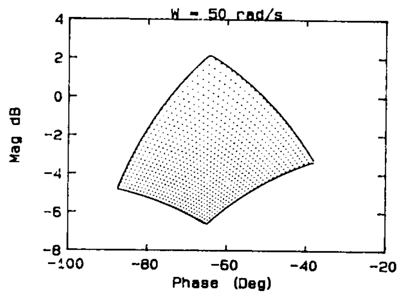

Fig. 4. Template boundary for example: $\omega = 50$ rad/s.

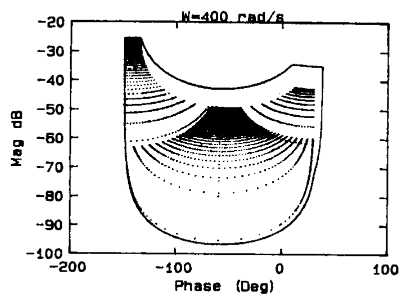

Fig. 5. Template boundary for example: $\omega = 400$ rad/s.

TABLE I
COMPUTATION TIMES FOR THE EXAMPLE

Template	Fast Algorithm	Grid Method
$\omega = 50$	10 s	40 min (900 points)
$\omega = 400$	10 s	60 min (3780 points)

Two frequency response template boundaries for (3.3) generated using the above algorithms are shown in Figs. 4 and 5 in Nichols format. To check the accuracy and speed of the new algorithm, the interiors of the boundaries shown in Figs. 4 and 5 have been filled with points generated using the grid method described earlier. Due to parameter dependence between q and r (i.e., because $q = r$) the generated boundaries are only outer bound approximations to the true templates. However, in this case approximation is seen to be quite good. (For a detailed discussion of this difference, see [4].)

The new algorithm was found to be quite fast. Rough computation times for this algorithm as applied to the above example are given in Table I. Approximate computation times using the grid method are provided for comparison. All computation was done on an IBM PC/AT with math coprocessor. The fast algorithm was written in Microsoft Quick Basic 4.0. The grid method algorithm was implemented using PC Matlab 2.2.

V. Conclusions

A procedure for fast machine computation of parametric rational functions is described. The algorithm has been implemented in Microsoft quick Basic 4.0. As noted above, execution times are minimal.

Acknowledgment

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A Note on Robust Stability Bounds

M. E. SEZER AND D. D. ŠILJAK

Abstract—**The purpose of this note is to comment on stability robustness bounds for linear systems with multiple uncertain parameters, which were obtained via Lyapunov functions. Simple examples are used to illustrate the effects of the choice of functions and corresponding majorizations.**

There has been a considerable number of papers (see [1]–[4] and references therein) dealing with the analysis of robust stability under structured perturbations using quadratic Lyapunov functions. Similar studies have been carried out independently in the context of connective stability of large scale systems using vector Lyapunov functions [5], [6]. In this note, we compare the two approaches using simple examples, and in particular, comment on the choice of functions, majorizations, and system decompositions.

Let us consider a linear system

$$\mathcal{S}: \dot{x} = Ax + \sum_{k=1}^{K} p_k E_k x \tag{1}$$

where $x \in \mathcal{R}^n$, A is a constant stable matrix (i.e., all eigenvalues have negative real parts), E_k's are constant matrices, and p_k's are uncertain (possibly, time-varying) parameters.

To obtain bounds on $|p_k|$ for stability of $\mathcal{S}$ we choose a quadratic Lyapunov function

$$V(x) = x^T H x \tag{2}$$

where H is the positive definite solution of the equation

$$A^T H + HA = -G \tag{3}$$

for some positive definite matrix G.

