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Brief communication

A note on stability of neutral delay-differential systems

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Abstract

In this note, asymptotic stability of linear neutral delay-differential systems is investigated. A delay-independent sufficient condition for the stability of the system is derived in terms of the spectral radius. The proposed criterion requires more relaxed assumption than those reported in the literature. The effectiveness of the method is illustrated in numerical examples. © 1999 The Franklin Institute. Published by Elsevier Science Ltd.

1. Introduction

Consider a linear neutral delay-differential system

$$\dot{x}(t) = Ax(t) + Bx(t-h) + C\dot{x}(t-h), \tag{1}$$

where $x(t)$ is an n-dimensional state vector, A, B and $C \in \mathscr{R}^{n \times n}$ are constant matrices, h is a positive constant delay, and the system matrix A is assumed to be a Hurwitz matrix. That is, all the eigenvalues of A have negative real parts. The system given in Eq. (1) often appears in the theory of automatic control or population dynamics. However, it is not easy to establish simple stability criteria for the system. In the literature, only a few stability analysis methods are investigated (e.g. [1–5]).

In this note, we present a new delay-independent sufficient condition for asymptotic stability of the system given in Eq. (1). In the work of [1, 2], the basic assumption such

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that the system matrix A has a negative matrix measure, is required to apply their stability criteria, while the proposed criterion allows more relaxed assumption that the system matrix is Hurwitz. Furthermore, the derived sufficient condition is expressed in terms of the spectral radius of the matrix which is the combination of the modulus matrices. Therefore, there is better possibility that the proposed criterion is less conservative than those in the literature [1, 2], which use the matrix norms and matrix measures.

The rest of this note is organized as follows. In Section 2, we state notations and well-known lemmas about matrix properties. In Section 3, a sufficient condition for the stability of the systems, in terms of spectral radius, is derived. To show the effectiveness of the proposed criterion, numerical examples are given in Section 4.

2. Preliminaries

To derive main result, we state some notations and lemmas. Let $\rho[R]$ denote the largest modulus of the eigenvalues of the matrix R, which is known as the spectral radius of R. $|R|$ denotes a matrix formed by taking the absolute value of every element of R, and it is called the modulus matrix of R. I denotes the identity matrix of appropriate order. $\lambda_M(R)$ denotes the maximum eigenvalue of matrix R. The relation $R \leqslant T$ represents that all the elements of matrices, R and T, satisfy $r_{ij} \leqslant t_{ij}$ for all i and j. Also, $\|R\| = [\lambda_M(R^T R)]^{1/2}$ denotes matrix norm of R and $\mu(R) = \frac{1}{2}\lambda_M(R^T + R)$ denotes matrix measure of R.

Also, the following lemma is used for main result.

Lemma 1. *Consider any $n \times n$ matrices R, T, and V.*
Part I [6]: *If $|R| \leqslant V$, then*

$$(a) \quad |RT| \leqslant |R||T| \leqslant V|T|,$$

$$(b) \quad |R + T| \leqslant |R| + |T| \leqslant V + |T|,$$

$$(c) \quad \rho[R] \leqslant \rho[|R|] \leqslant \rho[V],$$

$$(d) \quad \rho[RT] \leqslant \rho[|R||T|] \leqslant \rho[V|T|],$$

$$(e) \quad \rho[R + T] \leqslant \rho[|R + T|] \leqslant \rho[|R| + |T|] \leqslant \rho[V + |T|].$$

Part II [7]: *If $\rho[R] < 1$, then* $\det(I \pm R) \neq 0$.
Part III [8]: *If $\|R\| < 1$, then $(I - R)^{-1}$ exists and*

$$(I - R)^{-1} = I + R + R^2 + \cdots.$$

3. Sufficient condition for asymptotic stability

In this section, we derive a sufficient condition for the asymptotic stability of the system given in Eq. (1). For the definition of asymptotic stability, refer to Definitions 1 and 2 in Ref. [1].

Let $F(s) = (sI - A)^{-1}$, and F_M be the matrix formed by taking the maximum magnitude of each element of $F(s)$ for $\Re s \geqslant 0$. Then, we have

Theorem 1. *Assume that* $\|C\| < 1$. *Then, the neutral delay-differential system given in Eq.* (1) *is asymptotically stable, if the following inequality is satisfied*:

$$\rho\left[F_M\left(|B| + \frac{|CA| + |CB|}{1 - \|C\|}\right)\right] < 1. \tag{2}$$

Proof. The characteristic equation of the system given in Eq. (1) is

$$\Delta(s) = \det[sI - A - (B + Cs)\exp(-hs)] = 0. \tag{3}$$

Since $\det[RT] = \det[R]\det[T]$ for any two $n \times n$ matrices R and T, so we have

$$\Delta(s) = \det[I - C\exp(-hs)]\det[sI - (I - C\exp(-hs))^{-1} \cdot (A + B\exp(-hs))], \tag{4}$$

where the matrix $(I - C\exp(-hs))^{-1}$ exists and $\det(I - C\exp(-hs)) \neq 0$ because $\|C\exp(-hs)\| \leqslant \|C\| < 1$ for $\Re s \geqslant 0$.

