

A note on some generalized curvature tensor

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Dedicated to the memory of Professor Krishan Lal Duggal (1929–2022)

Abstract. For any semi-Riemannian manifold (M, g) we define some generalized curvature tensor E as a linear combination of Kulkarni-Nomizu products formed by the metric tensor, the Ricci tensor and its square of given manifold. That tensor is closely related to quasi-Einstein spaces, Roter spaces and some Roter type spaces.¹

1. INTRODUCTION

Let (M, g) be a semi-Riemannian manifold. We denote by g , R , S , κ and C , the metric tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of (M, g) , respectively. Further, let $A \wedge B$ be the Kulkarni-Nomizu product of symmetric $(0, 2)$ -tensors A and B . Now we can define the $(0, 2)$ -tensors S^2 and S^3 , the $(0, 4)$ -tensors $R \cdot S$, $C \cdot S$ and $Q(A, B)$, and the $(0, 6)$ -tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$ and $Q(A, T)$, where T is a generalized curvature tensor. For precise definitions of the symbols used, we refer to Section 2 of this paper, as well as to [34, Section 1], [37, Section 1], [38, Chapter 6] and [45, Sections 1 and 2].

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 2$, is said to be an *Einstein manifold* [2], or an *Einstein space*, if at every point of M its Ricci tensor S is proportional to g , i.e.,

$$(1.1) \quad S = \frac{\kappa}{n} g$$

on M , assuming that the scalar curvature κ is constant when $n = 2$. According to [2, p. 432] this condition is called the *Einstein metric condition*.

Let (M, g) be a semi-Riemannian manifold of dimension $\dim M = n \geq 3$. We set

$$(1.2) \quad E = g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}_g(S^2)}{2(n-1)} g \wedge g.$$

It is easy to check that the tensor E is a generalized curvature tensor. Further, we define the subsets \mathcal{U}_R and \mathcal{U}_S of M by $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\}$ and $\mathcal{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$, respectively, where $G = \frac{1}{2} g \wedge g$. If $n \geq 4$ then we define the set $\mathcal{U}_C \subset M$ as the set of all points of (M, g) at which $C \neq 0$. We note that if $n \geq 4$ then $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$ (see, e.g., [24]).

An extension of the class of Einstein manifolds form quasi-Einstein, 2-quasi-Einstein and partially Einstein manifolds. A semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, is said to be a *quasi-Einstein manifold*, or a *quasi-Einstein space*, if

$$(1.3) \quad \text{rank}(S - \alpha g) = 1$$

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on $\mathcal{U}_S \subset M$, where α is some function on \mathcal{U}_S . It is known that every non-Einstein warped product manifold $\overline{M} \times_F \tilde{N}$ with a 1-dimensional $(\overline{M}, \overline{g})$ base manifold and a 2-dimensional manifold (\tilde{N}, \tilde{g}) or an $(n-1)$ -dimensional Einstein manifold (\tilde{N}, \tilde{g}) , $\dim M = n \geq 4$, and a warping function F , is a quasi-Einstein manifold (see, e.g., [7, 34]). A Riemannian manifold (M, g) , $\dim M = n \geq 3$, whose Ricci tensor has an eigenvalue of multiplicity $n-1$ is a non-Einstein quasi-Einstein manifold (cf. [23, Introduction]). We mention that quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces (see, e.g., [27, 34] and references therein). Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in [29, 40, 43, 61] (see also [27] and references therein). Quasi-Einstein manifolds satisfying some pseudosymmetry type curvature conditions were investigated recently in [1, 7, 24, 31, 42]. Quasi-Einstein hypersurfaces in conformally flat semi-Riemannian manifolds were studied in [55, 78]. In those papers quasi-Einstein hypersurfaces were called *pseudo-Einstein hypersurfaces* (see also [66, 71]). Similarly, in [50, 86] quasi-Einstein semi-Riemannian manifolds (hypersurfaces) were called *pseudo-Einstein manifolds (hypersurfaces)*.

Let (M, g) , $\dim M = n \geq 3$, be a semi-Riemannian manifold. We note that (1.3) holds at a point $x \in \mathcal{U}_S \subset M$ if and only if $(S - \alpha g) \wedge (S - \alpha g) = 0$ at x , i.e.,

$$(1.4) \quad \frac{1}{2} S \wedge S - \alpha g \wedge S + \frac{\alpha^2}{2} g \wedge g = 0$$

at x (cf. [61, Proposition 2.1]). From (1.4), by a suitable contraction, we get immediately

$$(1.5) \quad S^2 = (\kappa - (n-2)\alpha) S + \alpha((n-1)\alpha - \kappa) g.$$

Using (1.1) we can easily check that the following equation is satisfied on any Einstein manifold (M, g)

$$(1.6) \quad g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}_g(S^2)}{2(n-1)} g \wedge g = 0,$$

i.e., $E = 0$ on M , where the tensor E is defined by (1.2). Moreover, as it was stated in [28, Lemma 2.1], (1.6) is satisfied on every quasi-Einstein manifold (M, g) , $n \geq 3$. The converse statement also is true. Precisely, from Proposition 2.1 it follows that if (M, g) , $n \geq 4$, is a semi-Riemannian manifold satisfying (1.6) on $\mathcal{U}_S \subset M$ then a condition of the form (1.3) holds on \mathcal{U}_S , where α is some function on this set. In Section 2 we also present another result related to the tensor E (see Proposition 2.2). Namely, we prove that if a generalized curvature tensor T is a linear combination of the tensors R , $S \wedge S$, $g \wedge S$, $g \wedge S^2$, and $g \wedge g$ then the Weyl tensor of T is a linear combination of the tensors C and E . The tensor E is determined by some Kulkarni-Nomizu products formed by g , S and S^2 , i.e., E is defined by (1.2). In the same way, we can define the $(0, 4)$ -tensor $E(A)$ corresponding to a symmetric $(0, 2)$ -tensor A

$$(1.7) \quad E(A) = g \wedge A^2 + \frac{n-2}{2} A \wedge A - \text{tr}_g(A) g \wedge A + \frac{(\text{tr}_g(A))^2 - \text{tr}_g(A^2)}{2(n-1)} g \wedge g.$$

The semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, will be called a *partially Einstein manifold*, or a *partially Einstein space* (cf. [5, Foreword], [82, p. 20]), if at every point $x \in \mathcal{U}_S \subset M$ its Ricci operator \mathcal{S} satisfies $S^2 = \lambda S + \mu Id_x$, or equivalently,

$$(1.8) \quad S^2 = \lambda S + \mu g,$$

where $\lambda, \mu \in \mathbb{R}$ and Id_x is the identity transformation of $T_x M$. Evidently, (1.5) is a special case of (1.8). Thus every quasi-Einstein manifold is a partially Einstein manifold. The converse statement is not true. Contracting (1.8) we get $\text{tr}_g(S^2) = \lambda \kappa + n \mu$. This together with (1.8) yields (cf. [25, Section 5])

$$S^2 - \frac{\text{tr}_g(S^2)}{n} g = \lambda \left(S - \frac{\kappa}{n} g \right).$$

In particular, a Riemannian manifold (M, g) , $\dim M = n \geq 3$, is a partially Einstein space if at every point $x \in \mathcal{U}_S \subset M$ its Ricci operator S has exactly two distinct eigenvalues κ_1 and κ_2 with multiplicities p and $n - p$, respectively, where $1 \leq p \leq n - 1$. Evidently, if $p = 1$, or $p = n - 1$, then (M, g) is a quasi-Einstein manifold.