We compute

$$\begin{aligned}\dot{V}(x) &= -x^T G x + x^T \left(\sum_{k=1}^{K} p_k F_k \right) x \\ &\leq -\left[1 - \sigma_M \left(\sum_{k=1}^{K} p_k G^{-1/2} F_k G^{-1/2} \right) \right] \|G^{1/2} x\| \end{aligned} \tag{4}$$

with respect to (1), where

$$F_k = E_k^T H + HE_k \tag{5}$$

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and $\sigma_M(\cdot)$ denotes the maximum singular value. From (4), a sufficient condition for stability of $\mathcal{S}$ is

$$\sigma_M\left(\sum_{k=1}^{K} p_k G^{-1/2} F_k G^{-1/2}\right) < 1. \tag{6}$$

The standard choice of G is $\bar{G} = I$ which is shown [1] to maximize the estimate $\sigma_m(G)/2\sigma_M(H)$ of the degree of stability of A, where $\sigma_m(\cdot)$ is the minimum singular value. This choice of $\bar{G}$ in (3) produces $\bar{H}$ and the best available robustness bounds [3] which can be obtained from (6) are

$$\sum_{k=1}^{K} \sigma_M(\bar{F}_k)|p_k| < 1$$

$$\max_k |p_k| < \sigma_M^{-1}\left(\sum_{k=1}^{K} |\bar{F}_k|\right)$$

$$\left(\sum_{k=1}^{K} p_k^2\right)^{1/2} < \sigma_M^{-1/2}\left(\sum_{k=1}^{K} \bar{F}_k^2\right) \tag{7}$$

where the matrices $\bar{F}_k$ are obtained from (5) for $H = \bar{H}$. In (7), $|\cdot|$ denotes a matrix formed by taking absolute values of the elements of the indicated matrix. The three bounds (7) correspond to the three standard norms: diamond, parallelepiped, and sphere, respectively.

Several comments concerning the bounds (7) are as follows.

1) The choice $\bar{G} = I$ is not the best to use in (6). Leaving G free, we obtain from (6) the bounds

$$\sum_{k=1}^{K} \sigma_M(G^{-1/2} F_k G^{-1/2})|p_k| < 1$$

$$\max_k |p_k| < \sigma_M^{-1}\left(\sum_{k=1}^{K} |G^{-1/2} F_k G^{-1/2}|\right)$$

$$\left(\sum_{k=1}^{K} p_k^2\right)^{1/2} < \sigma_m^{1/2}(G)\sigma_M^{-1/2}\left(\sum_{k=1}^{K} G^{-1/2} F_k^2 G^{-1/2}\right). \tag{8}$$

Using additional majorizations in (8) we can make G appear explicitly as

$$\sum_{k=1}^{K} \sigma_M(F_k)|p_k| < \sigma_m(G)$$

$$\max_k |p_k| < \sigma_m(G)\sigma_M^{-1}\left(\sum_{k=1}^{K} |F_k|\right)$$

$$\left(\sum_{k=1}^{K} p_k^2\right)^{1/2} < \sigma_m(G)\sigma_M^{-1/2}\left(\sum_{k=1}^{K} F_k^2\right). \tag{9}$$

Obviously, the bounds (9) are less conservative than those of (7). We illustrate this fact by a simple example.

Let $K = 2$, and

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, E_1 = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}. \tag{10}$$

The three sets of bounds on uncertain parameters are computed as

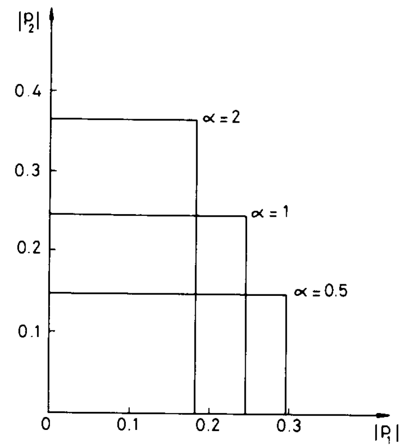

Fig. 1. Effect of scaling on the rectangular robustness region of the system (10).

follows. For $\bar{G} = I$ we get from (7)

$$2.8028|p_1| + 1.6180|p_2| < 1$$

$$\max\{|p_1|, |p_2|\} < 0.2469$$

$$(p_1^2 + p_2^2)^{1/2} < 0.3159. \tag{11}$$

Choosing

$$G = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \tag{12}$$

we calculate

$$2.5275|p_1| + 1.5000|p_2| < 1$$

$$\max\{|p_1|, |p_2|\} < 0.2490$$

$$(p_1^2 + p_2^2)^{1/2} < 0.3378 \tag{13}$$

using (8), and

$$2.5616|p_1| + 1.5000|p_2| < 1$$

$$\max\{|p_1|, |p_2|\} < 0.2478$$

$$(p_1^2 + p_2^2)^{1/2} < 0.3369 \tag{14}$$

using (9). Expectedly, the bounds (13) and (14) are better than (11), with (14) slightly worse than (13).