Therefore, if we can show that

$$\det[sI - (I - C\exp(-hs))^{-1} \cdot (A + B\exp(-hs))] \neq 0 \quad \text{for } \Re s \geqslant 0, \tag{5}$$

then,

$$\Delta(s) \neq 0 \;\; \Re s \geqslant 0. \tag{6}$$

Here, Eqs. (5) and (6) guarantee the asymptotic stability of the system given in Eq. (1) by Hu [2] (Theorem 1).

For simplicity, let $\xi = \exp(-hs)$ and $T = \xi C$. Then, using the inequality $(I - T)^{-1} = I + (I - T)^{-1}T$, the left-hand side of Eq. (5) becomes

$$\begin{aligned}
&\det[sI - (I - T)^{-1}(A + \xi B)] \\
&\quad = \det[sI - (I + (I - T)^{-1}T)(A + \xi B)] \\
&\quad = \det[(sI - A) - \xi B - (I - T)^{-1}(TA + \xi TB)] \\
&\quad = \det[sI - A]\det[I - (sI - A)^{-1}(\xi B + (I - T)^{-1}(TA + \xi TB))] \\
&\quad = \det[sI - A]\det[I - F(s)(\xi B + (I - T)^{-1}(TA + \xi TB))].
\end{aligned} \tag{7}$$

Therefore, Eq. (5) can be rewritten as

$$\det[sI - A]\det[I - F(s)(\xi B + (I - T)^{-1}(TA + \xi TB))] \neq 0. \tag{8}$$

Since A is a Hurwitz matrix, $\det[sI - A] \neq 0$ for $\Re s \geqslant 0$. So, Eq. (8) is further simplified as

$$\det[I - F(s)(\xi B + (I - T)^{-1}(TA + \xi TB))] \neq 0 \quad \text{for } \Re s \geqslant 0. \tag{9}$$

Now, if we can show that $\rho[F(s)(\xi B + (I - T)^{-1}(TA + \xi TB))] < 1$ for $\Re s \geqslant 0$, then, by Part II of Lemma I, Eq. (9) is satisfied.

Using Part I and III of Lemma I, and the inequalities $\|T\| \leqslant \|C\|$ and $|\xi CB| \leqslant |CB|$ for $\Re s \geqslant 0$, we obtain

$$\begin{aligned}
&\rho[F(s)(\xi B + (I - T)^{-1}(TA + \xi TB))] \\
&\quad \leqslant \rho[|F(s)| \cdot (|\xi B| + |(I - T)^{-1}(TA + \xi TB)|)] \\
&\quad \leqslant \rho[|F(s)| \cdot (|\xi B| + |(I - T)^{-1}| \cdot |(TA + \xi TB)|)] \\
&\quad = \rho[|F(s)| \cdot (|\xi B| + |(I + T + T^2 + \cdots)| \cdot (|TA| + |\xi TB|))] \\
&\quad \leqslant \rho[F_M \cdot (|B| + \|I + T + T^2 + \cdots\| I \cdot (|TA| + |TB|))] \\
&\quad \leqslant \rho[F_M \cdot (|B| + (\|I\| + \|T\| + \|T\|^2 + \cdots) I \cdot (|CA| + |CB|))] \\
&\quad \leqslant \rho[F_M \cdot (|B| + (1 + \|C\| + \|C\|^2 + \cdots) I \cdot (|CA| + |CB|))] \\
&\quad = \rho[F_M \cdot (|B| + \frac{1}{1 - \|C\|} \cdot (|CA| + |CB|))] < 1.
\end{aligned} \tag{10}$$

Thus, this completes the proof. □

Remark 1. Since the system matrix A is Hurwitz, the matrix F_M always exists and it can obtained for some s on imaginary axis by the maximum modulus theorem.

