by all such balls. Any element of  $\mathcal{A}$  is of the form

$$\bigcup_{1 \leq i \leq m} \bigcap_{1 \leq k \leq n_i} A_{ik} \quad (1)$$

where  $A_{ik}$  is a ball or its complement and  $\{m, n_1, \dots, n_m\}$  are positive integers. A step function is a linear combination of characteristic functions determined by elements of  $\mathcal{A}$ . Hence a step function  $\phi$  has the form  $\sum d_i \chi_{A_i}$  where each  $d_i$  is real and  $A_i \in \mathcal{A}$ . Since  $\mathcal{A}$  is an algebra, the  $\{A_i\}$  can be taken to be pairwise disjoint. It is easy to see that if  $\phi$  and  $\psi$  are step functions, then so are  $\phi + \psi$ ,  $\phi - \psi$ ,  $\inf\{\phi, \psi\}$ , and  $\sup\{\phi, \psi\}$ .

DEFINITION. A bounded real valued function  $f$  defined on  $K$  is Riemann integrable if for each positive  $\epsilon$  there exist step functions  $\phi$  and  $\psi$  such that  $\phi \leq f \leq \psi$  and  $\int \psi d\mu - \int \phi d\mu < \epsilon$ .

Given a bounded real valued function  $f$  defined on  $K$ , the upper envelope  $h$  of  $f$  is the function defined by

$$h(x) = \inf_{\delta > 0} \sup_{y \in B_\delta(x)} f(y) \quad x \in K$$

and the lower envelope  $g$  of  $f$  is defined by

$$g(x) = \sup_{\delta > 0} \inf_{y \in B_\delta(x)} f(y) \quad x \in K$$

It is well known that  $h$  is upper semicontinuous,  $g$  is lower semicontinuous,  $g(x) \leq f(x) \leq h(x)$  for each  $x$ , and  $g(x) = h(x)$  if and only if  $f$  is continuous at  $x$  (see Royden [1, p.49]).

**THEOREM.** Suppose  $\mu(B_r(x)) = \mu(\bar{B}_r(x))$  for each  $x$  in  $K$  and for each positive  $r$ . A bounded real valued function  $f$  defined on  $K$  is Riemann integrable if and only if the set of points at which  $f$  is discontinuous has  $\mu$ -measure zero.

Proof. Let  $h$  be the upper envelope of  $f$  and  $g$  its lower envelope. Let  $\psi$  be any step function that exceeds  $f$ . Since each member of  $\mathcal{A}$  can be expressed in the form depicted in (1), the condition on the balls of  $K$  implies that each member of  $\mathcal{A}$  is the union of an open set and a set of  $\mu$ -measure zero. It follows that  $\psi$  can be represented as

$$\sum_{j=1}^n a_j \chi_{A_j}$$

where (i)  $A_j$  is an open set for  $1 \leq j \leq m$  (ii)  $\mu(A_j) = 0$  for  $m < j \leq n$  (iii)  $\{A_1, A_2, \dots, A_n\}$  partition  $K$ .

Let  $x \in \bigcup_{j=1}^m A_j$ . Since  $\psi$  is constant near  $x$ , there exists  $\delta > 0$  such that  $\psi(x) \geq \sup_{y \in B_\delta(x)} f(y)$  so that  $\psi(x) \geq h(x)$ . Hence,  $\mu\{x: \psi(x) < h(x)\} = 0$ , and we have  $\int \psi \, d\mu \geq \int h \, d\mu$ . We now construct a decreasing sequence of step functions converging pointwise to  $h$  so that

$\inf \{ \int \psi \, d\mu : \psi \geq f \text{ and } \psi \text{ is a step function} \} = \int h \, d\mu.$