2) While a choice of G affects the size of robustness regions in the parameter space, the shape of the regions defined by the last two inequalities in each set of bounds can be controlled by scaling the uncertain parameters p_k and the corresponding matrices E_k. Letting

$$p_k' = p_k/\alpha_k, \; E_k' = \alpha_k E_k \tag{15}$$

where $\alpha_k > 0$ and $\sum_{k=1}^{K} \alpha_k = 1$, the last two bounds in (8) become

$$\max_k |p_k/\alpha_k| < \sigma_M^{-1}\left(\sum_{k=1}^{K} \alpha_k |G^{-1/2} F_k G^{-1/2}|\right)$$

$$\left(\sum_{k=1}^{K} p_k^2/\alpha_k^2\right)^{1/2} < \sigma_m^{1/2}(G)\sigma_M^{-1/2}\left(\sum_{k=1}^{K} \alpha_k^2 G^{-1/2} F_k^2 G^{-1/2}\right). \tag{16}$$

To demonstrate the use of scaling we consider the same example (10) with $\bar{G} = I$. Letting $\alpha = \alpha_2/\alpha_1$, we use (16) to get

$$\max\{|p_1|, |p_2|/\alpha\} < 2/[2 + \alpha + (13 + 8\alpha + 5\alpha^2)^{1/2}]$$

$$p_1^2 + p_2^2/\alpha^2 < 4/[17 + 6\alpha^2 + (208 + 64\alpha^2 + 20\alpha^4)^{1/2}]. \quad (17)$$

The effect of scaling on the rectangular robustness region of the system in (10) corresponding to the second inequality in (11) is shown in Fig. 1 for three different choices of α.

3) We can further reduce the conservativeness of robustness bounds by introducing more flexibility in the majorization process. For a nonsingular matrix T the inequality in (4) can be replaced by

$$\dot{V}(x) \leq -\left[\sigma_m(T^T GT) - \sigma_M\left(\sum_{k=1}^{K} p_k T^T F_k T\right)\right] \|T^{-1}x\|. \quad (18)$$

The corresponding stability condition becomes

$$\sigma_M\left(\sum_{k=1}^{K} p_k T^T F_k T\right) < \sigma_m(T^T GT) \quad (19)$$

which reduces to (6) when $T = G^{-1/2}$. Additional flexibility introduced by the matrix T is equivalent to the flexibility provided by a coordinate change of the state space of $\mathcal{S}$. Obviously, a transformation $x = T\tilde{x}$ used in [3], [4], and [7] amounts to a choice of $\tilde{G} = T^T GT$ in the new coordinate frame. In [4], $\tilde{G}$ is fixed as I and a diagonal T was searched which produces the best bounds. It is clear that with the restriction of $\tilde{G} = I$ any improvement due to a diagonal transformation can be captured by a proper choice of a diagonal G in the original coordinates. In fact, no transformation is necessary if one does not restrict the choice of G.

4) From our experience in vector Lyapunov functions [6], we know that the majorization process can be improved by exploiting the special structural features of $\mathcal{S}$. When A has, or can be transformed into, a block-diagonal structure, then the perturbations appear as interconnections among the blocks.

We assume that $\mathcal{S}$ is decomposed into N interconnected subsystems as

$$\mathcal{S}: \dot{x}_i = A_i x_i + \sum_{j=1}^{N} A_{ij} x_j, \qquad i = 1, 2, \cdots, N \quad (20)$$

where $x_i \in \mathcal{R}^{n_i}$, $\sum_i^N n_i = n$, A_i are constant stable matrices, and the interconnection matrices are represented as

$$A_{ij} = \sum_{k=1}^{K} p_k E_k^{ij}. \quad (21)$$

We again use a quadratic Lyapunov function

$$v_i(x_i) = x_i^T H_i x_i \quad (22)$$

for the subsystem

$$\mathcal{S}_i: \dot{x}_i = A_i x_i \quad (23)$$

where H_i is the positive definite solution of the equation

$$A_i^T H_i + H_i A_i + -G_i \quad (24)$$

for some positive definite G_i.

