A NOTE ON STRICTLY CYCLIC SHIFTS ON $I_{ m p}$

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<u>ABSTRACT</u>. In this paper the author shows that a well known sufficient condition for strict cyclicity of a weighted shift on ℓ_p is not a necessary condition for any p with 1 .

1. INTRODUCTION.

For $1 \leq p < \infty$ let ℓ_p be the Banach space of absolutely p-summable sequences of complex numbers. Let S_α denote the weighted shift on ℓ_p with weight sequence $\alpha = \{\alpha_n\}_1^\infty$ defined by $S_\alpha \begin{bmatrix} \frac{\infty}{2} \alpha_n x_n e_n \end{bmatrix} = \frac{\infty}{n-1} \alpha_n x_{n-1} e_n$. Let $\beta_0 = 1$ and $\beta_n = \alpha_1 \alpha_2 \dots \alpha_n$ for all $n \geq 1$. (For more detail we refer the reader to [2] and [3]).

Mary Embry [1] showed that for p = 1 the weighted shift \boldsymbol{S}_{α} is strictly cyclic if and only if

$$\sup_{n,m} \left| \frac{\beta_{n+m}}{\beta_n \beta_m} \right| < \infty.$$
 (1.1)

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Edward Kerlin and Alan Lambert [2] considered the natural extension to (1.1) for the case 1 \infty with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sup_{n} \sum_{m=0}^{n} \left| \frac{\beta_n}{\beta_m \beta_{n-m}} \right|^q < \infty$$
 (1.2)

and showed (1.2) implies S_{α} is strictly cyclic on ℓ_p . They also proved that (1.2) is necessary if the weight sequence α is eventually decreasing. This strongly suggested that (1.2) is a necessary and sufficient condition for strict cyclicity. However, we will show in this paper that (1.2) is not a necessary condition for S_{α} to be strictly cyclic for any p with 1 .

To preserve the clarity of the proof we will consider the cases 1 \leq 2 and 2 \infty separately.

(a) Let $1 and let q be such that <math>\frac{1}{p} + \frac{1}{q} = 1$. Let $\{n_k\}_1^{\infty}$ be a sequence of rapidly increasing positive integers, e.g., choose $n_1 = 10$ and $n_k = (10n_{k-1})$ for k > 1.

We now define the weight sequence $\alpha = \{\alpha_i\}_1^\infty$ by $\alpha_1 = \alpha_2 = \dots = \alpha_{n_1} = 1$ and for k > 1

$$\alpha_{i} = \begin{cases} n_{k}^{-1} & \text{if } n_{k-1} < i \leq n_{k} - n_{k-1} \\ \\ n_{k-1}^{-1} & \text{if } n_{k} - n_{k-1} < i \leq n_{k}. \end{cases}$$

Clearly, $\alpha_{n_{k-1}+1} = \alpha_{n_{k-1}+2} = \dots = \alpha_{n_k-n_{k-1}}$ and $\alpha_{n_k-\ell+1} \leq \alpha_\ell$ for $1 \leq \ell \leq \frac{n_k}{2}$. Thus, for $0 < m < n_k$

$$\frac{\beta_{n_k}}{\beta_m \beta_{n_k-m}} = \frac{\alpha_{n_k-m+1} \alpha_{n_k-m+2} \cdots \alpha_{n_k}}{\alpha_1 \alpha_2 \cdots \alpha_m} \geq \frac{\alpha_{n_k-n_{k-1}+1} \cdots \alpha_{n_k}}{\alpha_1 \alpha_2 \cdots \alpha_{n_{k-1}}} \geq (\alpha_{n_k})^{n_{k-1}}.$$

Therefore,

$$\begin{array}{ll} n_k^{-1} & \left| \frac{\beta_{n_k}}{\beta_m \beta_{n_k - m}} \right|^q \geq n_k (\alpha_{n_k})^{q n_{k-1}} = n_k n_{k-1}^{-q n_{k-1}} \rightarrow \infty \text{ as } k \rightarrow \infty. \end{array}$$

Hence,

$$\sup_{n} \sum_{m=0}^{n} \left| \frac{\beta_{n}}{\beta_{m} \beta_{n-m}} \right|^{q} = \infty.$$

We now show that S_{α} is strictly cyclic. It is known that $S_{\alpha}[2]$ is strictly cyclic if and only if

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^{n} \frac{\beta_{n}}{\beta_{m}\beta_{n-m}} x_{m} y_{n-m} \right|^{p} < \infty \text{ for all } x, y \in \ell_{p}.$$
 (1.3)

Obviously, for 0 < m < n

$$\frac{\beta_n}{\beta_m \beta_{n-m}} = \frac{\alpha_{n-m+1} \cdots \alpha_n}{\alpha_1 \alpha_2 \cdots \alpha_m} \leq \alpha_n$$

and $\frac{\beta_n}{\beta_m \beta_{n-m}} = 1$ for m = 0 or m = n.

Thus,

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^{n} \frac{\beta_{n}}{\beta_{m}\beta_{n-m}} x_{m} y_{n-m} \right|^{p} \leq \sum_{n=0}^{\infty} \left\{ \left| x_{0} y_{n} \right| + \left| y_{0} x_{n} \right| + \sum_{m=1}^{n-1} \alpha_{n} \left| x_{m} y_{n-m} \right| \right\}^{p}$$

$$\leq \sum_{n=0}^{\infty} \left\{ \left| x_{0} y_{n} \right| + \left| y_{0} x_{n} \right| + \alpha_{n} \left\| x \right\|_{2} \left\| y \right\|_{2} \right\}^{p} < \infty$$

$$\text{since } \alpha, x, y \in \ell_{p}.$$

Hence (1.3) holds and \boldsymbol{S}_{α} is strictly cyclic.

