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A NOTE ON SOLITARY WAVE SOLUTIONS OF THE LEIBOVICH-ROBERTS EQUATION*

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Abstract. The propagation of weakly nonlinear, weakly dispersive sound waves in a magnetic cylinder satisfies an equation originally discussed in a limiting form by Leibovich (1970) and in general form by Roberts (1985). We show here that the resulting Leibovich–Roberts equation possesses nonlinear solitary wave behaviors akin to Benjamin–Ono waves in a slab. We also show how the structure of the solitary waves can be determined using a variational principle.

I. Introduction. When a layered inhomogeneity occurs in a system, the possibility always arises that the inhomogeneity acts to duct information, thereby causing a solitary wave to propagate. Such phenomena have received considerable attention in recent years, particularly in the scientific arena of fluid flow. A particularly lucid summary is provided by Roberts (1985) of work to date on such solitary waves. To a large extent the available knowledge on solitary wave behaviors and on solitons has been carried out for planar geometries (waves in a slab)—perhaps because of the lack of the usual appearance of some scaling factor related to the divergence of wave fronts as always seems to occur in other than planar geometries. Yet it is becoming increasingly necessary in many disciplines that a greater degree of attention be paid to solitary wave behavior in nonplanar geometries; for instance, the theoretical work of Leibovich (1970) and the experimental work of Pritchard (1970) on waves in a cylindrical vortex core. More recently, Roberts (1985) has persuasively argued that, in the solar physics case, cylindrical magnetic flux tubes are capable of supporting solitary wave behavior, and that such behavior is, perhaps, one origin for spicules seen in the solar chromosphere. For a disturbance, $v(z,t)$, propagating in the direction, z, at time, t, along the symmetry axis of a cylindrical geometry, Leibovich obtained a propagation equation of the form,

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial z} + \beta v\frac{\partial v}{\partial z} + \alpha\frac{\partial^3}{\partial z^3}\left(P\int_{-\infty}^{\infty}\frac{v(s,t)\,ds}{|z-s|}\right) = 0, \tag{1}$$

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where c, β, and α are constants, and P denotes principal value. In the case of a magnetic flux tube, Roberts (1985) obtained an equation of similar appearance,

$$\frac{\partial v}{\partial t}+c\frac{\partial v}{\partial z}+\beta v\frac{\partial v}{\partial z}+\alpha\frac{\partial^3}{\partial z^3}\int_{-\infty}^{\infty}\frac{v(s,t)\,ds}{[\lambda^2+(z-s)^2]^{1/2}}=0, \tag{2}$$

where c, β, α, and λ^2 are positive constants, Δ. Here λ provides a direct measure of the ratio of the sound speed to the Alfven speed (see Roberts, 1985).

It is clear that, in some sense, Eq. (2) is a generalization of Eq. (1) since, if the limit exists, solutions of (1) can be obtained from solutions to (2) by setting $\lambda \to 0$.

Leibovich (1970) and Leibovich and Randall (1972) examined the solitary wave behavior of Eq. (1) numerically using asymptotic expansion techniques. They found that a solitary wave is present, which is qualitatively similar in character to solutions of the Benjamin–Ono equation. This behavior might have been anticipated since the Benjamin–Ono equation describes solitary wave behavior in a slab.

What is not so clear is the question of modifications to the solitary wave behavior when the presence of a finite λ is allowed for—as in Roberts' case. The question we address here is directed toward uncovering properties of the solitary wave behavior for Eq. (2)—which we have taken the liberty of calling the Leibovich–Roberts (LR) equation. Section II of the paper reduces the LR equation to a simpler form based upon the supposition that a solitary wave behavior exists, and it also examines some of the global and local properties of the reduced equation. Section III introduces a nonlinear variational principle related to the reduced equation, and develops the adjoint field equation, as well as the globally conserved quantities from an application of Noether's theorem. A simple illustration of the use of the nonlinear variational principle to construct solutions to the LR equation concludes the section. A discussion and our conclusions are given in Section IV.

