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A NOTE ON SOFT NEAR-RINGS AND IDEALISTIC SOFT NEAR-RINGS

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Abstract

Molodtsov introduced the theory of soft sets, which can be seen as an effective mathematical tool to deal with uncertainties, since it is free from the difficulties that the usual theoretical approaches have troubled. In this paper, we apply the definitions proposed by Ali et al. [M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, *On some new operations in soft set theory*, Comput. Math. Appl. 57 (2009), 1547–1553] to the concept of soft near-rings and substructures of soft near-rings, proposed by Atagün and Sezgin [A. O. Atagün and A. Sezgin, *Soft Near-rings*, submitted] and show them with illustrating examples. Moreover, we investigate the properties of idealistic soft near-rings with respect to the near-ring mappings and we show that the structure is preserved under the near-ring epimorphisms. Main purpose of this paper is to extend the study of soft near-rings from a theoretical aspect.

1 Introduction

In order to solve miscellaneous problems in environmental science, medical science, social science, economics and engineering, methods in classical mathematics may not be successfully used because of various uncertainties arising in these fields. Thus, there have been a great deal of research and applications in the literature concerning some special tools like probability theory, fuzzy set theory [21,22], rough set theory [15,16], vague set theory [9], interval mathematics [5,10], and intuitionistic fuzzy set theory [6,23]. But each of these theories has its advantages as well as limitations in dealing with uncertainties as mentioned by Molodtsov [14]. In 1999, Molodtsov introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties. It associates a set with a set of parameters and thus free from the difficulties effecting existing methods. Soft set theory has rich potential applications most of which have already been demonstrated by Molodtsov [14] in many fields such

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as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory.

Soft set theory has attracted much attention since its introduction by Molodtsov. Maji et al. [13] investigated the applications of soft set theory to a decision making problem. Chen et al. [7] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. Kong et al. [11] further studied the problem of the reduction of soft sets and fuzzy soft sets by introducing a definition for normal parameter reduction. In theoretical aspects, Maji et al. [12] defined and studied several operations on soft sets. But some of the definitions, such as the intersection of two soft sets, in [12] was realized to be a not clearly defined notion, which suffers from many problems; hence some related critiques and additional properties of soft sets was investigated in [3,20]. Ali et al. [2] gave some new notions of soft sets and Sezgin and Atagün [18] discussed the basic properties of operations on soft sets such as intersection, extended intersection, restricted union, restricted difference defined in [2] and they illustrated their interconnections between each other. Soft set theory has continued to experience tremendous growth in the mean of algebraic structures since Aktaş and Çağman [1] defined and studied soft groups, soft subgroups, normal soft subgroups, soft homomorphisms, adopting the definition of soft sets in [14]. Sezgin and Atagün [19] corrected some of the problematic cases in [1], introduced the concepts of normalistic soft group and normalistic soft group homomorphism. Applying the definition of soft set, Atagün and Sezgin [3] introduced and studied soft subrings and soft ideals of a ring, soft subfields of a field and soft submodule of a left R-module. Feng et al. [8] introduced and investigated soft semirings and soft semiring homomorphism. Atagün and Sezgin [4] introduced the notions of soft near-rings, soft sub-nearrings, soft (left, right) ideals, (left, right) idealistic soft near-rings and soft near-ring homomorphisms with corresponding examples.

In this paper, we further extend the study of soft near-rings. In particular, we investigate the properties of idealistic soft near-rings with respect to near-ring epimorphisms by corresponding examples.

2 Preliminaries

By a near-ring, we shall mean an algebraic system $(N, +, .)$, where

- $(N, +)$ forms a group (not necessarily abelian)
- $(N, .)$ forms a semi-group and
- $(a + b)c = ac + bc$ for all $a, b, c \in N$ (i.e. we study on right near-rings.)

Throughout this paper, N will always denote a right near-ring. A subgroup M of N with $MM \subseteq M$ is called a subnear-ring of N. A normal subgroup I of N is called a right ideal if $IN \subseteq I$ and denoted by $I \lhd_r N$. It is called a left ideal if $n(s + i) - ns \in I$ for all $n, s \in N$ and $i \in I$ and denoted by $I \lhd_\ell N$. If such a normal subgroup I is both left and right ideal in N, then it is called an ideal in

N and denoted by $I \triangleleft N$. Homomorphisms, epimorphisms, isomorphisms etc. for near-rings are defined in the usual way. For all undefined concepts and notions we refer to Pilz [17].

Molodtsov [14] defined the soft set in the following manner:

Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$.

Definition 2.1 ([14]) A pair (F, A) is called a soft set over U, where F is a mapping given by

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-elements of the soft set (F, A), or as the set of ε-approximate elements of the soft set. To illustrate this idea, Molodtsov investigate several examples in [14]. These examples were also discussed in [1,12].

Maji et al. [12] studied the theoretical aspect of soft set and introduced several binary operations such as intersection, union, AND-operation, and OR-operation of soft sets and investigated them in more detail. Feng et al. [8] defined the bi-intersection of two soft sets as an alternative to the definition of soft sets intersection by Maji et al. [12]. Based on analysis of several operations on soft sets introduced in [12], Ali et al. [2] introduced some new operations on soft sets which we apply to the concept of soft near-rings throughout this paper.

Definition 2.2 ([12]) The *intersection* of two soft sets (F, A), (G, B) over a common universe set U is the soft set (H, C), where $C = A \cap B$, and $\forall e \in C$, $H(e) = F(e)$ or $G(e)$, (as both are the same set). We write $(F, A)\widetilde{\cap}(G, B) = (H, C)$.

