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A NOTE ON STRICTLY CYCLIC SHIFTS ON ℓ_p

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ABSTRACT. In this paper the author shows that a well known sufficient condition for strict cyclicity of a weighted shift on ℓ_p is not a necessary condition for any p with $1 < p < \infty$.

1. INTRODUCTION.

For $1 \leq p < \infty$ let ℓ_p be the Banach space of absolutely p-summable sequences of complex numbers. Let S_α denote the weighted shift on ℓ_p with weight sequence $\alpha = \{\alpha_n\}_1^\infty$ defined by $S_\alpha\left[\sum_{n=0}^{\infty} x_n e_n\right] = \sum_{n=1}^{\infty} \alpha_n x_{n-1} e_n$. Let $\beta_0 = 1$ and $\beta_n = \alpha_1\alpha_2\ldots\alpha_n$ for all $n \geq 1$. (For more detail we refer the reader to [2] and [3]).

Mary Embry [1] showed that for p = 1 the weighted shift S_α is strictly cyclic if and only if

$$\sup_{n,m} \left|\frac{\beta_{n+m}}{\beta_n \beta_m}\right| < \infty. \tag{1.1}$$

Edward Kerlin and Alan Lambert [2] considered the natural extension to (1.1) for the case $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right|^q < \infty \tag{1.2}$$

and showed (1.2) implies S_α is strictly cyclic on ℓ_p. They also proved that (1.2) is necessary if the weight sequence α is eventually decreasing. This strongly suggested that (1.2) is a necessary and sufficient condition for strict cyclicity. However, we will show in this paper that (1.2) is not a necessary condition for S_α to be strictly cyclic for any p with $1 < p < \infty$.

2. PROOF.

To preserve the clarity of the proof we will consider the cases $1 < p \leq 2$ and $2 < p < \infty$ separately.

(a) Let $1 < p \leq 2$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{n_k\}_1^\infty$ be a sequence of rapidly increasing positive integers, e.g., choose $n_1 = 10$ and $n_k = (10n_{k-1})^{10qn_{k-1}}$ for $k > 1$.

We now define the weight sequence $\alpha = \{\alpha_i\}_1^\infty$ by $\alpha_1 = \alpha_2 = \ldots = \alpha_{n_1} = 1$ and for $k > 1$

$$\alpha_i = \begin{cases} n_k^{-1} & \text{if } n_{k-1} < i \leq n_k - n_{k-1} \\ \\ n_{k-1}^{-1} & \text{if } n_k - n_{k-1} < i \leq n_k. \end{cases}$$

Clearly, $\alpha_{n_{k-1}+1} = \alpha_{n_{k-1}+2} = \ldots = \alpha_{n_k - n_{k-1}}$ and $\alpha_{n_k - \ell+1} \leq \alpha_\ell$ for $1 \leq \ell \leq \frac{n_k}{2}$.

Thus, for $0 < m < n_k$

$$\frac{\beta_{n_k}}{\beta_m \beta_{n_k - m}} = \frac{\alpha_{n_k-m+1}\alpha_{n_k-m+2} \cdots \alpha_{n_k}}{\alpha_1 \alpha_2 \ldots \alpha_m} \geq \frac{\alpha_{n_k - n_{k-1}+1} \ldots \alpha_{n_k}}{\alpha_1 \alpha_2 \ldots \alpha_{n_{k-1}}} \geq (\alpha_{n_k})^{n_{k-1}}.$$

Therefore,

$$\sum_{m=1}^{n_k-1} \left|\frac{\beta_{n_k}}{\beta_m\beta_{n_k-m}}\right|^q \geq n_k(\alpha_{n_k})^{qn_{k-1}} = n_k n_{k-1}^{-qn_{k-1}} \to \infty \text{ as } k \to \infty.$$

Hence,

$$\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q = \infty.$$

We now show that S_α is strictly cyclic. It is known that S_α [2] is strictly cyclic if and only if

$$\sum_{n=0}^{\infty} \left|\sum_{m=0}^{n} \frac{\beta_n}{\beta_m\beta_{n-m}} x_m y_{n-m}\right|^p < \infty \text{ for all } x,y \in \ell_p. \tag{1.3}$$

Obviously, for $0 < m < n$

$$\frac{\beta_n}{\beta_m\beta_{n-m}} = \frac{\alpha_{n-m+1} \cdots \alpha_n}{\alpha_1\alpha_2 \cdots \alpha_m} \leq \alpha_n$$

and $\frac{\beta_n}{\beta_m\beta_{n-m}} = 1$ for $m = 0$ or $m = n$.