(b) Let 2 \infty and let $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{n_k\}_1^{\infty}$ be a sequence of rapidly increasing integers, e.g., choose $n_1 = 10$ and n_k such that

$$\sum_{n=1}^{n_k} \frac{1}{n} \ge (10n_{k-1})^{10n_{k-1}} \text{ for } k > 1.$$

Define $\{d_i\}_{1}^{\infty}$ such that $\prod_{i=1}^{m} d_i = n$ for $n = 1, 2, \cdots$ and define $\{s_k\}_{1}^{\infty}$ by $s_1 = 10$ and $s_k = 2s_{k-1} + 2n_k$ for k > 1.

We now define the weight sequence $\alpha = \{\alpha_i\}_1^{\infty}$ by $\alpha_1 = \dots = \alpha_{s_1} = 1$ and for k > 1,

$$\alpha_{i} = \begin{cases} n_{k}^{-2} & \text{if } s_{k-1} < i \le s_{k-1} + n_{k} \\ n_{k}^{-2} & s_{k} - s_{k-1} - i + 1 \end{cases}$$

$$if s_{k-1} + n_{k} < i \le s_{k} - s_{k-1} \\ n_{k-1}^{-2} & \text{if } s_{k} - s_{k-1} < i \le s_{k}.$$

Now, for $s_{k-1} < m < \frac{s_k}{2}$,

$$\frac{\beta_{s_k}}{\beta_m\beta_{s_k-m}} = \frac{\alpha_{s_k-m+1} \cdots \alpha_{s_k}}{\alpha_1\alpha_2 \cdots \alpha_m} \ge \frac{\alpha_{s_k-m+1} \cdots \alpha_{s_k-s_{k-1}}}{\alpha_{s_{k-1}+1} \cdots \alpha_m} n_{k-1}^{-2s_{k-1}} \ge n_{k-1}^{-2s_{k-1}} \prod_{i=1}^{m-s_{k-1}} d_i$$

$$= n_{k-1}^{-2s_{k-1}} (m-s_{k-1})^{-\frac{1}{q}}.$$

Thus.

$$\begin{array}{c|c} s_k \\ \Sigma \\ m=0 \end{array} \left| \begin{array}{c|c} \beta s_k \\ \hline \beta_m \beta_{s_k-m} \end{array} \right|^q \begin{array}{c} -2qs_{k-1} & n_k \\ \geq n_{k-1} & \Sigma & i^{-1} \\ & i=1 \end{array} \right| \begin{array}{c|c} -10n_{k-1} & n_k \\ \Sigma & i^{-1} \rightarrow \infty \text{ as } k \rightarrow \infty. \end{array}$$

Hence,

$$\sup_{n} \sum_{m=0}^{n} \left| \frac{\beta_{n}}{\beta_{m} \beta_{n-m}} \right|^{q} = \infty.$$

We now show that

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^{n} \frac{\beta_{n}}{\beta_{m}\beta_{n-m}} x_{m} y_{n-m} \right|^{p} < \infty \quad \text{for all x,y } \epsilon \ \ell_{p}.$$

If
$$s_{k-1} < n < s_k - s_{k-1}$$
 and $0 < m < n$ then, $\frac{\beta_n}{\beta_m \beta_{n-m}} \le n_k^{-2}$.

Let $h = min\{m,n-m\}$ then, for $s_k - s_{k-1} \le n \le s_k$ and $0 \le m \le n$,

$$\frac{\beta_n}{\beta_m \beta_{n-m}} \leq n_{k-1}^{-2} h^{-\frac{1}{q}}.$$

Thus,

Let
$$\delta = \frac{p}{p-2}$$
 then $\delta > q$ and $\frac{1}{\delta} + \frac{2}{p} = 1$. Let $M = \begin{bmatrix} \infty & 2^{-\frac{\delta}{q}} \end{bmatrix}^{\frac{1}{\delta}} \left[||\mathbf{x}||_{p}^{p} + ||\mathbf{y}||_{p}^{p} \right]^{\frac{2}{p}} < \infty$.

Then,

$$\leq \sum_{k=2}^{\infty} \sum_{n=s_{k}-s_{k-1}} \left(\sum_{m=1}^{\infty} \left(\sum_{m=1}^{\infty}$$

$$\stackrel{\circ}{\leq} \sum_{k=2}^{\infty} \sum_{n=s}^{s_k} n_{k-1}^{-2p} M^p < \infty.$$

Combining (1.4) and (1.5) we obtain that (1.3) is satisfied. QED

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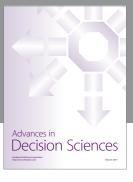
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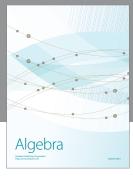
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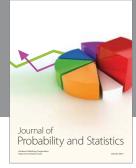
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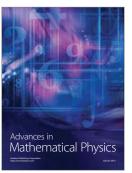


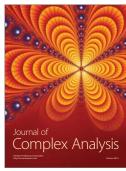


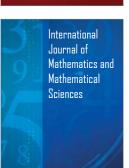


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