In contrast with the above definition of soft set intersection, Feng et al. [8] alternatively defined the following binary operation, called *bi-intersection* of two soft sets.

Definition 2.3 ([8]) The bi-intersection of two soft sets (F, A) and(G, B) over a common universe U is defined to be the soft set (H, C), where $C = A \cap B$ and $H : C \to P(U)$ is a mapping given by $H(x) = F(x) \cap G(x)$ for all $x \in C$. This is denoted by $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$.

In [3,19], it is pointed out that *intersection* of two soft sets is not a well-defined notion, which makes it impossible to check the validity of the equalities for some of the assertions in [12]. Therefore, Ali et al. [2] introduced two new definitions for intersection, called *extended intersection* and *the restricted intersection* as following:

Definition 2.4 ([2]) Let (F, A) and (G, B) be two soft sets over a common universe U. The *extended intersection* of (F, A) and (G, B) is defined to be the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \left\{ \begin{array}{ll} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cap G(e) & \text{if } e \in A \cap B. \end{array} \right\}$$

This relation is denoted by $(F, A) \sqcap_{\varepsilon} (G, B) = (H, C)$.

Definition 2.5 ([2]) Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The *restricted intersection* of (F, A) and (G, B) is denoted by $(F, A) \Cap (G, B)$, and is defined as $(F, A) \Cap (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.

Definition 2.6 ([12]) If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) *AND* (G, B)" denoted by $(F, A) \wedge (G, B)$ is defined by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$.

Definition 2.7 ([12]) If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) *OR* (G, B)" denoted by $(F, A) \vee (G, B)$ is defined by $(F, A) \vee (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cup G(y)$ for all $(x, y) \in A \times B$.

Definition 2.8 ([12]) Let (F, A) and (G, B) be two soft sets over a common universe U. The *union* of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions: (i) $C = A \cup B$; (ii) for all $e \in C$,

$$H(e) = \left\{ \begin{array}{ll} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{array} \right\}$$

As a generalization, Feng et al. [8] introduced the below definition for soft sets in the following way:

Definition 2.9 ([8]) Let $(F_i, A_i)_{i \in I}$ be a nonempty family of soft sets over a common universe U. The union of these soft sets is defined to be the soft set (G, B) such that $B = \bigcup_{i \in I} A_i$ and for all $x \in B$, $G(x) = \bigcup_{i \in I(x)} F_i(x)$ where $I(x) = \{i \in I | x \in A_i\}$. In this case we write $\widetilde{\bigcup}_{i \in I}(F_i, A_i) = (G, B)$.

Definition 2.10 ([8]) Let $(F_i, A_i)_{i \in I}$ be a nonempty family of soft sets over a common universe set U. The *AND-soft set* $\bigwedge_{i \in I}(F_i, A_i)$ of these soft sets is defined to be the soft set (H, B) such that $B = \prod_{i \in I} A_i$ and $H(x) = \bigcap_{i \in I(x)} F_i(x)$ for all $x = (x_i)_{i \in I} \in B$.

Note that if $A_i = A$ and $F_i = F$ for all $i \in I$, then $\bigwedge_{i \in I}(F_i, A_i)$ is denoted by $\bigwedge_{i \in I}(F, A)$. In this case, $\prod_{i \in I} A_i = \prod_{i \in I} A$ means the direct power A^I.

Definition 2.11 Let $(F_i, A_i)_{i\in I}$ be a nonempty family of soft sets over a common universe set U. The *restricted intersection* of these soft sets is defined to be the soft set (G, B) such that $B = \bigcap_{i\in I} A_i \neq \emptyset$ and for all $x \in B$, $G(x) = \bigcap_{i\in I(x)} F_i(x)$ for all $x \in B$. In this case we write $\Cap_{i\in I}(F_i, A_i) = (G, B)$.

From now on, let N be a near-ring and A be a nonempty set. R will refer to an arbitrary binary relation between an element of A and an element of N, that is, R is a subset of $A \times N$ without otherwise specified. A set-valued function $F : A \to P(N)$ can be defined as $F(x) = \{y \in N \mid (x, y) \in R\}$ for all $x \in A$. Then the pair (F, A) is a soft set over N, which is derived from the relation R.

The concept of a support is defined for both fuzzy sets and formal power series in the literature. A similar notion for soft sets is defined in [8]. For a soft set (F, A), the set $Supp(F, A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called the support of the soft set (F, A). The null soft set is a soft set with an empty support, and a soft set (F, A) is non-null if $Supp(F, A) \neq \emptyset$ [8].

In [4], Atagün and Sezgin defined the soft near-ring as following:

Definition 2.12 ([4]) Let (F, A) be a non-null soft set over a near-ring N. Then (F, A) is called a *soft near-ring* over N if $F(x)$ is a subnear-ring of N for all $x \in Supp(F, A)$.