In Section 3 we present definitions of some classes of semi-Riemannian manifolds determined by curvature conditions of pseudosymmetry type. Investigations of semi-Riemannian manifolds satisfying some particular curvature conditions of pseudosymmetry type lead to Roter spaces (see Proposition 4.1). Roter spaces form an important subclass of the class of non-conformally flat and non-quasi-Einstein partially Einstein manifolds of dimension ≥ 4 . Namely, a non-quasi-Einstein and non-conformally flat semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$, satisfying on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the following equation

$$(1.9) \quad R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \frac{\eta}{2} g \wedge g,$$

where ϕ, μ and η are some functions on this set, is called a *Roter type manifold*, or a *Roter manifold*, or a *Roter space* (see, e.g., [6, Section 15.5], [22, 34, 35, 38]). Equation (1.9) is called a *Roter equation* (see, e.g., [28, Section 1]). In Section 4 we present results on such manifolds. For instance, every Roter space (M, g) , $\dim M = n \geq 4$, satisfies among others the following pseudosymmetry type curvature condition on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ (see Theorem 4.2)

$$(1.10) \quad C \cdot R - R \cdot C = Q(S, C) - \frac{\kappa}{n-1} Q(g, C).$$

Let (M, g) , $\dim M = n \geq 4$, be a non-partially-Einstein and non-conformally flat semi-Riemannian manifold. If its Riemann-Christoffel curvature R is at every point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ a linear combination of the Kulkarni-Nomizu products formed by the tensors $S^0 = g$ and $S^1 = S, \dots, S^{p-1}, S^p$, where p is some natural number ≥ 2 , then (M, g) is called a *generalized Roter type manifold*, or a *generalized Roter manifold*, or a *generalized Roter type space*, or a *generalized Roter space*. For instance, when $p = 2$, we have

$$(1.11) \quad R = \frac{\phi_2}{2} S^2 \wedge S^2 + \phi_1 S \wedge S^2 + \frac{\phi}{2} S \wedge S + \mu_1 g \wedge S^2 + \mu g \wedge S + \frac{\eta}{2} g \wedge g,$$

where $\phi, \phi_1, \phi_2, \mu_1, \mu$ and η are functions on $\mathcal{U}_S \cap \mathcal{U}_C$. Because (M, g) is a non-partially Einstein manifold, at least one of the functions μ_1, ϕ_1 and ϕ_2 is a non-zero function. Equation (1.11) is called a *Roter type equation* (see, e.g., [28, Section 1]). We refer to [28, 33, 34, 35, 42, 73, 74, 75, 76, 77] for results on manifolds (hypersurfaces) satisfying (1.11).

As it was stated in [28, Lemma 2.2] (see Proposition 4.4), if (M, g) , $\dim M = n \geq 4$, is a Roter space satisfying (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then on this set

$$(1.12) \quad C = \frac{\phi}{n-2} \left(g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}_g(S^2)}{2(n-1)} g \wedge g \right).$$

In Section 5 we recall results on some warped product manifolds with 2-dimensional base manifold obtained in [28].

In Section 6 we state that on every essentially conformally symmetric manifold the following equation is satisfied

$$(1.13) \quad \begin{aligned} FC &= \frac{n-2}{2(n-2)} S \wedge S \\ &= \frac{1}{n-2} \left(g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}_g(S^2)}{2(n-1)} g \wedge g \right). \end{aligned}$$

In Section 7 we recall some known results on hypersurfaces M , $\dim M \geq 4$, isometrically immersed in a conformally flat spaces. In particular, we mention that at every point of M its Weyl conformal curvature tensor C and the $(0,4)$ -tensor $E(H)$, formed for the second fundamental tensor H of M , are linearly dependent (see (7.3)).

In the last section we investigate non-Einstein and non-conformally flat hypersurfaces M , $\dim M \geq 4$, isometrically immersed in semi-Riemannian spaces of constant curvature satisfying some curvature conditions of pseudosymmetry type. Under some additional assumptions imposed on the second fundamental tensor H of M we obtain equations involved with the tensor E .

2. PRELIMINARIES.

Throughout this paper, all manifolds are assumed to be connected paracompact manifolds of class C^∞ . Let (M, g) , $\dim M = n \geq 3$, be a semi-Riemannian manifold, and let ∇ be its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on M . We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X, Y)$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y$ and

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

respectively, where $X, Y, Z \in \Xi(M)$ and A is a symmetric $(0,2)$ -tensor on M . The Ricci tensor S , the Ricci operator \mathcal{S} and the scalar curvature κ of (M, g) are defined by

$$S(X, Y) = \text{tr}\{Z \rightarrow \mathcal{R}(Z, X)Y\}, \quad g(\mathcal{S}X, Y) = S(X, Y), \quad \kappa = \text{tr } \mathcal{S},$$

respectively. The endomorphism $\mathcal{C}(X, Y)$ is defined by

$$\mathcal{C}(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y)Z.$$

Now the $(0,4)$ -tensor G , the Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M, g) are defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4)$ and

$$\begin{aligned} R(X_1, X_2, X_3, X_4) &= g(\mathcal{R}(X_1, X_2)X_3, X_4), \\ C(X_1, X_2, X_3, X_4) &= g(\mathcal{C}(X_1, X_2)X_3, X_4), \end{aligned}$$

respectively, where $X_1, X_2, \dots \in \Xi(M)$. For a symmetric $(0,2)$ -tensor A we denote by \mathcal{A} the endomorphism related to A by $g(\mathcal{A}X, Y) = A(X, Y)$. The $(0,2)$ -tensors A^p , $p = 2, 3, \dots$, are defined by $A^p(X, Y) = A^{p-1}(\mathcal{A}X, Y)$, assuming that $A^1 = A$. In this way, for $A = S$ and $\mathcal{A} = \mathcal{S}$ we get the tensors S^p , $p = 2, 3, \dots$, assuming that $S^1 = S$.

Let \mathcal{B} be a tensor field sending any $X, Y \in \Xi(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X, Y)$, and let B be the $(0, 4)$ -tensor associated with \mathcal{B} by

$$(2.1) \quad B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4).$$

The tensor B is said to be a *generalized curvature tensor* if the following two conditions are fulfilled: $B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2)$ and

$$B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) = 0.$$

For \mathcal{B} as above, let B be again defined by (2.1). We extend the endomorphism $\mathcal{B}(X, Y)$ to a derivation $\mathcal{B}(X, Y) \cdot$ of the algebra of tensor fields on M , assuming that it commutes with contractions and $\mathcal{B}(X, Y) \cdot f = 0$ for any smooth function f on M . Now for a $(0, k)$ -tensor field T , $k \geq 1$, we can define the $(0, k+2)$ -tensor $B \cdot T$ by

$$\begin{aligned} (B \cdot T)(X_1, \dots, X_k, X, Y) &= (\mathcal{B}(X, Y) \cdot T)(X_1, \dots, X_k) \\ &= -T(\mathcal{B}(X, Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, \mathcal{B}(X, Y)X_k). \end{aligned}$$

If A is a symmetric $(0, 2)$ -tensor then we define the $(0, k+2)$ -tensor $Q(A, T)$ by

$$\begin{aligned} Q(A, T)(X_1, \dots, X_k, X, Y) &= (X \wedge_A Y \cdot T)(X_1, \dots, X_k) \\ &= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k). \end{aligned}$$

In this manner we obtain the $(0, 6)$ -tensors $B \cdot B$ and $Q(A, B)$.

Substituting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$ we get the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$, $Q(g, R)$, $Q(S, R)$, $Q(g, C)$, $Q(S, C)$, and $Q(g, S)$, $Q(g, S^2)$, $Q(S, S^2)$.

For a symmetric $(0, 2)$ -tensor A and a $(0, k)$ -tensor T , $k \geq 2$, we define their *Kulkarni-Nomizu tensor* $A \wedge T$ by (see, e.g., [24, Section 2])

$$\begin{aligned} (A \wedge T)(X_1, X_2, X_3, X_4; Y_3, \dots, Y_k) \\ &= A(X_1, X_4)T(X_2, X_3, Y_3, \dots, Y_k) + A(X_2, X_3)T(X_1, X_4, Y_3, \dots, Y_k) \\ &\quad - A(X_1, X_3)T(X_2, X_4, Y_3, \dots, Y_k) - A(X_2, X_4)T(X_1, X_3, Y_3, \dots, Y_k). \end{aligned}$$

It is obvious that the following tensors are generalized curvature tensors: R , C and $A \wedge B$, where A and $B = T$ are symmetric $(0, 2)$ -tensors. We have

$$(2.2) \quad C = R - \frac{1}{n-2}g \wedge S + \frac{\kappa}{(n-2)(n-1)}G,$$

$$(2.3) \quad G = \frac{1}{2}g \wedge g,$$

and (see, e.g., [24, Lemma 2.2(i)])

$$(2.4) \quad \begin{aligned} (a) \quad Q(A, A \wedge B) &= -\frac{1}{2}Q(B, A \wedge A), \\ (b) \quad A \wedge Q(A, B) &= -\frac{1}{2}Q(B, A \wedge A). \end{aligned}$$

By an application of (2.4)(a) we obtain on M the identities

$$(2.5) \quad Q(g, g \wedge S) = -Q(S, G), \quad Q(S, g \wedge S) = -\frac{1}{2}Q(g, S \wedge S).$$

Further, by making use of (2.2), (2.3) and (2.5), we get immediately

$$\begin{aligned}
 Q(g, C) &= Q(g, R) - \frac{1}{n-2} Q(g, g \wedge S) + \frac{\kappa}{(n-2)(n-1)} Q(g, G) \\
 (2.6) \quad &= Q(g, R) - \frac{1}{n-2} Q(g, g \wedge S) = Q(g, R) + \frac{1}{n-2} Q(S, \frac{1}{2} g \wedge g),,
 \end{aligned}$$

$$\begin{aligned}
 Q(S, C) &= Q(S, R) - \frac{1}{n-2} Q(S, g \wedge S) + \frac{\kappa}{(n-2)(n-1)} Q(S, G) \\
 (2.7) \quad &= Q(S, R) + \frac{1}{2(n-2)} Q(g, S \wedge S) - \frac{\kappa}{(n-2)(n-1)} Q(g, g \wedge S).
 \end{aligned}$$

From (2.4) (a) it follows immediately that $Q(g, g \wedge g) = 0$. Thus we have

$$(2.8) \quad Q(g, E) = Q(g, g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S),$$

where the tensor E is defined by (1.2).