II. The solitary wave Leibovich–Roberts equation.

a. *General Reduction.* On the assumption that a solitary wave exists to Eq. (2), write $\xi = z - Vt$ with

$$v(z,t)=\phi(z-Vt)=\phi(\xi). \tag{3}$$

Then we can write Eq. (2) in the form

$$-V\frac{\partial\phi}{\partial\xi}+c\frac{\partial\phi}{\partial\xi}+\beta\phi\frac{\partial\phi}{\partial\xi}+\alpha\frac{\partial^3}{\partial\xi^3}\int_{-\infty}^{\infty}\frac{\phi(\zeta)\,d\zeta}{[\lambda^2+(\zeta-\xi)^2]^{1/2}}=0 \tag{4}$$

with $\zeta = s - Vt$.

Equation (4) has an immediate first integral,

$$\frac{1}{2}\beta\phi^2+(c-V)\phi+\alpha\frac{\partial^2}{\partial\xi^2}\int_{-\infty}^{\infty}\frac{\phi(\zeta)\,d\zeta}{[\lambda^2+(\zeta-\xi)^2]^{1/2}}=-\Lambda, \tag{5}$$

where Λ is a constant of integration.

Set $w_* = (c-V)(2\beta)^{-1/2}$, $\gamma = \Lambda - (c-V)^2/(2\beta)$, and $w = (\beta/2)^{1/2}\phi + w_*$ when w satisfies the equation

$$w^2+\gamma=-\alpha\left(\frac{2}{\beta}\right)^{1/2}\frac{\partial^2}{\partial\xi^2}\int_{-\infty}^{\infty}\frac{w(\zeta)\,d\zeta}{[\lambda^2+(\zeta-\xi)^2]^{1/2}}. \tag{6}$$

By integrating Eq. (6) over the domain $-\infty \leq \xi \leq \infty$ we see that γ must be negative for a solution to exist, i.e., $\Lambda < (c-V)^2/(2\beta)$.

When this is true set $\gamma = -\Gamma^2$ and $w = \Gamma(1+f)$ to obtain from Eq. (6) the equation for f in the form

$$f^2 + 2f = -\frac{\alpha}{\Gamma}\left(\frac{2}{\beta}\right)^{1/2} \frac{\partial^2}{\partial \xi^2} \int_{-\infty}^{\infty} \frac{f(\zeta)\, d\zeta}{[\lambda^2 + (\zeta - \xi)^2]^{1/2}}. \tag{7}$$

Now rescale the independent variable by setting $\xi = \lambda x$, $\zeta = \lambda y$, $(\beta/2)^{1/2} a = \frac{\alpha}{\Gamma \lambda^2}$ when

$$f^2 + 2f = -a \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{f(y)\, dy}{[1 + (y-x)^2]^{1/2}} \tag{8}$$

with

$$\int_{-\infty}^{\infty} f^2\, dx = -2 \int_{-\infty}^{\infty} f\, dx. \tag{9}$$

It is apparent by inspection of Eq. (8) that $f(x)$ is symmetric: $f(-x) = f(x)$, so that we can also write the constraint equation (9) in the form

$$\int_0^{\infty} f(x)^2\, dx = -2 \int_0^{\infty} f(x)\, dx. \tag{10}$$

b. *Asymptotic Properties.* For $|x| \to \infty$, we see from Eq. (8) that $f \propto |x|^{-3}$ so that the integrals in (10) are indeed bounded. For $|x| \to 0$ we can use the symmetry property, $f(x) = f(-x)$, to write

$$f(x) \cong f_0 + f_1 x^2 + f_2 x^4 + \cdots. \tag{11}$$

A zeroth-order attempt at obtaining a structural form for $f(x)$ might then proceed as follows:

Let

$$f(x) \cong f_0 + f_1 x^2 \quad \text{in } |x| < x_* \tag{12a}$$

and let

$$f(x) \cong (f_0 + f_1 x_*^2)(x_*/x)^3 \text{ in } |x| \geq x_* \tag{12b}$$

so that $f(x)$ is continuous on $x = x_*$. Use the integral constraint (10) to obtain the connection

$$\frac{2}{5}(f_1 x_*^2)^2 + f_1 x_*^2 \left(\frac{5}{3} + \frac{16}{15} f_0\right) + 3 f_0 \left(1 + \frac{2}{5} f_0\right) = 0. \tag{13}$$