A standard Lyapunov function [8] for $\mathcal{S}$ is

$$V(x) = \sum_{i=1}^{N} d_i v_i(x_i) \quad (25)$$

where $d_i > 0$.

We compute with respect to $\mathcal{S}$ the derivative

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^{N} d_i \left(-x_i^T G_i x_i + 2x_i^T H_i \sum_{j=1}^{N} A_{ij} x_j\right) \\ &\leq -\sum_{i=1}^{N} d_i \Bigg\{ \|G_i^{1/2} x_i\| - 2\|G_i^{1/2} x_i\| \\ &\quad \cdot \sum_{j=1}^{N} \sigma_M(G_i^{-1/2} H_i A_{ij} G_j^{-1/2}) \|G_j^{1/2} x_j\| \Bigg\} \\ &= u^T(x)(W^T D + DW)u(x) \end{aligned} \quad (26)$$

where $u(x) = (\|G_1^{1/2}x_1\|, \|G_2^{1/2}x_2\|, \cdots, \|G_N^{1/2}x_N\|)^T$, $D = \text{diag}\{d_1, d_2, \cdots, d_N\}$, and the elements of the aggregate matrix $W = (w_{ij})$ are defined as

$$w_{ij} = \begin{cases} 1 - 2\xi_{ii}, & i = j \\ -2\xi_{ij}, & i \neq j \end{cases} \quad (27)$$

and

$$\begin{aligned} \xi_{ij} &= \sigma_M(G_i^{-1/2} H_i A_{ij} G_j^{-1/2}) \\ &= \sigma_M\left(\sum_{k=1}^{K} p_k G_i^{-1/2} H_i E_k^{ij} G_j^{-1/2}\right). \end{aligned} \quad (28)$$

A sufficient condition for stability of $\mathcal{S}$ is that W be an M-matrix [7], which corresponds to inequality (6) when no decompositions are used. In fact, we can use the three norms of (1) to majorize further ξ_{ij}'s and extract the uncertain parameters p_k as in (8).

To illustrate the use of vector Lyapunov functions we use the same example (10). This example is interesting because we can eliminate the effect of the choice of G_i's, and thus attribute any gain in robustness bounds to the vector approach. The aggregate matrix is obtained as

$$W = \begin{bmatrix} 1 - 2|p_1| & -2|p_2| \\ -3|p_1| & 1 - |p_2| \end{bmatrix} \quad (29)$$

which is an M-matrix if and only if

$$2|p_1| + |p_2| + 4|p_1||p_2| < 1. \quad (30)$$

The region determined by inequality (30) turns out to be the *exact* robustness region for the system (10). Of course, this kind of result should not be expected in more complex cases where the choice of G_i's may be crucial in obtaining the least conservative robustness bounds. We should note, however, that in the case of a single uncertain parameter, the exact estimates of robustness bounds are available by an entirely different approach relying on characteristic polynomials [9].

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Further Comments on "A Multivariable Generalized Self-Tuning Controller with Decoupling Design"

QUAN-GEN ZHOU

Abstract—**Results for updating the weighting matrices presented in the paper[1] are shown to be problematic and generally difficult to use.**

In the recent note [1], it was pointed out that the weighting matrices used in the paper[1] for nonminimum phase systems would result in cancellation of the process zeros. This difficulty may be avoided in case perfect dynamic decoupling is no longer pursued [2]. However, this note is to show that whether for nonminimum phase systems or for minimum phase systems, the results presented in the paper[1] are not well-founded and generally difficult to use. The notation adopted here is the same as in the paper.[1]

From the Appendix of the paper,[1] it can be seen that as a prerequisite for the derivation of the k-step-ahead predictor, the roots of det $\tilde{C}(a)$ should lie outside the unit disk. In another respect, substituting the control law into the controlled system model and using (A.6) and (A.7) in the paper,[1] the closed-loop system is determined by the following equation:

$$\begin{aligned}&\tilde{C}(z^{-1})(P(z^{-1})+Q(z^{-1})B^{-1}(z^{-1})A(z^{-1}))y(t)\\&=\tilde{C}(z^{-1})R(z^{-1})w(t-k)+\tilde{C}(z^{-1})(F(z^{-1})\\&+Q(z^{-1})B^{-1}(z^{-1})C(z^{-1}))\xi(t).\end{aligned} \tag{1}$$