Let A_1, A_2 and B be symmetric $(0, 2)$ -tensors. We have (see, e.g., [7, Lemma 2.1(i)] and references therein)

$$(2.9) \quad A_1 \wedge Q(A_2, B) + A_2 \wedge Q(A_1, B) + Q(B, A_1 \wedge A_2) = 0.$$

From (2.9) we get easily (see also [24, Lemma 2.2(iii)] and references therein)

$$Q(B, A_1 \wedge A_2) + Q(A_1, A_2 \wedge B) + Q(A_2, B \wedge A_1) = 0.$$

Let A be a symmetric $(0, 2)$ -tensor and T a $(0, k)$ -tensor, $k = 2, 3, \dots$. The tensor $Q(A, T)$ is called the *Tachibana tensor* of A and T , or the Tachibana tensor for short (see, e.g., [36]). Using the tensors g, R and S we can define the following $(0, 6)$ -Tachibana tensors: $Q(S, R), Q(g, R), Q(g, g \wedge S)$ and $Q(S, g \wedge S)$. We can check, by making use of (2.4)(a) and (2.5), that other $(0, 6)$ -Tachibana tensors constructed from g, R and S may be expressed by the four Tachibana tensors mentioned above or vanish identically on M .

Let T be a generalized curvature tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$. We denote by $\text{Ric}(T)$, $\kappa(T)$ and $\text{Weyl}(T)$ the Ricci tensor, the scalar curvature and the Weyl tensor of the tensor T , respectively. We refer to [24, Section 2], [25, Section 3] or [31, Section 3] for definitions of the considered tensors. In particular, we have

$$(2.10) \quad \text{Weyl}(T) = T - \frac{1}{n-2} g \wedge \text{Ric}(T) + \frac{\kappa(T)}{2(n-2)(n-1)} g \wedge g.$$

Let A be a symmetric $(0, 2)$ -tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$. Let $E(A)$ be the tensor defined by (1.7). It is easy to check that $\text{Ric}(E(A))$ is a zero tensor. Therefore, we also have $\kappa(E(A)) = 0$. Any generalized curvature tensor T defined on a 3-dimensional semi-Riemannian manifold (M, g) is expressed by $T = g \wedge \text{Ric}(T) - (\kappa(T)/4)g \wedge g$ [56, p. 48] (see also [21, Lemma 2 (ii)]). Thus we see that the tensor $T = E(A)$ on any 3-dimensional semi-Riemannian manifold (M, g) is a zero tensor. In particular, on any 3-dimensional semi-Riemannian manifold (M, g) we have $E = 0$.

Let A be a symmetric $(0, 2)$ -tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$. We denote by \mathcal{U}_A the set of points of M at which $A \neq \frac{\text{tr}_g(A)}{n} g$.

Proposition 2.1. *Let A be a symmetric $(0, 2)$ -tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$. (i) (cf. [28, Lemma 2.1]) If the following condition is satisfied on $\mathcal{U}_A \subset M$*

$$(2.11) \quad \text{rank}(A - \alpha g) = 1$$

then

$$(2.12) \quad g \wedge A^2 + \frac{n-2}{2} A \wedge A - \text{tr}_g(A) g \wedge A + \frac{(\text{tr}_g(A))^2 - \text{tr}_g(A^2)}{2(n-1)} g \wedge g = 0$$

and

$$(2.13) \quad A^2 - \frac{\text{tr}_g(A^2)}{n} = (\text{tr}_g(A) - (n-2)\alpha) \left(A - \frac{\text{tr}_g(A)}{n} g \right)$$

on \mathcal{U}_A , where α is some function on \mathcal{U}_A .

(ii) If (2.12) is satisfied on $\mathcal{U}_A \subset M$ then

$$(2.14) \quad A^2 - \frac{\text{tr}_g(A^2)}{n} g = \rho \left(A - \frac{\text{tr}_g(A)}{n} g \right)$$

and

$$(2.15) \quad \left(A - \frac{\text{tr}_g(A) - \rho}{n-2} g \right) \wedge \left(A - \frac{\text{tr}_g(A) - \rho}{n-2} g \right) = 0$$

on \mathcal{U}_A , where ρ is some function on \mathcal{U}_A .

Proof. (i) (cf. the proof of [28, Lemma 2.1]) Equation (2.11) yields [61, Proposition 2.2]

$$(2.16) \quad \frac{1}{2} A \wedge A = \alpha g \wedge A - \frac{\alpha^2}{2} g \wedge g.$$

This, by suitable contractions yields

$$(2.17) \quad A^2 - \text{tr}_g(A) A = -(n-2)\alpha A - \alpha \text{tr}_g(A) g + (n-1)\alpha^2 g,$$

$$(2.18) \quad \text{tr}_g(A^2) - (\text{tr}_g(A))^2 = -2(n-1)\alpha \text{tr}_g(A) + n(n-1)\alpha^2.$$

Now using (2.16), (2.17) and (2.18) we can easily check that (2.12) and (2.13) hold on \mathcal{U}_A .

(ii) (cf. the proof of [20, Lemma 3.4]) In the local coordinates (2.12) reads

$$(2.19) \quad \begin{aligned} & g_{hk} A_{ij}^2 + g_{ij} A_{hk}^2 - g_{hj} A_{ik}^2 - g_{ik} A_{hj}^2 + (n-2)(A_{hk} A_{ij} - A_{hj} A_{ik}) \\ & - \text{tr}_g(A) (g_{hk} A_{ij} + g_{ij} A_{hk} - g_{hj} A_{ik} - g_{ik} A_{hj}) \\ & + \frac{(\text{tr}_g(A))^2 - \text{tr}_g(A^2)}{n-1} (g_{hk} g_{ij} - g_{hj} g_{ik}) = 0. \end{aligned}$$

Contracting (2.19) with $A^{ij} = A_{rs} g^{ri} g^{sj}$ and $A^k_l = A_{rl} g^{rk}$ we find

$$(2.20) \quad \begin{aligned} A^3 &= \frac{3\text{tr}_g(A)}{n} A^2 + \left(\frac{(n^2 - 3n + 3)\text{tr}_g(A^2)}{(n-1)n} - \frac{(\text{tr}_g(A))^2}{n-1} \right) A \\ &+ \left(\frac{(\text{tr}_g(A))^3}{(n-1)n} - \frac{\text{tr}_g(A) \text{tr}_g(A^2)}{n-1} + \frac{\text{tr}_g(A^3)}{n} \right) g, \end{aligned}$$

$$\begin{aligned}
& A_{hl}A_{ij}^2 - A_{il}A_{hj}^2 + g_{ij}A_{hl}^3 - g_{hj}A_{il}^3 + (n-2)(A_{ij}A_{hl}^2 - A_{hj}A_{il}^2) \\
& - \text{tr}_g(A)(A_{hl}A_{ij} - A_{il}A_{hj} + g_{ij}A_{hl}^2 - g_{hj}A_{il}^2) \\
(2.21) \quad & + \frac{(\text{tr}_g(A))^2 - \text{tr}_g(A^2)}{n-1}(g_{ij}A_{hl} - g_{hj}A_{il}) = 0,
\end{aligned}$$

respectively. From (2.21), by symmetrization in l, j , we obtain

$$\begin{aligned}
& Q(g, A^3) + (n-3)Q(A, A^2) - \text{tr}_g(A)Q(g, A^2) \\
(2.22) \quad & + \frac{(\text{tr}_g(A))^2 - \text{tr}_g(A^2)}{n-1}Q(g, A) = 0.
\end{aligned}$$

Applying (2.20) into (2.22) we get

$$(n-3) \left(Q(A, A^2) - \frac{\text{tr}_g(A)}{n}Q(g, A^2) + \frac{\text{tr}_g(A^2)}{n}Q(g, A) \right) = 0,$$

which yields

$$Q \left(A - \frac{\text{tr}_g(A)}{n}g, A^2 - \frac{\text{tr}_g(A^2)}{n}g \right) = 0.$$

From this, in view of [48, Lemma 2.4 (ii)], it follows that (2.14) holds on \mathcal{U}_A . Now (2.12) and (2.14), by an application of [58, Lemma 3.1], lead to (2.15), completing the proof of (ii).