To obtain two remaining connections between f_0, f_1, and x_* we then use the approximate forms (12) in Eq. (8) and evaluate the result for both large and small x. Thus at small x we obtain

$$f_0^2 + 2 f_0 = -2a \int_0^{\infty} \frac{(2y^2 - 1) f(y)\, dy}{(1+y^2)^{5/2}} \text{ as } x \to 0 \tag{14a}$$

while, as $x \to \infty$, Eq. (8) yields

$$(f_0 + f_1 x_*^2) x_*^3 = -4a \int_0^{\infty} f(y)\, dy, \text{ as } x \to \infty. \tag{14b}$$

Using the approximate forms (12) to evaluate the right-hand sides of Eqs. (14) we obtain from Eq. (14a) that

$$f_0^2 + 2f_0 \cong -2a\left\{(f_0 - f_1)(s_*^3 - s_*) - 3f_1 s_* + f_1 \ln\left(\frac{1-s_*}{1+s_*}\right)\right.$$
$$\left. +(f_0 + f_1 x_*^2)x_*^3 \int_{x_*}^{\infty} \frac{(2y^2-1)\,dy}{y^3(1+y^2)^{5/2}}\right\} \qquad (15a)$$

while Eq. (14b) yields

$$f_0 + f_1 x_*^2 \cong -\frac{4a}{x_*^2}\left[\frac{3f_0}{2} + \frac{5f_1}{6}x_*^2\right] \qquad (15b)$$

with $s_* = \sin\theta_*$ and $x_* \equiv \tan\theta_*$.

Equations (13), (15a), and (15b) provide three constraints connecting the unknown parameters f_0, f_1, and x_*. Hence it is possible to unravel approximate values for these parameters for a given value of a. An alternative is to specify the match point x_* and to then ask for the values of f_0, f_1, and a consistent with the chosen x_*.

Improvements can be made by increasing the number of terms in the two series presentations of $f(x)$ at small and large x, satisfying the global constraint condition (10), and then evaluating the integral equation (8) as $x \to 0$ and $x \to \infty$ term by term, thereby producing a series of parameter equations which give increasingly accurate values for the parameters as the number of terms increases.

This classical approach to the nonlinear equation (8) is arranged to satisfy the equation exactly to a better and better degree in the number of derivatives of $f(x)$ at all x, eventually producing a well-fitting solution to the equation.

By way of contrast, variational principles to obtaining solutions of equations abandon the idea of fitting the original equation perfectly in favor of obtaining a global behavior to the solution which approximates to the true solution, sometimes overshooting, sometimes undershooting the true equation behavior. As more terms are added to the trial functions in the variational, the global behavior more and more closely satisfies the original equation. We shall return to this point in the next section of the paper where functions adjoint to the original nonlinear equation (8) play a role. It is appropriate first to consider bifurcation points of Eq. (8) since there is a close connection between bifurcated solutions to Eq. (8) and the adjoint functions of the next section.

c. *Bifurcation properties.* Suppose that, as the single real positive parameter a is increased in Eq. (8), a solution is $f = f_0(x)$ for $a < a_0$ and that in $a > a_0$, the solution bifurcates so that in $a = a_0 + \varepsilon$ (with $\varepsilon > 0$) we have a second solution $f_2 = f_0(x) + \varepsilon f_1(x, \varepsilon)$ which must reduce to $f_2 = f_0$ as $\varepsilon \to 0$. Then by following the technique given by Tricomi (1957), it is not difficult to show that as $\varepsilon \to 0$ the requirement that $a = a_0$ be a bifurcation value is reduced to the requirement that a nontrivial solution exist to the homogeneous integral equation

$$f_1(x) = \frac{-a_0}{2(1+f_0(x))}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{f_1(y)\,dy}{[1+(y-x)^2]^{1/2}} \qquad (16)$$

where $f_1(x)$ is the limit form of $f_1(x, \varepsilon)$ as $\varepsilon \to 0$.