Let  $N$  be a fixed positive integer. Let  $\{B_{r_1}(x_1), \dots, B_{r_m}(x_m)\}$  be a cover of  $K$  by balls of radius at most  $1/N$  such that if  $y \in B_{r_i}(x_i)$ , then  $h(y) < h(x_i) + 1/N$ . Now let  $\theta_N: K \rightarrow \mathbb{R}$  be the step function described by  $\theta_N(x) = \inf \{h(x_i) + 1/N : x \in B_{r_i}(x_i)\}$ . Define  $\psi_N$  to be  $\theta_N$ . Given any positive integer  $p$ , define  $\theta_{N+p}$  as above, and let  $\psi_{N+p}$  be  $\inf \{\theta_{N+p}, \psi_{N+p-1}\}$ . Clearly, for each  $p$   $\psi_{N+p}$  is a step function, and  $\psi_{N+p} \geq \psi_{N+p+1} \geq h$ . To establish the pointwise convergence, suppose to the contrary that for some  $x_0$  in  $K$  and  $\epsilon > 0$  we have for each  $p$

$$\psi_{N+p}(x_0) > h(x_0) + 2\epsilon$$

Pick  $n$  so large that  $1/n < \epsilon$ . There exists a point  $x_n$  such that  $d(x_0, x_n) < 1/n$  and  $\psi_n(x_0) \leq h(x_n) + 1/n$ . Clearly,  $h(x_n) > h(x_0) + \epsilon$  which violates the upper semicontinuity of  $h$ . Hence,  $\{\psi_n\}$  is the desired sequence.

Using the above technique we can show in the same manner that  $\int g \, d\mu = \sup \{ \int \phi \, d\mu : \phi \leq f \text{ and } \phi \text{ is a step function} \}$ . The proof is now completed by observing the equivalence of the following statements:  
 (i)  $f$  is Riemann integrable (ii)  $\int g \, d\mu = \int h \, d\mu$  (iii)  $f$  is continuous except at a set of points of  $\mu$ -measure zero.

A simple example shows that the theorem need not hold if our condition on the balls of the metric space is omitted. Let  $K$  be the closed unit disc in the plane with the usual metric. If  $B$  is a Borel subset of  $K$ , define  $\mu(B)$  to be  $\mu_1(B \cap \{(x,y) : x^2 + y^2 = 1\}) + \mu_2(B \cap \{(x,y) : x^2 + y^2 < 1\})$

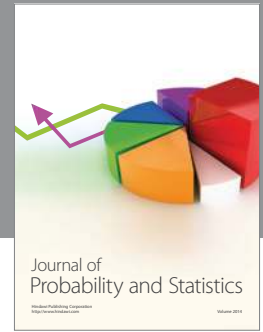
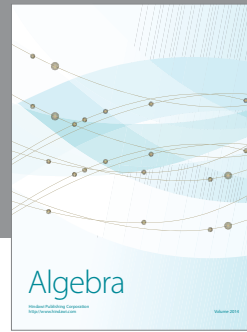
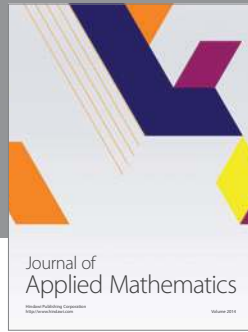
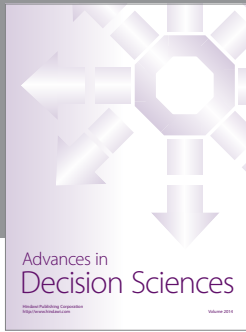
$< 1 \}$  where  $\mu_2$  is two dimensional Lebesgue measure and  $\mu_1$  is one dimensional Lebesgue measure, considering the circle as having measure  $2\pi$ . Then the characteristic function of the unit circle is Riemann integrable (being a step function), but its set of discontinuities has measure  $2\pi$ .

#### REFERENCES

1. H. L. Royden. Real Analysis, Macmillan, New York, 1968.

KEY WORDS AND PHRASES. *Riemann integrable functions on a compact metric space, Compact metric space with Borel measure.*

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