Proposition 2.2. *Let T be a generalized curvature tensor on a semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$. If the following condition is satisfied at a point $x \in M$*

$$(2.23) \quad T = \alpha_1 R + \frac{\alpha_2}{2} S \wedge S + \alpha_3 g \wedge S + \alpha_4 g \wedge S^2 + \frac{\alpha_5}{2} g \wedge g$$

then

$$(2.24) \quad \text{Weyl}(T) = \alpha_1 C + \frac{\alpha_2}{n-2} E$$

at this point, where the tensor E is defined by (1.2) and $\alpha_1, \dots, \alpha_5 \in \mathbb{R}$.

Proof. From (2.23), by a suitable contraction, we get immediately

$$(2.25) \quad \text{Ric}(T) = (\alpha_1 + \alpha_2\kappa + (n-2)\alpha_3)S + ((n-2)\alpha_4 - \alpha_2)S^2 + \alpha_6 g,$$

where α_6 is some real number. Now using (1.2), (2.2), (2.3), (2.10), (2.23) and (2.25) we get

$$\begin{aligned}
\text{Weyl}(T) &= T - \frac{1}{n-2}g \wedge \text{Ric}(T) + \frac{\kappa(T)}{2(n-2)(n-1)}g \wedge g \\
&= \alpha_1 R + \frac{\alpha_2}{2}S \wedge S + \alpha_3 g \wedge S + \alpha_4 g \wedge S^2 + \frac{\alpha_7}{2}g \wedge g \\
&\quad - \frac{1}{n-2}(\alpha_1 + \alpha_2\kappa + (n-2)\alpha_3)g \wedge S - \frac{1}{n-2}((n-2)\alpha_4 - \alpha_2)g \wedge S^2 \\
&= \alpha_1 R + \frac{\alpha_2}{2}S \wedge S - \frac{\alpha_1 + \alpha_2\kappa}{n-2}g \wedge S + \frac{\alpha_2}{n-2}g \wedge S^2 + \frac{\alpha_7}{2}g \wedge g \\
&= \alpha_1 \left(R - \frac{1}{n-2}g \wedge S \right) + \frac{\alpha_2}{n-2} \left(g \wedge S^2 + \frac{n-2}{2}S \wedge S - \kappa g \wedge S \right) + \frac{\alpha_7}{2}g \wedge g \\
&= \alpha_1 C + \frac{\alpha_2}{n-2}E + \frac{\alpha_8}{2}g \wedge g,
\end{aligned}$$

i.e.,

$$(2.26) \quad \text{Weyl}(T) = \alpha_1 C + \frac{\alpha_2}{n-2} E + \frac{\alpha_8}{2} g \wedge g,$$

where α_7 and α_8 are some real numbers. From (2.26), by suitable contraction, we get immediately $\alpha_8 = 0$, and in a consequence (2.24), completing the proof.

3. PSEUDOSYMMETRY TYPE CURVATURE CONDITIONS

It is well-known that if a semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, is locally symmetric then $\nabla R = 0$ on M (see, e.g., [70, Chapter 1.5]). This implies the following integrability condition $\mathcal{R}(X, Y) \cdot R = 0$ in short $R \cdot R = 0$. Semi-Riemannian manifold satisfying the last condition is called *semisymmetric* (see, e.g., [3, Chapter 8.5.3], [4, Chapter 20.7], [70, Chapter 1.6], [80, 84]). Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. A semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, is said to be *pseudosymmetric* if the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point of M (see, e.g., [3, Chapter 8.5.3], [4, Chapter 20.7], [6, Section 15.1], [38, Chapter 6], [70, Chapter 12.4], [24, 27, 38, 49, 63, 64, 75, 81, 83, 84, 85] and references therein). This is equivalent to

$$(3.1) \quad R \cdot R = L_R Q(g, R)$$

on $\mathcal{U}_R \subset M$, where L_R is some function on \mathcal{U}_R . Every semisymmetric manifold is pseudosymmetric. The converse statement is not true (see, e.g., [49]). We note that (3.1) implies

$$(3.2) \quad R \cdot S = L_R Q(g, S)$$

and

$$(3.3) \quad R \cdot C = L_R Q(g, C).$$

Conditions (3.1), (3.2) and (3.3) are equivalent on the set $\mathcal{U}_S \cap \mathcal{U}_C$ of any warped product manifold $\overline{M} \times_F \tilde{N}$, with $\dim \overline{M} = \dim \tilde{N} = 2$ (see, e.g., [34] and references therein).

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, is called *Ricci-pseudosymmetric* if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent at every point of M (see, e.g., [3, Chapter 8.5.3], [6, Section 15.1], [27]). This is equivalent on \mathcal{U}_S to

$$(3.4) \quad R \cdot S = L_S Q(g, S),$$

where L_S is some function on \mathcal{U}_S . Every warped product manifold $\overline{M} \times_F \tilde{N}$ with a 1-dimensional manifold $(\overline{M}, \overline{g})$ and an $(n-1)$ -dimensional Einstein semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 3$, and a warping function F , is a Ricci-pseudosymmetric manifold, see, e.g., [7, Section 1] and [34, Example 4.1].

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$, is said to be *Weyl-pseudosymmetric* if the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M [24, 27]. This is equivalent on \mathcal{U}_C to

$$(3.5) \quad R \cdot C = L_1 Q(g, C),$$

where L_1 is some function on \mathcal{U}_C . We can easily check that on every Einstein manifold (M, g) , $\dim M \geq 4$, (3.5) turns into

$$R \cdot R = L_1 Q(g, R).$$

For a presentation of results on the problem of the equivalence of pseudosymmetry, Ricci-pseudosymmetry and Weyl-pseudosymmetry we refer to [27, Section 4].

A semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$, is said to be a *manifold with pseudosymmetric Weyl tensor* (to have a *pseudosymmetric conformal Weyl tensor*) if the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M (see, e.g., [6, Section 15.1], [24, 27, 34]). This is equivalent on \mathcal{U}_C to

$$(3.6) \quad C \cdot C = L_C Q(g, C),$$

where L_C is some function on \mathcal{U}_C . Every warped product manifold $\overline{M} \times_F \tilde{N}$, with $\dim \overline{M} = \dim \tilde{N} = 2$, satisfies (3.6) (see, e.g., [24, 27, 34] and references therein). Thus in particular, the Schwarzschild spacetime, the Kottler spacetime and the Reissner-Nordström spacetime satisfy (3.6). Semi-Riemannian manifolds with pseudosymmetric Weyl tensor were studied among others in [24, 42, 50].

Warped product manifolds $\overline{M} \times_F \tilde{N}$, of dimension ≥ 4 , satisfying on $\mathcal{U}_C \subset \overline{M} \times_F \tilde{N}$, the condition

$$(3.7) \quad R \cdot R - Q(S, R) = L Q(g, C),$$

where L is some function on \mathcal{U}_C , were studied among others in [10]. In that paper necessary and sufficient conditions for $\overline{M} \times_F \tilde{N}$ to be a manifold satisfying (3.7) are given. Moreover, in that paper it was proved that any 4-dimensional warped product manifold $\overline{M} \times_F \tilde{N}$, with a 1-dimensional base $(\overline{M}, \overline{g})$, satisfies (3.7) [10, Theorem 4.1].

We refer to [7, 21, 24, 27, 31, 34, 38, 42, 75] for details on semi-Riemannian manifolds satisfying (3.1) and (3.4)-(3.7), as well other conditions of this kind, named pseudosymmetry type curvature conditions. We also refer to [42, Section 3] for a recent survey on manifolds satisfying such curvature conditions. It seems that the condition (3.1) is the most important condition of that family of curvature conditions (see, e.g., [34]). The Schwarzschild spacetime, the Kottler spacetime, the Reissner-Nordström spacetime, as well as the Friedmann-Lemaître-Robertson-Walker spacetimes are the “oldest” examples of pseudosymmetric warped product manifolds (see, e.g., [34, 38, 49, 75]). We finish this section with the following remarks.