One particular solution of Eq. (8) satisfying the constraint equation (10) is $f_0(x) \equiv 0$. Hence, bifurcation points exist whenever a nontrivial solution exists to

$$f_1(x) = -\frac{a_0}{2}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{f_1(y)\,dy}{[1+(y-x)^2]^{1/2}}. \tag{17}$$

Now Eq. (17) is of the convolutional type. Taking the Fourier transform of both sides (with real transform variable k conjugate to x), we obtain

$$f_1(k)\left[1 - \frac{a_0}{\pi}k^2 K_0(k)\right] = 0 \tag{18}$$

where $K_0(k)$ is the standard modified Bessel function. Hence for $f_1(k) \neq 0$ bifurcation points exist whenever we can satisfy

$$k^2 K_0(k) = \pi/a_0. \tag{19}$$

Now the left-hand side of Eq. (19), which is positive, has a single peak value of around unity so that if $a_0 \leq O(\pi^{-1})$ we cannot satisfy Eq. (19) and no bifurcation of the $f_0 = 0$ solution occurs. As a_0 is increased, eventually two, and only two, real values of k, k_1, and k_2 say, satisfy Eq. (19) so that the bifurcation consists of the addition of two sine wave solutions in the neighborhood of any bifurcation point.

For other solutions, $f_0(x)$, different bifurcation properties and points exist due to the denominator $1 + f_0(x)$ in Eq. (16).

III. A variational principle approach.

a. *General considerations.* As we have remarked already, the essence of a variational approach is to seek for approximate satisfaction of the original equation in a global, rather than a local, sense (Morse and Feshbach, 1953). The problem of constructing extremal variationals to nonlinear equations has been well addressed by Becker (1964) and we follow his general method of approach here.

Consider the Lagrangian functional,

$$L = \int_{-\infty}^{\infty}\mathscr{L}(f, f^+, x)\,dx, \tag{20}$$

where

$$\mathscr{L}(f, f^+, x) = f^+(f^2 + 2f) - a\frac{\partial f^+}{\partial x}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{f(y)\,dy}{[1+(y-x)^2]^{1/2}}. \tag{21}$$

If L is extremized with respect to arbitrary variations in f^+ we recover Eq. (8). If L is extremized with respect to variations in f, we determine the equation to be satisfied by the adjoint function, f^+, as

$$f^+ = -\frac{a}{2(1+f)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{f^+(y)\,dy}{[1+(y-x)^2]^{1/2}}. \tag{22}$$

We see immediately that the adjoint function, f^+, satisfies the identical linear equation as that satisfied by the bifurcation solution (Eq. (16)), the difference being that the adjoint function satisfies Eq. (22) for *all* values of the parameter, a, while the bifurcation solution satisfies the same equation only in the limit as the parameter a tends to the bifurcation value a_0.

To provide the equivalent of a Wronskian requirement on the function f and its adjoint f^+, we multiply Eq. (8) by $f^+(x)$, Eq. (22) by $f(x)$ and subtract to obtain

$$f^2(x)f^+(x) = -a\left[f(x)\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{f^+(y)\,dy}{[1+(y-x)^2]^{1/2}} - f^+(x)\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{f(y)\,dy}{[1+(y-x)^2]^{1/2}}\right]. \tag{23}$$

Upon integrating Eq. (23) in $-\infty \le x \le \infty$, and assuming, with Becker (1964), the usual convergence of end-point integration values on the right-hand side of (23) we obtain

$$\int_{-\infty}^{\infty} f^2(x)f^+(x)\,dx = 0. \tag{24}$$

Hence, $f^+(x)$ is the adjoint function to $f(x)$, belonging to the same parameter value a, provided Eq. (24) is satisfied.

We note also that we can write the Lagrangian density in the form

$$\mathscr{L}\left(f, f^+, \frac{\partial f}{\partial x}, \frac{\partial f^+}{\partial x}, x\right) = f^+(f^2+2f) - \frac{\partial f^+}{\partial x}M(x) - a\frac{\partial f}{\partial x}M^+(x) \tag{25}$$

with

$$[M(x), M^+(x)] = \int_{-\infty}^{\infty}\frac{[\partial f(y)/\partial y, \partial f^+(y)/\partial y]}{[1+(y-x)^2]^{1/2}}\,dy$$

and where M and M^+ are regarded as explicit functions of x. We note that we can then write Eq. (8) in the form

$$\frac{\partial \mathscr{L}}{\partial f^+} - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial(\partial f^+/\partial x)}\right) = 0 \tag{26}$$

and Eq. (22) in the form

$$\frac{\partial \mathscr{L}}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial(\partial f/\partial x)}\right) = 0. \tag{27}$$