Example 2.13 ([4]) Let us consider the additive group $(\mathbb{Z}_6, +)$. Under a multiplication defined by following table, $(\mathbb{Z}_6, +, .)$ is a (right) near-ring.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	2
5	3	5	1	3	5	1

Let (F, A) be a soft set over $\mathbb{Z}_6$, where $A = \mathbb{Z}_6$ and $F : A \to P(\mathbb{Z}_6)$ is a set-valued function defined by

$$F(x) = \{y \in \mathbb{Z}_6 \mid xRy \Leftrightarrow xy \in \{0, 3\}\}$$

for all $x \in A$. Then $F(0) = F(3) = \mathbb{Z}_6$ and $F(1) = F(2) = F(4) = F(5) = \{0, 3\}$ are subnear-rings of $\mathbb{Z}_6$. Hence (F, A) is a soft near-ring over $\mathbb{Z}_6$.
Let (G, A) be a soft set over $\mathbb{Z}_6$, where $G : A \to P(\mathbb{Z}_6)$ is defined by

$$G(x) = \{y \in \mathbb{Z}_6 \mid xy \in \{1, 2, 3\}\}$$

for all $x \in A$. Then $G(1) = \{0, 1, 3, 4\}$ is not a subnear-ring of $\mathbb{Z}_6$ and hence (G, A) is not a soft near-ring over $\mathbb{Z}_6$.

Definition 2.14 ([4]) Let (F, A) and (G, B) be soft near-rings over N. Then the soft near-ring (F, A) is called a *soft subnear-ring* of (G, B) if it satisfies:

i) $A \subset B$

ii) $F(x)$ is a subnear-ring of $G(x)$ for all $x \in Supp(F, A)$.

Proposition 2.15 ([4]) Let (F, A) and (G, A) be soft near-rings over N. Then we have the following:

a) If $F(x) \subset G(x)$ for all $x \in A$, then (F, A) is a soft subnear-ring of (G, A).

b) $(F, A)\widetilde{\sqcap}(G, A)$ is a soft subnear-ring of both (F, A) and (G, A) if it is non-null.

Definition 2.16 ([4]) Let (F, A) be a soft near-ring over N. A non-null soft set (G, I) over N is called a *soft left (resp. right) ideal* of (F, A) denoted by $(G, I)\widetilde{\triangleleft}_\ell(F, A)$ (resp. $(G, I)\widetilde{\triangleleft}_r(F, A)$) if it satisfies:

i) $I \subset A$

ii) $G(x) \triangleleft_\ell F(x)$ (resp. $G(x) \triangleleft_r F(x)$) for all $x \in Supp(G, I)$.

If (G, I) is both soft left and soft right ideal of (F, A), then it is said that (G, I) is a soft ideal of (F, A) and denoted by $(G, I)\widetilde{\triangleleft}(F, A)$.

Definition 2.17 ([4]) Let (F, A) be a soft near-ring over N. If $F(x) \triangleleft_\ell N$ (resp. $F(x) \triangleleft_r N$, $F(x) \triangleleft N$) for all $x \in Supp(F, A)$, then (F, A) is called a *left idealistic (resp. right idealistic, idealistic) soft near-ring* over N.

3 Some operations applied to soft near-rings and substructures of soft near-rings

Proposition 3.1 Let (F, A), (G, A) and (H, B) be soft near-rings over N. Then we have the following:

a) $(F, A)\Cap(H, B)$ is a soft subnear-ring of both (F, A) and (H, B) if it is non-null.

b) $(F, A)\sqcap_\varepsilon(G, A)$ is a soft subnear-ring of both (F, A) and (G, A) if it is non-null.

Proof. **a)** Since $A \cap B \subset A$ (and $A \cap B \subset B$), the first condition of Definition 2.14 is satisfied. Let $(F, A)\Cap(H, B) = (T, C)$, where $C = A \cap B$ and $T(x) = F(x) \cap H(x)$ for all $x \in C$. Since $T(x) = F(x) \cap H(x) \subset F(x)$ and $T(x) = F(x) \cap H(x) \subset H(x)$ for all $x \in A \cap B$, the proof is completed from Proposition 2.15 a).

b) Let $(F, A) \sqcap_\varepsilon (G, A) = (Q, A)$ where $Q(x) = F(x) \cap G(x)$ for all $x \in A$. Since $Q(x) = F(x) \cap G(x) \subset F(x)$ and $Q(x) = F(x) \cap G(x) \subset G(x)$ for all $x \in A$, the proof is completed from Proposition 2.15 a).

Theorem 3.2 Let (F, A) be a soft near-ring over N and $(F_i, A_i)_{i\in I}$ be a nonempty family of soft subnear-rings of (F, A). Then we have the following:

a) $⋒_{i\in I}(F_i, A_i)$ is a soft subnear-ring of (F, A), if it is non-null.

b) $\bigwedge_{i\in I}(F_i, A_i)$ is a soft subnear-ring of $\bigwedge_{i\in I}(F, A)$, if it is non-null.

c) If $\{A_i \mid i \in I\}$ are pairwise disjoint, i.e., $i \neq j$ implies $A_i \cap A_j = \emptyset$ then $\widetilde{\bigcup}_{i\in I}(F_i, A_i)$ is soft subnear-ring over (F, A).