Remark 3.1. (i) In view of [20, Lemma 3.2 (ii)], we can state that the following identity is satisfied on every semi-Riemannian manifold (M, g) , $\dim M = n \geq 3$, with vanishing Weyl conformal curvature tensor C

$$(3.8) \quad R \cdot R - Q(S, R) = \frac{1}{(n-2)^2} Q(g, g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S).$$

From (3.8), by (2.8), we get

$$R \cdot R - Q(S, R) = \frac{1}{(n-2)^2} Q(g, E),$$

where the tensor E is defined by (1.2). In particular, if $n = 3$ then $E = 0$ on M .

(ii) As it was stated in [20, Theorem 3.1] on every 3-dimensional semi-Riemannian manifold (M, g) the identity $R \cdot R = Q(S, R)$ is satisfied.

(iii) From (i) it follows that on every semi-Riemannian conformally flat manifold (M, g) , $\dim M = n \geq 4$, the conditions: $R \cdot R = Q(S, R)$ and (1.6) are equivalent.

Remark 3.2. Let (M, g) , $\dim M = n \geq 4$, be a semi-Riemannian manifold.

(i) [34, Theorem 3.4 (i)] The following identity is satisfied on $\mathcal{U}_C \subset M$

$$(3.9) \quad C \cdot R + R \cdot C = R \cdot R + C \cdot C - \frac{1}{(n-2)^2} Q(g, g \wedge S^2 - \frac{\kappa}{n-1} g \wedge S).$$

(ii) If (3.7) holds on $\mathcal{U}_C \subset M$ then (3.9) turns into

$$(3.10) \quad \begin{aligned} C \cdot R + R \cdot C &= C \cdot C + Q(S, R) + LQ(g, C) \\ &- \frac{1}{(n-2)^2} Q(g, g \wedge S^2 + \frac{n-2}{2} S \wedge S - \frac{\kappa}{n-1} g \wedge S). \end{aligned}$$

Moreover, from (3.10), by an application of (2.7) and (2.8), we get on $\mathcal{U}_C \subset M$

$$(3.11) \quad C \cdot R + R \cdot C = C \cdot C + Q(S, C) + LQ(g, C) - \frac{1}{(n-2)^2} Q(g, E),$$

where the tensor E is defined by (1.2).

(iii) (cf. [34, Theorem 3.4 (iii)]) If (3.6) and (3.7) hold on $\mathcal{U}_C \subset M$ then (3.11) turns into

$$C \cdot R + R \cdot C = Q(S, C) + (L_C + L) Q(g, C) - \frac{1}{(n-2)^2} Q(g, E).$$

4. ROTER SPACES

Some results of [24, 39, 54] (cf. [34, Section 1]) we can present in the following proposition.

Proposition 4.1. Let (M, g) , $\dim M = n \geq 4$, be a non-conformally flat and non-Einstein semi-Riemannian manifold.

(i) [54, Theorem 3.1, Theorem 3.2 (ii)] If (3.1) and (3.6) hold on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then at every point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ (1.3) or (1.9) is satisfied.

(ii) [39, Theorem 3.1, Theorem 3.2 (ii)] If (3.1) and (3.7) hold on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then at every point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ (1.3) or (1.9) is satisfied.

(iii) (cf. [24, Proposition 3.2, Theorem 3.3, Theorem 4.4]) If (3.6), (3.7) and $R \cdot S = Q(g, D)$, for some symmetric $(0, 2)$ -tensor D , hold on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then at every point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ (1.3) or (1.9) is satisfied.

We recall that a non-quasi-Einstein and non-conformally flat semi-Riemannian manifold (M, g) , $\dim M = n \geq 4$, satisfying (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ is called a *Roter type manifold*, or a *Roter manifold*, or a *Roter space* (see, e.g., [6, Section 15.5], [22, 34, 35, 38]).

Roter spaces and in particular Roter hypersurfaces in semi-Riemannian spaces of constant curvature were studied in: [8, 22, 24, 31, 40, 44, 46, 47, 60, 67, 68]. In particular, (3.1) and (3.4)-(3.7) are satisfied on such manifolds. More precisely, we have

Theorem 4.2. (see, e.g., [27, 34], [45, eq. (28)]) *If (M, g) , $\dim M = n \geq 4$, is a semi-Riemannian Roter space satisfying (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then on this set we have:*

$$\begin{aligned}
S^2 &= \alpha_1 S + \alpha_2 g, & \alpha_1 &= \kappa + \frac{(n-2)\mu - 1}{\phi}, & \alpha_2 &= \frac{\mu\kappa + (n-1)\eta}{\phi}, \\
R \cdot C &= L_R Q(g, C), & L_R &= \frac{1}{\phi} ((n-2)(\mu^2 - \phi\eta) - \mu), \\
R \cdot R &= L_R Q(g, R), \\
R \cdot S &= L_R Q(g, S), \\
R \cdot R &= Q(S, R) + L Q(g, C), & L &= L_R + \frac{\mu}{\phi} = \frac{n-2}{\phi} (\mu^2 - \phi\eta), \\
C \cdot C &= L_C Q(g, C), & L_C &= L_R + \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \alpha_1 \right), \\
C \cdot R &= L_C Q(g, R), \\
C \cdot S &= L_C Q(g, S), \\
C \cdot R + R \cdot C &= Q(S, C) + \left(L + L_C - \frac{1}{(n-2)\phi} \right) Q(g, C), \\
R \cdot C - C \cdot R &= \left(\frac{1}{\phi} \left(\mu - \frac{1}{n-2} \right) + \frac{\kappa}{n-1} \right) Q(g, R) + \left(\frac{\mu}{\phi} \left(\mu - \frac{1}{n-2} \right) - \eta \right) Q(S, G),
\end{aligned}$$

(1.10) and

$$R \cdot C - C \cdot R = Q \left(\left(\frac{\mu\kappa}{n-1} + \eta \right) g + \left(\frac{1}{n-2} - \mu - \frac{\phi\kappa}{n-1} \right) S, g \wedge S \right).$$

Remark 4.3. (i) *In the standard Schwarzschild coordinates $(t; r; \theta; \phi)$, and the physical units ($c = G = 1$), the Reissner-Nordström-de Sitter ($\Lambda > 0$), and Reissner-Nordström-anti-de Sitter ($\Lambda < 0$) spacetimes are given by the line element (see, e.g., [79])*

$$\begin{aligned}
(4.1) \quad ds^2 &= -h(r) dt^2 + h(r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \\
h(r) &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^3,
\end{aligned}$$

where M , Q and Λ are non-zero constants.

(ii) [26, Section 6] (see also [8, Remark 2 (ii)], [41, Remark 2.1 (ii)]) *The metric (4.1) satisfies (1.9) with*

$$\begin{aligned}
\phi &= \frac{3}{2}(Q^2 - Mr)r^4 2Q^{-4}, & \mu &= \frac{1}{2}(Q^4 + 3Q^2\Lambda r^4 - 3\Lambda Mr^5)Q^{-4}, \\
\eta &= \frac{1}{12}(3Q^6 + 4Q^4\Lambda r^4 - 3Q^4Mr + 9Q^2\Lambda^2 r^8 - 9\Lambda^2 Mr^9)r^{-4}Q^{-4}.
\end{aligned}$$

If we set $\Lambda = 0$ in (4.1) then we obtain the line element of the Reissner-Nordström spacetime, see, e.g., [62, Section 9.2] and references therein. It seems that the Reissner-Nordström spacetime is the oldest example of the Roter warped product space.

(iii) *In [41] a particular class of Roter warped product spaces was determined such that every manifold of that class admits a non-trivial geodesic mapping onto some Roter warped product*

space. Moreover, both geodesically related manifolds are pseudosymmetric of constant type.

(iv) An algebraic classification of the Roter type 4-dimensional spacetimes is given in [8].

(v) Some comments on pseudosymmetric manifolds (also called Deszcz symmetric spaces), as well as Roter spaces, are given in [9, Section 1] (see also [8, Remark 2 (iii)], [41, Remark 2.1 (iii)]): "From a geometric point of view, the Deszcz symmetric spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms." and "From an algebraic point of view, Roter spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms." For further comments we refer to [84].

We finish this section with the following results.