From the form of the Lagrangian density (25) we also have canonical "momenta" defined by

$$\Pi^+ = \frac{\partial \mathscr{L}}{\partial(\partial f/\partial x)} \equiv -M^+(x)a, \tag{28a}$$

$$\Pi = \frac{\partial \mathscr{L}}{\partial(\partial f^+/\partial x)} \equiv -aM(x) \tag{28b}$$

so that we can construct a Hamiltonian density

$$\begin{aligned}\mathscr{H} &= \Pi\frac{\partial f}{\partial x} + \Pi^+\frac{\partial f^+}{\partial x} - \mathscr{L}\\ &= -f^+(f^2+2f)\end{aligned} \tag{29}$$

and a Hamiltonian H through

$$H = \int_{-\infty}^{\infty}\mathscr{H}(x)\,dx = -\int_{-\infty}^{\infty} f^+(f^2+2f)\,dx. \tag{30}$$

If the adjoint function constraint (24) is used, then the Hamiltonian reduces to

$$H = -2\int_{-\infty}^{\infty} ff^+\,dx. \tag{31}$$

b. *Conserved quantities.* The construction of conserved quantities is based on symmetry arguments. The solitary wave propagation equation (8) and the adjoint function equation (22) should not depend on how the origin of the "spatial" coordinate, x, is chosen.

So if, in the Lagrangian, the coordinate x is replaced by $x+\xi$ where ξ is arbitrary, but infinitesimal, no difference should ensue in the Lagrangian. But since the regime of integration is over all values of x, this implies that the Lagrangian density should remain unchanged under such a shift in the origin of coordinate measurements. Now if we had obtained a solution to Eqs. (8) and (22) (subject to the constraint conditions (10) and (24)) then the Lagrangian density would be expressible as an explicit function of x:

$$\mathscr{L} = \mathscr{L}(x). \tag{32}$$

Then changing the origin of coordinates will change the Lagrangian density:

$$\mathscr{L}(x+\xi) = \mathscr{L}(x) + \xi\frac{d\mathscr{L}(x)}{dx} + O(\xi^2). \tag{33}$$

But since f and f^+ are also functions of x it follows from (25) that

$$\begin{aligned}\xi\frac{d\mathscr{L}(x)}{dx} &= \xi\frac{\partial\mathscr{L}}{\partial f}\frac{\partial f}{\partial x} + \xi\frac{\partial\mathscr{L}}{\partial f^+}\frac{\partial f^+}{\partial x}\\ &\quad + \xi\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial f}{\partial x}\right)}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \xi\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial f^+}{\partial x}\right)}\frac{\partial}{\partial x}\left(\frac{\partial f^+}{\partial x}\right)\\ &\quad + \xi\left(\frac{\partial\mathscr{L}}{\partial M}\frac{\partial M}{\partial x} + \frac{\partial\mathscr{L}}{\partial M^+}\frac{\partial M^+}{\partial x}\right).\end{aligned} \tag{34}$$

We can now use the field equations (26) and (27) to eliminate $\frac{\partial\mathscr{L}}{\partial f}$ and $\frac{\partial\mathscr{L}}{\partial f^+}$ in Eq. (34). Upon so doing we obtain

$$\begin{aligned}\xi\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\Pi + \frac{\partial f^+}{\partial x}\Pi^+ - \mathscr{L}\right] &= -\xi\left\{\frac{\partial\mathscr{L}}{\partial M}\frac{\partial M}{\partial x} + \frac{\partial\mathscr{L}}{\partial M^+}\frac{\partial M^+}{\partial x}\right\}\\ &= +\xi a\left[\frac{\partial f^+}{\partial x}\frac{\partial M}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial M^+}{\partial x}\right].\end{aligned} \tag{35}$$

But this must hold true at all values of ξ. Thus

$$\begin{aligned}\frac{\partial\mathscr{H}}{\partial x} &= -a\left[\frac{\partial f^+}{\partial x}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{f(y)\,dy}{[1+(y-x)^2]^{1/2}} + \frac{\partial f}{\partial x}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{f^+(y)\,dy}{[1+(y-x)^2]^{1/2}}\right]\\ &\equiv -\frac{\partial}{\partial x}((f^2+2f)f^+).\end{aligned} \tag{36}$$