Proof. **a)** By Definition 2.11, let $⋒_{i\in I}(F_i, A_i) = (G, B)$, where $B = \bigcap_{i\in I} A_i \neq \emptyset$ and $G(x) = \bigcap_{i\in I} F_i(x)$ for all $x \in B$. Firstly, check that the parameter set of $⋒_{i\in I}(F_i, A_i)$, namely $B = \bigcap_{i\in I} A_i$ is a subset of the parameter set of (F, A), namely A. Suppose that the soft set (G, B) is non-null. If $x \in Supp(G, B)$, then $G(x) = \bigcap_{i\in I} F_i(x) \neq \emptyset$. It follows that for all $i \in I$, the nonempty set $F_i(x)$ is a subnear-ring of $F(x)$, since (F_i, A_i) is a family of soft subnear-rings of (F, A). Hence $G(x)$ is a subnear-ring of $F(x)$ for all $x \in Supp(G, B)$. This completes the proof.

b) By Definition 2.10, let $\bigwedge_{i\in I}(F_i, A_i) = (G, B)$, where $B = \prod_{i\in I} A_i$ and $G(x) = \bigcap_{i\in I} F_i(x)$ for all $x = (x_i)_{i\in I} \in B$. Since $B = \prod_{i\in I} A_i \subset \prod_{i\in I} A$, the first condition of Definition 2.14 is satisfied. Suppose that the soft set (G, B) is non-null. If $x = (x_i)_{i\in I} \in Supp(G, B)$, then $G(x) = \bigcap_{i\in I} F_i(x) \neq \emptyset$. Thus the nonempty set $F_i(x)$ is a subnear-ring of $F(x)$ since (F_i, A_i) is a family of soft subnear-rings of (F, A) for all $i \in I$. Hence $G(x)$ is a subnear-ring of $F(x)$ for all $x \in Supp(G, B)$, as required.

c) By Definition 2.9, we can write $\widetilde{\bigcup}_{i\in I}(F_i, A_i) = (G, B)$. Then $B = \bigcup_{i\in I} A_i$ and for all $x \in B$, $G(x) = \bigcup_{i\in I(x)} F_i(x)$ where $I(x) = \{i \in I \mid x \in A_i\}$. Firstly, check the parameter sets. The parameter set of $\widetilde{\bigcup}_{i\in I}(F_i, A_i) = (G, B)$, namely $B = \bigcup_{i\in I} A_i$ is a subset of the parameter set (F, A), namely A. Note first that (G, B) is non-null since $Supp(G, B) = \bigcup_{i\in I} Supp(F_i, A_i) \neq \emptyset$. Let $x \in Supp(G, B)$. Then $G(x) = \bigcup_{i\in I(x)} F_i(x) \neq \emptyset$, and so we have $F_{i_0}(x) \neq \emptyset$ for some $i_0 \in I(x)$. But by hypothesis, we know that $\{A_i \mid i \in I\}$ are pairwise disjoint. Hence the above i_0 is in fact unique. And so $G(x)$ coincides with $F_{i_0}(x)$. Furthermore, since (F_{i_0}, A_{i_0}) is a soft subnear-ring of (F, A), the nonempty set $F_{i_0}(x)$ is a subnear-ring of $F(x)$ for all $x \in Supp(G, B)$. Thus, the proof is completed.

Proposition 3.3 Let (F, A) be a soft near-ring over N and $(F_i, A_i)_{i\in I}$ be a nonempty family of soft subnear-rings of (F, A). Then $⋒_{i\in I}(F_i, A_i)$ is a soft subnear-ring of (F_i, A_i) for each $i \in I$, if it is non-null.

Proof. Let $⋒_{i\in I}(F_i, A_i) = (H, C)$, where $C = \bigcap_{i\in I} A_i \neq \emptyset$ and $H(x) = \bigcap_{i\in I} F_i(x)$ for all $x \in C$. The parameter set of $⋒_{i\in I}(F_i, A_i)$, that is, $\bigcap_{i\in I} A_i$ is a subset of the parameter set of $(F_i, A_i)_{i\in I}$, namely A_i (for all $i \in I$). Suppose that (H, C)

is a non-null soft set over N. If $x \in Supp(H, C)$, then $H(x) = \bigcap_{i \in I} F_i(x) \neq \emptyset$. Thus $\emptyset \neq F_i(x)$ are subnear-rings of N, for all $i \in I$. Therefore $H(x) = \bigcap_{i \in I} F_i(x)$ are subnear-rings of N. Moreover, since $\bigcap_{i \in I} F_i(x) \subset F_i(x)$, for all $i \in I$ and for all $x \in \bigcap_{i \in I} A_i$, from Proposition 2.15 a), $⋒_{i \in I}(F_i, A_i)$ is a soft subnear-ring of (F_i, A_i) for each $i \in I$, as required.

Theorem 3.4 Let (F, A) and (G, B) be soft near-rings over N. Then $(F, A) \sqcap_\varepsilon (G, B)$ is a soft near-ring over N, if it is non-null.

Proof. By Definition 2.4, we can write $(F, A) \sqcap_\varepsilon (G, B) = (H, C)$, where $C = A \cup B$ and

$$H(x) = \left\{ \begin{array}{ll} F(x) & \text{if } x \in A \setminus B, \\ G(x) & \text{if } x \in B \setminus A, \\ F(x) \cap G(x) & \text{if } x \in A \cap B \end{array} \right\}$$

for all $x \in C$. Suppose that (H, C) is a non-null soft set over N. Let $x \in Supp(H, C)$. If $x \in A \setminus B$, then $H(x) = F(x) \neq \emptyset$ is a subnear-ring of N, if $x \in B \setminus A$, then $H(x) = G(x) \neq \emptyset$ is a subnear-ring of N. And if $x \in A \cap B$, $H(x) = F(x) \cap G(x) \neq \emptyset$. Thus $\emptyset \neq F(x)$ and $\emptyset \neq G(x)$ are both subnear-rings of N, so are their intersection. This follows that (H, C) is a soft near-ring over N.