Proposition 4.4. [28, Lemma 2.2] *If (M, g) , $\dim M = n \geq 4$, is a Roter space satisfying (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then (1.12) holds on this set, i.e., $(n - 2)C = \phi E$, where the tensor E is defined by (1.2).*

Propositions 2.1, 4.1 and 4.4 lead to the following

Proposition 4.5. *Let (M, g) , $\dim M = n \geq 4$, be a non-conformally flat and non-Einstein semi-Riemannian manifold. If (3.1) and (3.6), or (3.1) and (3.7), or (3.6), (3.7) and $R \cdot S = Q(g, D)$, for some symmetric $(0, 2)$ -tensor D , hold on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then $E = \lambda C$ on $\mathcal{U}_S \cap \mathcal{U}_C$, where the tensor E is defined by (1.2) and λ is some function on this set.*

5. WARPED PRODUCT MANIFOLDS WITH 2-DIMENSIONAL BASE MANIFOLD

Proposition 2.1 and Proposition 4.4 imply

Proposition 5.1. [28, Proposition 2.3] *If (M, g) , $\dim M = n \geq 4$, is a semi-Riemannian manifold satisfying (1.3) or (1.9) at every point of $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then the following equation is satisfied on this set*

$$(5.1) \quad \tau C = g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}_g(S^2)}{2(n-1)} g \wedge g,$$

where τ is some function on $\mathcal{U}_S \cap \mathcal{U}_C$.

Proposition 5.1, [34, Theorem 7.1 (ii)] and [44, Theorem 4.1] imply

Theorem 5.2. [28, Theorem 2.4] *Let $\overline{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$, an $(n-2)$ -dimensional semi-Riemannian manifold (\tilde{N}, \tilde{g}) , $n \geq 4$, a warping function F , and let (\tilde{N}, \tilde{g}) be a space of constant curvature when $n \geq 5$. Then (5.1) holds on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \tilde{N}$.*

Example 5.3. [28, Example 2.1] (i) *Let $\mathbb{S}^k(r)$ be a k -dimensional standard sphere of radius r in \mathbb{E}^{k+1} , $k \geq 1$. It is well-known that the Cartesian product $\mathbb{S}^1(r_1) \times \mathbb{S}^{n-1}(r_2)$ of spheres $\mathbb{S}^1(r_1)$ and $\mathbb{S}^{n-1}(r_2)$, $n \geq 4$, and more generally, the warped product manifold $\mathbb{S}^1(r_1) \times_F \mathbb{S}^{n-1}(r_2)$ of spheres $\mathbb{S}^1(r_1)$ and $\mathbb{S}^{n-1}(r_2)$, $n \geq 4$, with a warping function F , is a conformally flat manifold.* (ii) *As it was stated in [60, Example 3.2], the Cartesian product $\mathbb{S}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$ of spheres $\mathbb{S}^p(r_1)$ and $\mathbb{S}^k(r_2)$ such that $2 \leq p \leq n-2$ and $(n-p-1)r_1^2 \neq (p-1)r_2^2$ is a non-conformally flat and non-Einstein manifold satisfying the Roter equation (1.9) on $\mathcal{U}_S \cap \mathcal{U}_C = \mathbb{S}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$.* (iii) [44, Example 4.1] *The warped product manifold $\mathbb{S}^p(r_1) \times_F \mathbb{S}^{n-p}(r_2)$, $2 \leq p \leq n-2$, with some special warping function F , satisfies on $\mathcal{U}_S \cap \mathcal{U}_C \subset \mathbb{S}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$ the Roter equation*

(1.9). Thus some warped product manifolds $\mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$ are Roter spaces.

(iv) Properties of pseudosymmetry type of warped products with 2-dimensional base manifold, a warping function F , and an $(n-2)$ -dimensional fibre, $n \geq 4$, assumed to be of constant curvature when $n \geq 5$, were determined in [34, Sections 6 and 7]. Evidently, warped product manifolds $\mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$, $n \geq 4$, are such manifolds. Let g , R , S , κ and C denote the metric tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of $\mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$, respectively. From [34, Theorem 7.1] it follows that on set V of all points of $\mathcal{U}_S \cap \mathcal{U}_C \subset \mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$ at which the tensor S^2 is not a linear combination of the tensors S and g , the Weyl tensor C is expressed by

$$(5.2) \quad C = \lambda \left(g \wedge S^2 + \frac{n-2}{2} S \wedge S - \kappa g \wedge S + \frac{\kappa^2 - \text{tr}_g(S^2)}{2(n-1)} g \wedge g \right),$$

where λ is some function on V . This, by (2.2), turns into

$$\begin{aligned} R &= \lambda g \wedge S^2 + \frac{n-2}{2} \lambda S \wedge S + \left(\frac{1}{n-2} - \kappa \lambda \right) g \wedge S \\ &\quad + \frac{1}{2(n-1)} \left((\kappa^2 - \text{tr}_g(S^2)) \lambda - \frac{\kappa}{n-2} \right) g \wedge g. \end{aligned}$$

Thus (1.11) is satisfied on V . Moreover, (1.9) holds at all points of $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus V$, at which (1.3) is not satisfied. From Lemma 2.2 it follows that (5.2) holds at all points of $\mathcal{U}_S \cap \mathcal{U}_C \subset \mathbb{S}^2(r_1) \times_F \mathbb{S}^{n-2}(r_2)$, $n \geq 4$, at which (1.3) is not satisfied. Finally, in view of Theorem 5.2, we can state that (5.1) holds on $\mathcal{U}_S \cap \mathcal{U}_C$.

6. ESSENTIALLY CONFORMALLY SYMMETRIC MANIFOLDS

Let (M, g) , $\dim M = n \geq 4$, be a semi-Riemannian manifold with parallel Weyl conformal curvature tensor, i.e. $\nabla C = 0$ on M . It is obvious that the last condition implies $R \cdot C = 0$. Moreover, let the manifold (M, g) be neither conformally flat nor locally symmetric. Such manifolds are called *essentially conformally symmetric manifolds*, e.c.s. manifolds/metrics, or ECS manifolds/metrics, in short (see, e.g., [11, 12, 14, 18, 19, 65]). E.c.s. manifolds are semisymmetric manifolds ($R \cdot R = 0$, [11, Theorem 9]) satisfying $\kappa = 0$ and $Q(S, C) = 0$ ([11, Theorems 7 and 8]). In addition,

$$(6.1) \quad FC = \frac{1}{2} S \wedge S$$

holds on M , where F is some function on M , called the *fundamental function* [12]. At every point of M we also have $\text{rank } S \leq 2$ [12, Theorem 5]. We mention that the local structure of e.c.s. manifolds is already determined. We refer to [13, 16] for the final results related to this subject. We also mention that certain e.c.s. metrics are realized on compact manifolds ([15, 17, 18, 19]).

Equation (6.1), by suitable contraction, leads immediately to $S^2 = \kappa S$, which by $\kappa = 0$, reduces to $S^2 = 0$. Evidently, $\text{tr}_g(S^2) = 0$. Now using (6.1) we get (1.13). Thus we have

Theorem 6.1. *Condition (5.1), with $\tau = (n-2)F$, is satisfied on every essentially conformally symmetric manifold (M, g) .*

7. HYPERSURFACES IN SEMI-RIEMANNIAN CONFORMALLY FLAT SPACES

Let M , $\dim M = n \geq 4$, be a hypersurface isometrically immersed in a semi-Riemannian conformally flat manifold N , $\dim N = n + 1$. Let g_{ad} , H_{ad} , $G_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}$ and C_{abcd} be the local components of the metric tensor g , the second fundamental tensor H , the $(0, 4)$ -tensor G and the Weyl conformal curvature tensor C of M , respectively. As it was stated in [48, eq. (20)] (see also [52, eq. (11)]) we have

$$(7.1) \quad \begin{aligned} C_{abcd} &= \varepsilon (H_{ad}H_{bc} - H_{ac}H_{bd}) - \frac{\varepsilon \operatorname{tr}(H)}{n-2} (g_{ad}H_{bc} + g_{bc}H_{bd} - g_{ac}H_{bd} - g_{bd}H_{ac}) \\ &+ \frac{\varepsilon}{n-2} (g_{ad}H_{bc}^2 + g_{bc}H_{bd}^2 - g_{ac}H_{bd}^2 - g_{bd}H_{ac}^2) + \mu G_{abcd}, \end{aligned}$$

where $\varepsilon = \pm 1$, $\operatorname{tr}(H) = g^{ad}H_{ad}$, $H_{ad}^2 = g^{bc}H_{ab}H_{cd}$ and μ is some function on M . From (7.1), by contraction we get easily

$$(7.2) \quad \mu = \frac{\varepsilon}{(n-2)(n-1)} ((\operatorname{tr}(H))^2 - \operatorname{tr}(H^2)),$$

where $\operatorname{tr}(H^2) = g^{ad}H_{ad}^2$. Now (7.1) and (7.2) yield

$$(7.3) \quad C = \frac{\varepsilon}{n-2} \left(g \wedge H^2 + \frac{n-2}{2} H \wedge H - \operatorname{tr}(H) g \wedge H + \frac{(\operatorname{tr}(H))^2 - \operatorname{tr}(H^2)}{2(n-1)} g \wedge g \right).$$

If $H = \frac{\operatorname{tr}(H)}{n}g$ at a point $x \in M$, i.e., M is umbilical at x , then from (7.3) it follows immediately that the tensor C vanishes at x . If at a non-umbilical point $x \in M$, we have $\operatorname{rank}(H - \alpha g) = 1$, for some $\alpha \in \mathbb{R}$, i.e., M is quasi-umbilical at x , then in view of Proposition 2.1 (i), the tensor C vanishes at x . Conversely, if at a non-umbilical point $x \in M$ the tensor C vanishes then in view of Proposition 2.1 (ii) we have $\operatorname{rank}(H - \alpha g) = 1$, for some $\alpha \in \mathbb{R}$. Thus we can present [48, Theorem 4.1] in the following form.