Equation (36) is equivalent to the momentum theorem of classical mechanics. It represents the rate at which the Hamiltonian density changes with spatial coordinate (the left-hand side of equation (36)) as a consequence of an applied "force" (the right-hand side of equation (36)).

c. *Application of the variational method.* Apart from its reduction of the LR solitary wave equation to an aesthetic form involving the adjoint function, and of its use in determining the structure of local and global conserved quantities by way of Noether's theorem, the variational method can also be used to approximate to solutions to the Leibovich–Roberts equation. For the application here we shall use the forms (20) and (21) together with the constraint conditions (10) and (24).

The sense of the argument, as detailed in Morse and Feshbach (1953) and Becker (1964), is to choose trial functions for f and f^+ which approximate whatever is known about the shapes of f and f^+ that has been gleaned from the true equations. These trial functions, chosen to satisfy exactly the constraint conditions, contain one or more "free" parameters. The trial functions are introduced into the Lagrangian (20), all integrals performed, leaving then an expression for the Lagrangian L which is dependent solely on the parameters in the trial functions. The Lagrangian is then extremized with respect to the parameters, thereby providing equations for a determination of the parameters. By repeated application with more and more trial functions, a better and better approximation is constructed to the exact solution to the true equation.

In our case, from Eq. (8) we know that f is of the form $f_0 + f_1 x^2 + \cdots$ at small x, and of the form $f \propto x^{-3}$ at large x (apart from the two uninteresting solutions $f \equiv 0$ and $f \equiv -2$).

Suppose then that we search for an approximate solution to Eq. (8) in the form

$$f(x) = \sum_{n=1}^{\infty} f_{n-1}[\psi(x)^3]^n \tag{37}$$

with $\psi(x) = (1 + x^2/x_*^2)^{-1/2}$ where the f_n and x_* are to be determined.

To be specific we shall carry through the computations for $f_{n-1} = 0$, $n > 1$—although inclusion of the higher-order terms involves only that more integrals be computed and not any qualitative change in the method of analysis.

Then with the ansatz

$$f = f_0(1 + x^2/x_*^2)^{-3/2}, \tag{38}$$

the constraint (10) yields the relation

$$f_0 = -\frac{32}{3\pi} \tag{39}$$

so that, provided $x_*^2 > 0$, we have that the approximate form for $f(x)$ is negative everywhere.

The constraint (24) requires that we choose $f^+(x)$ such that

$$\int_0^{\infty} f^+(x) \frac{dx}{(1 + x^2/x_*^2)^3} = 0. \tag{40}$$

For the choice

$$f^+(x) = [\psi(x)]^2 + b[\psi(x)]^4, \tag{41}$$

we obtain that $b = -8/7$ permits satisfaction of the constraint (40). Use of (38), (39), (41), and (42) in (20) enables us to write the Lagrangian in the form

$$\frac{L}{f_0} = \int_{-\infty}^{\infty} dx \left\{ [\psi(x)]^2 \left[1 - \frac{8}{7}[\psi(x)]^2\right] [\psi(x)]^3 (f_0[\psi(x)]^3 + 2) \right.$$
$$\left. -a4[\psi(x)] \left[1 - \frac{16}{7}[\psi(x)]^3\right] \frac{\partial \psi}{\partial x} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{[\psi(y)]^3\, dy}{[1 + (y - x)^2]^{1/2}} \right\}. \tag{42a}$$