Example 3.5 Let $N = (\mathbb{Z}_6, +, .)$ be the near-ring in Example 2.13, and let (F, A) be a soft set over N, where $A = \{1, 2\}$ and assume that $F : A \to P(N)$ is a set-valued function defined by

$$F(x) = \{y \in N \mid xRy \Leftrightarrow xy \in \{0, 3\}\}$$

for all $x \in A$. Then $F(1) = F(2) = \{0, 3\}$, which is a subnear-ring of N. Hence (F, A) is a soft near-ring over N.

Let (G, B) be a soft set over N, where $B = \{0, 2\}$ and suppose that $G : B \to P(N)$ is a set-valued function defined by

$$G(x) = \{y \in N \mid xRy \Leftrightarrow xy \in \{0, 2, 4\}\}$$

for all $x \in B$. Then $G(0) = G(2) = N$, which is a subnear-ring of N. Hence (G, B) is a soft near-ring over N.

Now let us consider $(F, A) \sqcap_\varepsilon (G, B)$. Then $(F, A) \sqcap_\varepsilon (G, B) = (H, A \cup B)$, where

$$H(x) = \left\{ \begin{array}{ll} F(x) & \text{if } x \in A \setminus B = \{1\}, \\ G(x) & \text{if } x \in B \setminus A = \{0\}, \\ F(x) \cap G(x) & \text{if } x \in A \cap B = \{2\} \end{array} \right\}$$

for all $x \in A \cup B$. Then $Supp(H, A \cup B) = \{0, 1, 2\}$ and $H(0) = \mathbb{Z}_6$, $H(1) = \{0, 3\}$, $H(2) = F(2) \cap G(2) = \{0, 3\}$. Since $H(x)$ is a subnear-ring of N for all $x \in Supp(H, A \cup B)$, $(F, A) \sqcap_\varepsilon (G, B)$ is a soft near-ring over N, as required.

Theorem 3.6 Let (G_1, I_1) and (G_2, I_2) be soft left ideals (resp. soft right ideals, soft ideals) of a soft near-ring (F, A) over a near-ring N. Then the soft set $(G_1, I_1) \sqcap_\varepsilon (G_2, I_2)$ is a soft left ideal (resp. soft right ideal, soft ideal) of (F, A) if it is non-null.

Proof. We give the proof for soft left ideals; the same proof can be seen for soft right ideals and hence for soft ideals. Assume that $(G_1, I_1)\widetilde{\triangleleft}_\ell(F, A)$ and $(G_2, I_2)\widetilde{\triangleleft}_\ell(F, A)$. By Definition 2.4, $(G_1, I_1) \sqcap_\varepsilon (G_2, I_2) = (G, I)$, where $I = I_1 \cup I_2$

$$G(x) = \left\{ \begin{array}{ll} G_1(x) & \text{if } x \in I_1 \setminus I_2, \\ G_2(x) & \text{if } x \in I_2 \setminus I_1, \\ G_1(x) \cap G_2(x) & \text{if } x \in I_1 \cap I_2 \end{array} \right\}$$

for all $x \in I$. Since $I_1 \subset A$ and $I_2 \subset A$, it is obvious that $I \subset A$. Suppose that the soft set (G, I) is non-null and let $x \in Supp(G, I)$. If $x \in I_1 \setminus I_2$, then $\emptyset \neq G_1(x) = G(x) \triangleleft_\ell F(x)$ and if $x \in I_2 \setminus I_1$, then $\emptyset \neq G_2(x) = G(x) \triangleleft_\ell F(x)$. And if $x \in I_1 \cap I_2$, then $\emptyset \neq G(x) = G_1(x) \cap G_2(x)$. Since $(G_1, I_1)\widetilde{\triangleleft}_\ell(F, A)$ and $(G_2, I_2)\widetilde{\triangleleft}_\ell(F, A)$ from assumption, we deduce that the nonempty sets $G_1(x)$ and $G_2(x)$ are both left ideals of $F(x)$. It follows that $G(x) \triangleleft_\ell F(x)$ for all $x \in Supp(G, I)$, since the intersection of left ideals is a left ideal in a near-ring. Therefore $(G_1, I_1) \sqcap_\varepsilon (G_2, I_2)\widetilde{\triangleleft}_\ell(F, A)$, as required.

Example 3.7 (cf., [17]) Let us consider the Klein-4 group $N = \{0, 1, 2, 3\}$. Under the operations defined by the following tables, $(N, +, .)$ is a (right) near-ring.

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	0	0	2
3	1	1	1	3

Let (F, A) be a soft set over N, where $A = N$ and assume that $F : A \to P(N)$ is a set-valued function defined by

$$F(x) = \{3\} \cup \{y \in N \mid xRy \Leftrightarrow xy \in \{0, 1\}\}$$

for all $x \in A$. Then we have $F(0) = F(1) = F(2) = F(3) = N$. It follows that (F, A) is a non-null soft set over N.

Let $I_1 = \{0, 3\}$ and $G_1 : I_1 \to P(N)$ be a set-valued function defined by

$$G_1(x) = \{0\} \cup \{y \in N \mid xRy \Leftrightarrow xy \in \{0, 2\}\}$$

for all $x \in I_1$. Then we have $G_1(0) = N$ and $G_1(3) = \{0\}$. It is easily seen that $G_1(0) \triangleleft F(0)$ and $G_1(3) \triangleleft F(3)$, hence $(G_1, I_1)\widetilde{\triangleleft}(F, A)$.