This implies that for the closed-loop system to be stable, the roots of det $\tilde{C}(a)$ should again lie outside the unit disk. Now making use of (A.7), (A.8), (A.4), and (A.2) in the paper,[1] we know

$$\det \tilde{C}(a)=\det C(a) \det P(a). \tag{2}$$

Since the roots of det $C(a)$ have been assumed to lie outside the unit disk, we therefore conclude that the roots of det $P(a)$ should lie outside the unit disk. However, when the techniques of Lang *et al.* in the paper[1] are carried out, the weighting matrix $P(z^{-1})$ is uniquely determined by (9) or (12) in the paper.[1] This obviously does not necessarily lead to satisfying the above requirement on $P(z^{-1})$, and most likely is the opposite, i.e., the roots of det $P(a)$ may be located inside the unit disk. In that case, the whole self-tuning algorithm can never work well and the closed-loop stability is certainly spoiled. From the above argument, it seems to us that the results presented in the paper[1] are problematic and may be difficult to use.

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Comments on "Tracking Control of a Flexible Robot Link"

ENRIQUE BARBIERI

Abstract—**In the paper[1] the model of a flexible slewing link is derived. The authors assert that when the equations of motion are written relative to a "desired coordinate frame," then a substantial linearization of the equations results. In this note, we raise some questions regarding the validity of that claim and show that in fact the coupling between the transverse displacement of the beam and the servomotor angle was not adequately accounted for.**

Consider the flexible slewing beam with a (point) mass m at the tip shown in Fig. 1. Three coordinate frames are defined:

$\{I\}$ =: inertial reference frame,

$\{D\}$ =: desired reference frame,[1]

$\{L\}$ =: local reference frame

where $u(x, t)$ and $y(s, t)$ are the transverse displacements of a point P (centroid of beam's cross section) on the beam relative to frames $\{D\}$ and $\{L\}$, respectively; $\theta(t)$ is the servomotor angle; $\theta_d(t)$ is the desired tracking angle; and $f_b(t)$ is the torque applied to the beam. The beam has length L, mass per unit length ρ, and bending stiffness product EI where E is Young's modulus of elasticity and I is the beam's area moment of inertia about the bending axis. The link rotates in a horizontal plane so that gravity effects are ignored.

The following standard assumptions are made. 1) The angular velocity is small so that the axial energy is neglected. 2) Beam displacements and strains are small. 3) The cross-sectional dimensions of the beam are much smaller than its length to neglect shear and rotatory inertia effects [1]. Also, to arrive at the model in the paper,[1] we include a term that introduces the centrifugal stiffening effect [2]. To simplify our analysis we initially do not include the mass at the tip of the beam and the structural damping term in the paper.[1] This, however, does not affect the argument presented below.

We write the system kinetic energy and work function expressions and use the extended Hamilton principle [2] to derive the following pair of nonlinear coupled integrodifferential equations and associated boundary conditions:

$$y_{tt}=-\frac{EI}{\rho}y_{ssss}+\left[y+\frac{\partial}{\partial s}\left(\frac{L^2-s^2}{2}y_s\right)\right]\dot{\theta}^2-s\ddot{\theta} \tag{1}$$

$$(J_b+J_h)\ddot{\theta}+\rho\int_0^L\left[sy_{tt}+\frac{d}{dt}\left[\left(y^2-\frac{L^2-s^2}{2}y_s^2\right)\dot{\theta}\right]\right]ds=f_b \tag{2}$$

$$y(0, t)=y_s(0, t)=y_{ss}(L, t)=y_{sss}(L, t)=0$$

where J_b is the beam's mass moment of inertia about the hub and J_h is a total rigid hub inertia.

The equations of motion in the $\{D\}$ coordinate frame are obtained by simply replacing $\theta(t)$ with $\theta_d(t)$ and $y(s, t)$ with $u(x, t)$. The boundary conditions are

$$u(0, t)=u_{xx}(L, t)=u_{xxx}(L, t)=0$$

$$u_x(0, t)=\tan[\theta(t)-\theta_d(t)].$$

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