Theorem 7.1. *Let M , $\dim M = n \geq 4$, be a hypersurface isometrically immersed in a semi-Riemannian conformally flat manifold N , $\dim N = n + 1$. At every non-umbilical point $x \in M$ the tensor C vanishes at x if and only if at x we have $\operatorname{rank}(H - \alpha g) = 1$, for some $\alpha \in \mathbb{R}$.*

Remark 7.2. *Let M , $\dim M = n \geq 4$, be a hypersurface isometrically immersed in a semi-Riemannian conformally flat manifold N , $\dim N = n + 1$.*

(i) *We assume that at all points of $\mathcal{U}_C \subset M$ the tensor H^2 is a linear combination of H and g , i.e.,*

$$(7.4) \quad H^2 = \alpha_1 H + \alpha_2 g$$

on \mathcal{U}_C , where α_1 and α_2 are some functions on this set. Now (7.3) turns into

$$(7.5) \quad \begin{aligned} C &= \frac{\varepsilon}{2} H \wedge H + \frac{\varepsilon(\alpha_1 - \operatorname{tr}(H))}{n-2} g \wedge H + \frac{\varepsilon}{n-2} \left(\alpha_2 + \frac{(\operatorname{tr}(H))^2 - \operatorname{tr}(H^2)}{2(n-1)} \right) g \wedge g \\ &= \frac{\alpha}{2} H \wedge H + \beta g \wedge H + \frac{\gamma}{2} g \wedge g, \end{aligned}$$

where

$$(7.6) \quad \alpha = \varepsilon, \quad \beta = \frac{\varepsilon(\alpha_1 - \operatorname{tr}(H))}{n-2}, \quad \gamma = \frac{\varepsilon}{n-2} \left(2\alpha_2 + \frac{(\operatorname{tr}(H))^2 - \operatorname{tr}(H^2)}{n-1} \right).$$

From (7.5) and (7.6), in view of [68, Theorem 3.1 (i)], we get

$$(7.7) \quad C \cdot C = (n-2) \left(\frac{\beta^2}{\alpha} - \gamma \right) Q(g, C) = (n-2)(\varepsilon\beta^2 - \gamma) Q(g, C)$$

on \mathcal{U}_C , with α , β and γ defined by (7.6). Thus M is a hypersurface with pseudosymmetric Weyl tensor. We also note that from (7.4) we get immediately $\alpha_2 = \frac{1}{n}(\text{tr}(H^2) - \alpha_1 \text{tr}(H))$ and

$$H^2 - \frac{\text{tr}(H^2)}{n} g = \alpha_1 \left(H - \frac{\text{tr}(H)}{n} g \right).$$

(ii) The above presented result, i.e., if (7.4) is satisfied at every point of $\mathcal{U}_C \subset M$ then (3.6) holds on this set, was already obtained in [53, Proposition 3.1]. We mention that Proposition 3.1 of [53] was proved without application of [68, Theorem 3.1 (i)].

(iii) We assume that the tensor H satisfies on $\mathcal{U}_C \subset M$

$$(7.8) \quad H^3 = \text{tr}(H) H^2 + \psi H,$$

where ψ is some function on this set and the $(0, 2)$ -tensor H^3 is defined by $H^3_{ad} = g^{bc} H^2_{ab} H_{cd}$. Then

$$(7.9) \quad C \cdot C = \left(\frac{\varepsilon}{(n-2)(n-1)} ((\text{tr}(H))^2 - \text{tr}(H^2)) + \frac{\varepsilon\psi}{n-2} \right) Q(g, C) - \frac{n-3}{n-2} Q(H^2, \frac{1}{2} H \wedge H)$$

on \mathcal{U}_C [51, eq. (10)], see also [53, the proof of Lemma 4.1]. We refer to [51, 53] for further results on hypersurfaces M in conformally flat manifold N satisfying (7.8).

(iv) Recently curvature properties of pseudosymmetry type of hypersurfaces isometrically immersed in a semi-Riemannian conformally flat manifold were investigated in [57] and [69].

8. HYPERSURFACES IN SEMI-RIEMANNIAN SPACE FORMS

Let now $N_s^{n+1}(c)$, $n \geq 4$, be a semi-Riemannian space of constant curvature with signature $(s, n+1-s)$, where $c = \frac{\tilde{\kappa}}{n(n+1)}$ and $\tilde{\kappa}$ is its scalar curvature. Let M , $\dim M = n \geq 4$, be a connected hypersurface isometrically immersed in $N_s^{n+1}(c)$. We denote by g , R , S , κ and C , the metric tensor, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of the hypersurface M , respectively. The Gauss equation of M in $N_s^{n+1}(c)$ reads (see, e.g., [32, 35, 36, 37, 74])

$$(8.1) \quad R - \frac{\tilde{\kappa}}{2n(n+1)} g \wedge g = \frac{\varepsilon}{2} H \wedge H, \quad \varepsilon = \pm 1.$$

From (8.1), by suitable contractions, we obtain

$$(8.2) \quad S - \frac{(n-1)\tilde{\kappa}}{n(n+1)} g = \varepsilon (\text{tr}(H) H - H^2),$$

$$(8.3) \quad \frac{\kappa}{n-1} - \frac{\tilde{\kappa}}{n+1} = \frac{\varepsilon}{n-1} ((\text{tr}(H))^2 - \text{tr}(H^2)).$$

Now using (8.1), (8.2) and (8.3) we get immediately

$$\begin{aligned}
Q(H^2, \frac{1}{2} H \wedge H) &= -Q(\text{tr}(H) H - H^2, \frac{1}{2} H \wedge H) \\
&= -Q(\varepsilon(\text{tr}(H) H - H^2), \frac{\varepsilon}{2} H \wedge H) \\
(8.4) \qquad &= -Q\left(S - \frac{(n-1)\tilde{\kappa}}{n(n+1)}g, R - \frac{\tilde{\kappa}}{2n(n+1)}g \wedge g\right).
\end{aligned}$$

We also recall that the curvatore condition of pseudosymmetry type (3.7) is satisfied on M . Precisely, we have on M [48, Proposition 3.1] (see also [37, eqs. (3.3) and (3.4)])

$$(8.5) \qquad R \cdot R - Q(S, R) = -\frac{(n-2)\tilde{\kappa}}{n(n+1)}Q(g, C).$$

Now (3.11), by (8.5), turns into ([37, Propopsition 4.7, eq. (4.36)])

$$(8.6) \qquad C \cdot R + R \cdot C = C \cdot C + Q(S, C) - \frac{(n-2)\tilde{\kappa}}{n(n+1)}Q(g, C) - \frac{1}{(n-2)^2}Q(g, E).$$

Let $\mathcal{U}_H \subset M$ be the set of all points at which the tensor H^2 is not a linear combination of the metric tensor g and the second fundamental tensor H of M . We have $\mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C \subset M$ (see, e.g., [30, 36, 37] or [59, p. 137]).