And let $I_2 = \{1, 3\}$ and $G_2 : I_2 \to P(N)$ be a set-valued function defined by

$$G_2(x) = \{0\} \cup \{y \in N \setminus I_2 \mid xRy \Leftrightarrow xy = 1\}$$

for all $x \in I_2$. Then we have $G_2(1) = G_2(3) = \{0, 2\}$. It can be easily illustrated that $\{0, 2\} \lhd N$. Since $G_2(1) \lhd F(1)$ and $G_2(3) \lhd F(3)$, $(G_2, I_2)\widetilde{\lhd}(F, A)$.

Now assume that $(G_1, I_1) \sqcap_\varepsilon (G_2, I_2) = (H, I)$, where $I = I_1 \cup I_2 = \{0, 1, 3\}$ and

$$H(x) = \left\{ \begin{array}{ll} G_1(x) & \text{if } x \in I_1 \setminus I_2 = \{0\}, \\ G_2(x) & \text{if } x \in I_2 \setminus I_1 = \{1\}, \\ G_1(x) \cap G_2(x) & \text{if } x \in I_1 \cap I_2 = \{3\} \end{array} \right\}$$

for all $x \in I$. Then $Supp(H, I) = \{0, 1, 3\}$ and $H(0) = N$, $H(1) = \{0, 2\}$, $H(3) = \{0\}$. We can easily see that $H(0) \lhd F(0)$, $H(1) \lhd F(1)$ and $H(3) \lhd F(3)$ for all $x \in Supp(H, I)$. Therefore $(G_1, I_1) \sqcap_\varepsilon (G_2, I_2)\widetilde{\lhd}(F, A)$.

Theorem 3.8 Let (F, A) and (G, B) be idealistic soft near-rings over N. Then $(F, A) \sqcap_\varepsilon (G, B)$ is an idealistic soft near-ring over N, if it is non-null.

Proof. By Definition 2.4, let $(F, A) \sqcap_\varepsilon (G, B) = (K, A \cup B)$, where

$$K(x) = \left\{ \begin{array}{ll} F(x) & \text{if } x \in A \setminus B, \\ G(x) & \text{if } x \in B \setminus A, \\ F(x) \cap G(x) & \text{if } x \in A \cap B \end{array} \right\}$$

for all $x \in A \cup B$. Suppose that $(K, A \cup B)$ is a non-null soft set over N. Let $x \in Supp(K, A \cup B)$. If $x \in A \setminus B$, then $\emptyset \neq K(x) = F(x) \lhd N$. If $x \in B \setminus A$, then $\emptyset \neq K(x) = G(x) \lhd N$ and if $x \in A \cap B$, then $K(x) = F(x) \cap G(x) \neq \emptyset$. Since $\emptyset \neq F(x) \lhd N$ and $\emptyset \neq G(x) \lhd N$, it follows that $K(x) \lhd N$ for all $x \in Supp(K, A \cup B)$. Therefore $(F, A) \sqcap_\varepsilon (G, B) = (K, A \cup B)$ is an idealistic soft near-ring over N.

Example 3.9 Let $N = (\mathbb{Z}_6, +, .)$ be the near-ring in Example 2.13 and (F, A) be a soft set over N, where $A = \{1, 2, 3\}$. Assume that $F : A \to P(N)$ is a set-valued function defined by

$$F(x) = \{0\} \cup \{y \in N \mid xRy \Leftrightarrow 3x = y\}$$

for all $x \in A$. Then $F(1) = F(2) = F(3) = \{0, 3\}$, which is a subnear-ring of N. Hence (F, A) is a soft near-ring over N. Moreover $F(x) \lhd N$ for all $x \in Supp(F, A)$, which means that (F, A) is an idealistic soft near-ring over N.

Let (G, B) be a soft set over N, where $B = \{0, 2, 4\}$ and suppose that $G : B \to P(N)$ is a set-valued function defined by

$$G(x) = \{y \in N \mid xRy \Leftrightarrow x0 = y\}$$

for all $x \in B$. Then $G(0) = G(2) = G(4) = \{0\}$, which is a sub-near-ring of N. Hence (G, B) is a soft near-ring over N. Furthermore $G(x) \lhd N$ for all

$x \in Supp(G, B)$, which implies that (G, B) is an idealistic soft near-ring over N.

Now let us consider $(F, A) \sqcap_{\varepsilon} (G, B)$. Assume that $(F, A) \sqcap_{\varepsilon} (G, B) = (H, A \cup B)$, where

$$H(x) = \left\{ \begin{array}{ll} F(x) & \text{if } \ x \in A \setminus B = \{1, 3\}, \\ G(x) & \text{if } \ x \in B \setminus A = \{0, 4\}, \\ F(x) \cap G(x) & \text{if } \ x \in A \cap B = \{2\} \end{array} \right\}$$

for all $x \in A \cup B$. Then $Supp(H, A \cup B) = \{0, 1, 2, 3, 4\}$ and $H(1) = H(3) = \{0, 3\}$, $H(0) = H(2) = H(4) = \{0\}$. We can easily see that $H(x) \lhd N$ for all $x \in Supp(H, A \cup B)$. Therefore $(F, A) \sqcap_{\varepsilon} (G, B) = (H, A \cup B)$ is an idealistic soft near-ring over N, as required.

Since every ideal of a near-ring N is a subnear-ring of N, we can deduce that every idealistic soft near-ring over N is a soft near-ring over N; however the following example shows that the converse is not true in general.