Remark 2. In the literature [1, 2], the sufficient conditions for stability of the system Eq. (1) are derived as

$$\text{Li [1]:} \quad \mu(A) + \frac{\|B\| + \|A\| \cdot \|C\|}{1 - \|C\|} < 0 \quad \text{for } \|C\| < 1, \tag{11}$$

$$\text{Hu [2]:} \quad \mu(A) + \|B\| + \frac{\|CA\| + \|CB\|}{1 - \|C\|} < 0 \quad \text{for } \|C\| < 1, \tag{12}$$

$$\begin{aligned}
\text{Hu [2]:} \quad & \mu(A) + \|B\| + \sum_{j=1}^{q} \{\|C^j A\| + \|C^j B\|\} \\
& + \frac{\|C^{q+1} A\| + \|C^{q+1} B\|}{1 - \|C\|} < 0 \quad \text{for } \|C\| < 1 \text{ and } q \geqslant 1,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\text{Hu [2]:} \quad & \mu(A^{-1}) + \|A^{-1} C\| \\
& + \frac{\|A^{-1} B A^{-1}\| + \|A^{-1} B A^{-1} C\|}{1 - \|A^{-1} B\|} < 0 \quad \text{for } \|A^{-1} B\| < 1.
\end{aligned} \tag{14}$$

The above sufficient conditions require the assumption that $\mu(A) < 0$ or $\mu(A^{-1}) < 0$, while Theorem I allows more relaxed assumption that A is a Hurwitz matrix. Furthermore, the stability criterion of Theorem I is expressed in terms of the spectral radius of the matrix which is the combination of the modulus matrices. Therefore, there is better possibility that the proposed criterion is less conservative than those in Eqs. (11)–(14), which use the matrix norms and matrix measure.

Remark III. If $C = 0$, the system Eq. (1) becomes a linear retarded delay-differential system, i.e.,

$$\dot{x}(t) = Ax(t) + Bx(t-h). \tag{15}$$

Then, by Theorem I, sufficient condition for the stability of system (15) is obtained as

$$\rho[F_M|B|] < 1. \tag{16}$$

4. Numerical examples

To demonstrate the application of the result, we give the following two examples.

Example 1. Consider the following system

$$\dot{x}(t) = Ax(t) + Bx(t-h) + C\dot{x}(t-h),$$

where

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

and α is a nonzero constant.

We now determine the stability bound in terms of α. Since $\mu(A) = 0.0811 > 0$ and $\mu(A^{-1}) = 0.0406 > 0$, the criteria of Li [1] and Hu [2] are not applicable. However, the system matrix A is Hurwitz, therefore Theorem I can be applied. The rational function matrix $F(s)$ and F_M are computed as

$$F(s) = \begin{bmatrix} \dfrac{2}{s^2+3s+2} & \dfrac{-2}{s^2+3s+2} \\ \dfrac{1}{s^2+3s+2} & \dfrac{s+3}{s^2+3s+2} \end{bmatrix}, \quad F_M = \begin{bmatrix} \dfrac{1}{3} & 1 \\ \dfrac{1}{2} & \dfrac{3}{2} \end{bmatrix}.$$

Then, using the inequality (2), we obtain the bound of α as

$$|\alpha| < 0.4.$$

Example 2. Consider the system

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} x(t-h) + \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix} \dot{x}(t-h).$$

It is clear that the system matrix A is Hurwitz and also the matrix measure $\mu(A)$ is negative. So, all the criteria can be applied.

With simple calculation, we obtain the bound of α for stability as

Li [1] (Eq. (11)): $|\alpha| < 0.4$

Hu [2] (Eq. (12)): $|\alpha| < 0.4$

Hu [2] (Eq. (13)): $|\alpha| < 0.4$ (In case of $q = 10$)

Hu [2] (Eq. (14)): $|\alpha| < 0.334$

Theorem I (Eq. (2)): $|\alpha| < 0.825$.

In the example, we can see that the sufficient condition of Theorem I is less conservative than others.

References

[1] L.M. Li, Stability of linear neutral delay-differential systems, Bull. Austral. Math. Soc. 38 (1988) 339–344.
[2] G. Di Hu, G.-Da Hu, Some simple stability criteria of neutral delay-differential systems, Appl. Math. Comput. 80 (1996) 257–271.
[3] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic Publishers, Boston, 1992.
[4] V. Kolmanovskii, A. Myshkis, Applied Theory of Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 1992.
[5] J. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer, New York, 1993.
[6] J.M. Ortega, Numerical Analysis, Academic Press, New York, 1972.
[7] K.H. Chen, Robust analysis and design of multiloop control systems, Ph.D. Dissertation, National Tsing-Hua University, ROC, 1989.
[8] P. Lancaster, M. Timenetsky, The Theory of Matrices, Academic Press, Orlando, FL, 1985.