Scaling x and y in terms of x_*, we can write

$$\frac{L}{f_0} = \int_{-\infty}^{\infty} du \left\{ [\phi(u)]^2 \left[1 - \frac{8}{7}[\phi(u)]^2\right] [\phi(u)]^3 x_* (f_0[\phi(u)]^3 + 2) \right.$$
$$\left. -4a\frac{\partial}{\partial u}\left\{[\phi(u)]^4 u \left(1 - \frac{16}{7}[\phi(u)]^3\right)\right\} \int_{-\infty}^{\infty} \frac{[\phi(v)]^3\, dv}{[1 + x_*^2(u-v)^2]^{1/2}} \right\} \quad (42b)$$

where $\phi(u) = (1+u^2)^{-1/2}$. Extremizing the Lagrangian L with respect to variations in x_* yields the equation to be satisfied by x_* in the form

$$4ax_* = -\frac{\int_{-\infty}^{\infty} du[\phi(u)]^5(1 - \frac{8}{7}[\phi(u)]^2)(2 + f_0[\phi(u)]^3)}{\int_{-\infty}^{\infty} du \frac{\partial}{\partial u} u[\phi(u)]^4(1 - \frac{16}{7}[\phi(u)]^3)] \int_{-\infty}^{\infty} \frac{[\phi(v)]^3(v-u)^2\, dv}{[1+x_*^2(v-u)^2]^{3/2}}}, \quad (43a)$$

i.e.,

$$4ax_* = \frac{-2}{35}\left\{\int_{-\pi/2}^{\pi/2} \sin\theta\cos\theta\left(1 - \frac{16}{7}\cos^3\theta\right) d\theta \left(\int_{-\pi/2}^{\pi/2} \frac{\sin\phi\cos^2\phi(\tan\theta - \tan\phi)^2\, d\phi}{[1 + x_*^2(\tan\theta - \tan\phi)^2]^{3/2}}\right)\right\}^{-1}. \quad (43b)$$

Now if $x_* \ll 1$, from Eq. (43b) we obtain

$$4ax_* \cong \frac{0.3}{35\pi(1 - \frac{128}{105\pi})} \approx 1/20 \quad (44)$$

so that $x_* \approx a/80$.

The requirement $x_* \ll 1$ can then be used to argue that $a \leq 80$ for satisfaction of (43b). Clearly for $a > 80$, we have to consider the opposite extreme of (43b) where, a priori, we choose $x_* \gg 1$—but the evaluation of this case is left as an exercise for the reader.

It is also clear that by increasing the number of terms f_{n-1} $(n > 1)$ in the approximate expression (37) for $f(x)$, we improve both the functional fit to the Leibovich–Roberts equation and we also systematically refine the values of the parameters. The lowest-order approximation yields

$$f \cong -\frac{32}{3\pi}(1 + x^2/x_*^2)^{-3/2} \quad (45)$$

with $x_* \approx 1/(80a)$ for $a < 80$.

IV. Discussion and conclusion. The structure of the generalized Leibovich–Roberts equation contains an intrinsic scale-factor which is not removable by any simple transformation. This scale-factor is intimately connected with the cylindrical nature of the solitary wave problem making the determination of solitary wave behavior more difficult than for the equivalent Benjamin–Ono equation in a flat slab.

Nevertheless we were able to reduce the Leibovich–Roberts equation to a solitary wave form, were able to obtain a first integral to it, and were able to categorize the overall structural behavior of solitary wave patterns. With this categorization to hand we then showed that a variational principle existed for the Leibovich–Roberts equation which admitted of adjoint functions, conserved quantities, and which could be used in an extremal manner to determine the structural and parametric form of the solitary wave solutions to the equation. We illustrated this latter point by a simple, specific, illustration.

The structure of the solitary wave pattern of the Leibovich–Roberts equation is long-range ($\propto x^{-3}$), as is the solitary wave pattern of the corresponding Benjamin–Ono equation ($\propto x^{-2}$). The major difference seems to be that while we know how to construct analytically exactly solitary wave solutions to the Benjamin–Ono equation, and to consider their interacting soliton characteristics, no such analytical method is yet known for the Leibovich–Roberts equation (even in the case where the scale-length is set to zero, recourse to numerical methods still seems to be necessary, as done by Leibovich (1970) over a decade and a half ago).

What we have shown in this paper is that a nonlinear variational principle, à la Becker (1964), provides a powerful and elegant method for coming to grips in a systematic manner with a determination of the solutions to the Leibovich–Roberts equation. It would seem that the major difficulty to producing a more accurate approximate structural solution than the one we have presented, is in the explicit analytic evaluation of the integrals in the Lagrangian. We suspect that numerical methods should be brought to bear to improve the resolution of the approximate behavior we have uncovered. And we would also be curious to see if more effort would uncover a method of providing exact analytic solutions.

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