Example 3.10 Let (F, A) be a soft near-ring over $N = \{0, 1, 2, 3\}$ in Example 3.7 and let $F : A \to P(N)$ be a set-valued function defined by

$$F(x) = \{0\} \cup \{y \in B \mid xy = 0\}$$

for all $x \in A$, where $A = N$ and $B = \{0, 1\}$. Then $F(0) = F(2) = \{0, 1\}$, $F(1) = F(3) = \{0\}$, which are all subnear-rings of N. Hence (F, A) is a soft near-ring over N. Nevertheless $F(0) = F(2) = \{0, 1\}$ is not a left ideal of N, neither is an ideal of N, since $2.(3 + 1) - 2.3 = 2 \notin \{0, 1\}$. It follows that (F, A) is not an idealistic soft near-ring over N, although it is a soft near-ring over N.

Proposition 3.11 Let (F, A) be a soft set over a near-ring N and $B \subset A$. If (F, A) is an idealistic soft near-ring over N, then so is (F, B), whenever (F, B) is non-null.

Proof. It is easy, hence omitted.

As can be seen from the following example, the converse of Proposition 3.11 is not true in general.

Example 3.12 Let (F, A) be the soft set given in Example 3.10. Remember that (F, A) is not an idealistic soft near-ring over N. However when we take $C = \{1, 3\} \subset A$, then $(F \mid_C, C)$ is an idealistic soft near-ring over N, where $F \mid_C$ is the restriction of F to C.

Definition 3.13 An idealistic soft near-ring (F, A) over a near-ring N is said to be *trivial* if $F(x) = \{0_N\}$ for all $x \in Supp(F, A)$.

An idealistic soft near-ring (F, A) over N is a said to be *whole* if $F(x) = N$ for all $x \in Supp(F, A)$.

Example 3.14 Let (F, A) be a soft near-ring over $N = \{0, 1, 2, 3\}$ given in Example 3.7, where $A = \{0, 1, 2\}$ and $F : A \to P(N)$ be a set-valued function defined by

$$F(x) = \{y \in N \mid xRy \Leftrightarrow 2x = y\}$$

for all $x \in A$. Then we have $F(0) = F(1) = F(2) = \{0\}$. Since $F(x) = \{0\}$ for all $x \in Supp(F, A) = \{0, 1, 2\}$, (F, A) is a trivial idealistic soft near-ring over N.

Let (G, B) be a soft near-ring over N, where $B = N$ and $G : B \to P(N)$ be a set-valued function defined by

$$G(x) = \{3\} \cup \{y \in N \mid xRy \Leftrightarrow xy \in \{0, 1\}\}$$

for all $x \in B$. Then we have $G(0) = G(1) = G(2) = G(3) = N$. Since $G(x) = N$ for all $x \in Supp(G, B)$, it follows that (G, B) is a whole idealistic soft near-ring over N.

Let (F, A) be a soft set over a near-ring N and let $f : N \to M$ be a mapping of near-rings. Then the soft set $(f(F), Supp(F, A))$ over M can be defined, where

$$f(F) : Supp(F, A) \to P(M)$$

is given by $f(F)(x) = f(F(x))$ for all $x \in Supp(F, A)$. It is also worth nothing that $Supp(F, A) = Supp(f(F), Supp(F, A))$.

Proposition 3.15 Let $f : N \to M$ be an epimorphism of near-rings. If (F, A) is an idealistic soft near-ring over N, then $(f(F), Supp(F, A))$ is an idealistic soft near-ring over M.

Proof. First of all, note that since (F, A) is an idealistic soft near-ring over N, by Definition 2.17 it has to be a non-null soft set over N, thus $(f(F), Supp(F, A))$ is a non-null soft set over M, too. We have $f(F)(x) = f(F(x)) \neq \emptyset$ for all $x \in Supp(f(F), Supp(F, A))$. Because of the fact that (F, A) is an idealistic soft near-ring over N, the nonempty set $F(x)$ is an ideal of N. Thus, we can conclude that its onto homomorphic image $f(F(x))$ is an ideal of M. So, $f(F(x))$ is an ideal of M for all $x \in Supp(f(F), Supp(F, A))$. It follows that $(f(F), Supp(F, A))$ is an idealistic soft near-ring over M.

Example 3.16 Let us consider the soft set $(F, A) \sqcap_{\varepsilon} (G, B) = (H, C)$ where $C = A \cup B = \{0, 1, 2\}$ over $N = \mathbb{Z}_6$ in Example 3.5. It is obvious that $Supp(H, C) = C$ and (H, C) is a non-null soft set over N, and for all $x \in Supp(H, C) = \{0, 1, 2\}$, $H(x) \lhd N$. Therefore (H, C) is an idealistic soft near-ring over N.

Now let us consider the subnear-ring $\{0, 2, 4\}$ of $(\mathbb{Z}_6, +, .)$. Let $f : \mathbb{Z}_6 \to \{0, 2, 4\}$ be the mapping defined by $f(x) = 4x$. Obviously, f is an epimorphism of near-rings.