We assume that the following conditions are satisfied on $\mathcal{U}_H \subset M$

$$(8.7) \qquad H^3 = \text{tr}(H) H^2 + \psi H + \rho g,$$

$$(8.8) \qquad C \cdot C = Q(g, T),$$

where T is a generalized curvature tensor and ψ and ρ some functions on \mathcal{U}_H . Now, in view of [37, Theorem 4.5], we obtain

$$\begin{aligned}
T &= \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\tilde{\kappa}}{n+1}\right)C + \frac{\lambda_1}{2}g \wedge g \\
(8.9) \qquad &- \frac{n-3}{(n-2)^2(n-1)}\left(g \wedge S^2 + \frac{n-2}{2}S \wedge S - \kappa g \wedge S\right)
\end{aligned}$$

on \mathcal{U}_H , where λ_1 is some function on this set. Using (1.2), (8.8) and (8.9) we get immediately

$$T = \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\tilde{\kappa}}{n+1}\right)C - \frac{n-3}{(n-2)^2(n-1)}E + \frac{\lambda}{2}g \wedge g$$

and

$$(8.10) \qquad C \cdot C = \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\tilde{\kappa}}{n+1}\right)Q(g, C) - \frac{n-3}{(n-2)^2(n-1)}Q(g, E)$$

on \mathcal{U}_H , where λ is some function on this set. In addition, if we assume that (3.6) holds on \mathcal{U}_H then from (7.9) it follows that

$$(8.11) \qquad \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\tilde{\kappa}}{n+1} - L_C\right)C = \frac{n-3}{(n-2)^2(n-1)}E + \frac{\lambda_2}{2}g \wedge g$$

on \mathcal{U}_H , where λ_2 is some function on this set. We note that (8.11), by a suitable contraction, yields $\lambda_2 = 0$, and in a consequence we obtain

$$(8.12) \quad \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\tilde{\kappa}}{n+1} - L_C \right) C = \frac{n-3}{(n-2)^2(n-1)} E.$$

From the above presented considerations it follows immediately the following result.

Theorem 8.1. *Let M be a non-Einstein and non-conformally flat hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If (3.6) and (8.7) are satisfied on $\mathcal{U}_H \subset M$ then (8.12) holds on \mathcal{U}_H .*

According to [30, Corollary 4.1], if on the subset \mathcal{U}_H of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, one of the tensors $R \cdot C$, $C \cdot R$ or $R \cdot C - R \cdot C$ is a linear combination of $R \cdot R$ and of a finite sum of tensors of the form $Q(A, T)$, where A is a symmetric $(0, 2)$ -tensor and T a generalized curvature tensor, then (8.7) holds on \mathcal{U}_H . In particular if one of the following conditions is satisfied on $\mathcal{U}_H \subset M$: $R \cdot C = Q(g, T_1)$, $C \cdot R = Q(g, T_2)$ or $R \cdot C - C \cdot R = Q(g, T_3)$, where T_1 , T_2 and T_3 are generalized curvature tensors, then (8.7) holds on \mathcal{U}_H . Now from Theorems 5.2, 5.3 and 5.4 of [36], in view of Proposition 2.2, it follows that

$$(8.13) \quad \text{Weyl}(T_1) = \left(\frac{\kappa + \varepsilon\psi}{n-1} - \frac{(n-1)\tilde{\kappa}}{n(n+1)} \right) C - \frac{1}{(n-2)(n-1)} E,$$

$$(8.14) \quad \text{Weyl}(T_2) = \left(\frac{\kappa + 2\varepsilon\psi}{n-1} - \frac{\tilde{\kappa}}{n+1} \right) C - \frac{n-3}{(n-2)^2(n-1)} E,$$

$$(8.15) \quad \text{Weyl}(T_3) = \left(\frac{\tilde{\kappa}}{n(n+1)} - \frac{\varepsilon\psi}{n-1} \right) C - \frac{1}{(n-2)^2(n-1)} E.$$

Thus we have

Theorem 8.2. *Let M be a non-Einstein and non-conformally flat hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (8.7) on $\mathcal{U}_H \subset M$, and let T_1 , T_2 and T_3 be generalized curvature tensors defined on \mathcal{U}_H . If one of the following conditions: $R \cdot C = Q(g, T_1)$, respectively, $C \cdot R = Q(g, T_2)$ and $R \cdot C - C \cdot R = Q(g, T_3)$, is satisfied on \mathcal{U}_H then (8.13), respectively (8.14) and (8.15), holds on \mathcal{U}_H .*

Finally, we assume that the tensor H satisfies (7.8) on $\mathcal{U}_H \subset M$. Now (7.9), by making use of (8.3) and (8.4), turns into

$$(8.16) \quad \frac{n-2}{n-3} C \cdot C = \rho Q(g, C) + Q \left(S - \frac{(n-1)\tilde{\kappa}}{n(n+1)} g, R - \frac{\tilde{\kappa}}{2n(n+1)} g \wedge g \right).$$

$$(8.17) \quad \rho = \frac{1}{n-3} \left(\frac{\kappa}{n-1} - \frac{\tilde{\kappa}}{n+1} + \varepsilon\psi \right).$$

From (8.16), by an application of (2.6), we obtain

$$\begin{aligned}
\frac{n-2}{n-3} C \cdot C &= \rho Q(g, R) + \frac{\rho}{2(n-2)} Q(S, g \wedge g) + Q\left(S - \frac{(n-1)\tilde{\kappa}}{n(n+1)} g, R - \frac{\tilde{\kappa}}{2n(n+1)} g \wedge g\right) \\
&= \frac{\rho}{2(n-2)} Q(S, g \wedge g) + \rho Q\left(g, R - \frac{\tilde{\kappa}}{2n(n+1)} g \wedge g\right) \\
&\quad + Q\left(S - \frac{(n-1)\tilde{\kappa}}{n(n+1)} g, R - \frac{\tilde{\kappa}}{2n(n+1)} g \wedge g\right) \\
&= Q\left(S, \frac{\rho}{2(n-2)} g \wedge g\right) + Q\left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \rho\right) g, R - \frac{\tilde{\kappa}}{2n(n+1)} g \wedge g\right) \\
&= Q\left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \rho\right) g, \frac{\rho}{2(n-2)} g \wedge g\right) \\
&\quad + Q\left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \rho\right) g, R - \frac{\tilde{\kappa}}{2n(n+1)} g \wedge g\right) \\
&= Q\left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \rho\right) g, R - \left(\frac{\tilde{\kappa}}{n(n+1)} - \frac{\rho}{n-2}\right) \frac{1}{2} g \wedge g\right).
\end{aligned}$$

Thus we see that if the tensor H satisfies (7.8) on $\mathcal{U}_H \subset M$ then

$$(8.18) \quad C \cdot C = \frac{n-3}{n-2} Q\left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \rho\right) g, R - \left(\frac{\tilde{\kappa}}{n(n+1)} - \frac{\rho}{n-2}\right) \frac{1}{2} g \wedge g\right)$$

on \mathcal{U}_H , where the function ρ is defined by (8.17).

In addition, we assume that (3.6) holds on $\mathcal{U}_H \subset M$. Now (8.16) turns into

$$(8.19) \quad \tau Q(g, C) - Q\left(S - \frac{(n-1)\tilde{\kappa}}{n(n+1)} g, R - \frac{\tilde{\kappa}}{2n(n+1)} g \wedge g\right) = 0,$$

$$(8.20) \quad \tau = \rho - \frac{n-2}{n-3} L_C.$$

From the presented above calculations it follows that (8.19) yields

$$(8.21) \quad Q\left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \tau\right) g, R - \left(\frac{\tilde{\kappa}}{n(n+1)} - \frac{\tau}{n-2}\right) \frac{1}{2} g \wedge g\right) = 0.$$

If

$$(8.22) \quad \text{rank}\left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \tau\right) g\right) = 1$$

at a point $x \in \mathcal{U}_H$ then in view of Proposition 2.1 $E = 0$ at x , where the tensor E is defined by (1.2). If

$$(8.23) \quad \text{rank}\left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \tau\right) g\right) > 1$$

at a point $x \in \mathcal{U}_H$ then by an application of [24, Proposition 2.4] (or, [31, Proposition 2.1]) it follows that the following equation is satisfied at x

$$(8.24) \quad \begin{aligned} & R - \left(\frac{\tilde{\kappa}}{n(n+1)} - \frac{\tau}{n-2} \right) \frac{1}{2} g \wedge g \\ &= \phi \left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \tau \right) g \right) \wedge \left(S - \left(\frac{(n-1)\tilde{\kappa}}{n(n+1)} - \tau \right) g \right), \quad \phi \in \mathbb{R}. \end{aligned}$$

This by Proposition 4.4 implies $(n-2)C = \phi E$. Thus we have

Theorem 8.3. *Let M be a non-Einstein and non-conformally flat hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.*

(i) *If (7.8) is satisfied on $\mathcal{U}_H \subset M$ then (8.18) holds on \mathcal{U}_H , where the function ρ is defined by (8.17) on this set.*

(ii) *If (3.6) and (7.8) are satisfied on $\mathcal{U}_H \subset M$ then (8.21) holds on \mathcal{U}_H , where the function τ is defined by (8.20) on this set. Moreover, $\lambda C = E$ on \mathcal{U}_H , where λ is some function on this set.*

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