Now we construct the soft set $(f(H), C)$ over $\{0, 2, 4\}$, where

$$f(H) : C \to P(\{0, 2, 4\})$$

is given by $f(H)(x) = f(H(x))$ for all $x \in C$. Then we have $f(H)(0) = f(H(0)) = \{0, 2, 4\}$, $f(H)(1) = f(H(1)) = \{0\}$, $f(H)(2) = f(H(2)) = \{0\}$. It is easy to see that $(f(H), C)$ is a non-null soft set over $\{0, 2, 4\}$ and $f(H)(x)$ is an ideal of $\{0, 2, 4\}$ for all $x \in Supp(f(H), C)$. Thus $(f(H), C)$ is an idealistic soft near-ring over $\{0, 2, 4\}$, as required.

Theorem 3.17 Let (F, A) be an idealistic soft near-ring over N and let $f : N \to M$ be an epimorphism of near-rings. Then

a) If $F(x) = Ker(f)$ for all $x \in Supp(F, A)$, then $(f(F), Supp(F, A))$ is a trivial idealistic soft near-ring over M.

b) If (F, A) is whole, then $(f(F), Supp(F, A))$ is a whole idealistic soft near-ring over M.

Proof. **a)** Assume that $F(x) = Ker(f)$ for all $x \in Supp(F, A)$. Then $f(F)(x) = f(F(x)) = 0_M$ for all $x \in Supp(F, A)$. That is to say $(f(F), Supp(F, A))$ is a trivial idealistic soft near-ring over M by Proposition 3.15 and Definition 3.13.

b) Suppose that (F, A) is whole. Then, $F(x) = N$ for all $x \in Supp(F, A)$. It follows that $f(F)(x) = f(F(x)) = F(N) = M$ for all $x \in Supp(F, A)$, which means that $(f(F), Supp(F, A))$ is a whole idealistic soft near-ring over M by Proposition 3.15 and Definition 3.13.

Now let us see the application of the above theorem with the following example:

Example 3.18 a) Let (F, A) be a soft set over $N = \mathbb{Z}_6$ given in Example 2.13, where $A = \{1, 2, 4, 5\}$ and $F : A \to P(\mathbb{Z}_6)$ is a set-valued function defined by

$$F(x) = \{y \in \mathbb{Z}_6 \mid xRy \Leftrightarrow xy \in \{0, 3\}\}$$

for all $x \in A$. Then we have $F(1) = F(2) = F(4) = F(5) = \{0, 3\}$. Since $F(x) \lhd N$ for all $x \in Supp(F, A) = A = \{1, 2, 4, 5\}$, (F, A) is an idealistic soft near-ring over $\mathbb{Z}_6$.

Now let us consider the near-ring epimorphism $f : \mathbb{Z}_6 \to \{0, 2, 4\}$ in Example 3.16. It is obvious that $Ker(f) = \{0, 3\}$. Then $F(x) = Ker(f)$ for all $x \in Supp(F, A)$. We need to show that $(f(F), A)$ is a trivial idealistic soft near-ring over $\{0, 2, 4\}$. To see this, we construct the soft set $(f(F), A)$ over $\{0, 2, 4\}$, where

$$f(F) : A \to P(\{0, 2, 4\})$$

is given by $f(F)(x) = f(F(x))$ for all $x \in A$. It follows that, $f(F)(1) = f(F(1)) = f(F)(2) = f(F(2)) = f(F)(4) = f(F(4)) = f(F)(5) = f(F(5)) = f(\{0, 3\}) = \{0\}$. It is easy to see that $(f(F), A)$ is an idealistic soft near-ring over $\{0, 2, 4\}$, furthermore $(f(F), A)$ is a trivial idealistic soft near-ring over $\{0, 2, 4\}$, as required.

b) Let (G, B) be a soft set over $N = \mathbb{Z}_6$ in Example 3.16, where $B = N$ and $G : B \to P(\mathbb{Z}_6)$ is a set-valued function defined by

$$G(x) = \{y \in \mathbb{Z}_6 \mid xRy \Leftrightarrow xy \in \{0, 2, 4\}\}$$

for all $x \in B$. Then, we have $G(0) = G(2) = G(4) = \mathbb{Z}_6$ and $G(1) = G(3) = G(5) = \emptyset$. It follows that $Supp(G, B) = \{0, 2, 4\}$. Since $G(x) \lhd N$ for all $x \in Supp(G, B)$, (G, B) is an idealistic near-ring over $\mathbb{Z}_6$, moreover since $G(x) = N$ for all $x \in Supp(G, B)$, (G, B) is whole.

Considering the near-ring epimorphism $f : \mathbb{Z}_6 \to \{0, 2, 4\}$ above, we need to show that $(f(G), Supp(G, B))$ is a whole idealistic soft near-ring over $\{0, 2, 4\}$. To illustrate this, we construct the soft set $(f(G), Supp(G, B))$ over $\{0, 2, 4\}$, where

$$f(G) : Supp(G, B) \to P(\{0, 2, 4\})$$

is given by $f(G)(x) = f(G(x))$ for all $x \in Supp(G, B)$. It follows that $f(G)(0) = f(G(0)) = f(G)(2) = f(G(2)) = f(G)(4) = f(G(4)) = \{0, 2, 4\}$. It is easy to see that $(f(G), Supp(G, B))$ is an idealistic soft near-ring over $\{0, 2, 4\}$, furthermore $(f(G), Supp(G, B))$ is a whole idealistic soft near-ring over $\{0, 2, 4\}$, as required.

4 Conclusion

In this paper, we have approached from a theoretical aspect to the concept of soft near-rings. We have applied some of the operations to soft near-rings and substructures of near-rings and investigated the properties of idealistic soft near-rings with respect to near-ring epimorphisms by corresponding examples.

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