Thus,

$$\sum_{n=0}^{\infty} \left|\sum_{m=0}^{n} \frac{\beta_n}{\beta_m\beta_{n-m}} x_m y_{n-m}\right|^p \leq \sum_{n=0}^{\infty} \{|x_0y_n| + |y_0x_n| + \sum_{m=1}^{n-1} \alpha_n|x_m y_{n-m}|\}^p$$

$$\leq \sum_{n=0}^{\infty} \{|x_0y_n| + |y_0x_n| + \alpha_n\|x\|_2\|y\|_2\}^p < \infty$$

since $\alpha,x,y \in \ell_p$.

Hence (1.3) holds and S_α is strictly cyclic.

(b) Let $2 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{n_k\}_1^\infty$ be a sequence of rapidly increasing integers, e.g., choose $n_1 = 10$ and n_k such that

$$\sum_{n=1}^{n_k} \frac{1}{n} \geq (10n_{k-1})^{10n_{k-1}} \text{ for } k > 1.$$

Define $\{d_i\}_1^\infty$ such that $\prod_{i=1}^{n} d_i = n^{-\frac{1}{q}}$ for $n = 1,2,\cdots$ and define $\{s_k\}_1^\infty$ by $s_1 = 10$ and $s_k = 2s_{k-1} + 2n_k$ for $k > 1$.

We now define the weight sequence $\alpha = \{\alpha_i\}_1^\infty$ by $\alpha_1 = \ldots = \alpha_{s_1} = 1$ and for $k > 1$,

$$\alpha_i = \begin{cases} n_k^{-2} & \text{if} \quad s_{k-1} < i \le s_{k-1}+n_k \\ n_k^{-2}\, d_{s_k-s_{k-1}-i+1} & \text{if} \quad s_{k-1}+n_k < i \le s_k - s_{k-1} \\ n_{k-1}^{-2} & \text{if} \quad s_k - s_{k-1} < i \le s_k. \end{cases}$$

Now, for $s_{k-1} < m < \frac{s_k}{2}$,

$$\frac{\beta_{s_k}}{\beta_m \beta_{s_k-m}} = \frac{\alpha_{s_k-m+1} \cdots \alpha_{s_k}}{\alpha_1\alpha_2 \cdots \alpha_m} \ge \frac{\alpha_{s_k-m+1} \cdots \alpha_{s_k-s_{k-1}}}{\alpha_{s_{k-1}+1} \cdots \alpha_m}\, n_{k-1}^{-2s_{k-1}} \ge n_{k-1}^{-2s_{k-1}} \prod_{i=1}^{m-s_{k-1}} d_i$$

$$= n_{k-1}^{-2s_{k-1}} (m-s_{k-1})^{-\frac{1}{q}}.$$

Thus,

$$\sum_{m=0}^{s_k} \left|\frac{\beta_{s_k}}{\beta_m \beta_{s_k-m}}\right|^q \ge n_{k-1}^{-2qs_{k-1}} \sum_{i=1}^{n_k} i^{-1} \ge n_{k-1}^{-10n_{k-1}} \sum_{i=1}^{n_k} i^{-1} \to \infty \text{ as } k \to \infty.$$

Hence,
$$\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right|^q = \infty.$$

We now show that

$$\sum_{n=0}^{\infty} \left|\sum_{m=0}^{n} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m}\right|^p < \infty \quad \text{for all } x,y \in \ell_p.$$

If $s_{k-1} < n < s_k - s_{k-1}$ and $0 < m < n$ then, $\quad \frac{\beta_n}{\beta_m \beta_{n-m}} \le n_k^{-2}$.

Let h = min{m,n−m} then, for $s_k - s_{k-1} \le n \le s_k$ and $0 < m < n$,

$$\frac{\beta_n}{\beta_m \beta_{n-m}} \le n_{k-1}^{-2}\, h^{-\frac{1}{q}} .$$

Thus,

$$\sum_{k=2}^{\infty} \sum_{n=s_{k-1}+1}^{s_k - s_{k-1} - 1} \left| \sum_{m=1}^{n-1} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p \le \sum_{k=2}^{\infty} \sum_{n=s_{k-1}+1}^{s_k - s_{k-1} - 1} \left(n_k^{-2} \sum_{m=1}^{n-1} |x_m y_{n-m}| \right)^p < \infty. \tag{1.4}$$

Let $\delta = \frac{p}{p-2}$ then $\delta > q$ and $\frac{1}{\delta} + \frac{2}{p} = 1$. Let $M = \left(\sum_{m=1}^{\infty} 2m^{-\frac{\delta}{q}} \right)^{\frac{1}{\delta}} \left(\| x \|_p^p + \| y \|_p^p \right)^{\frac{2}{p}} < \infty.$

Then,

$$\sum_{k=2}^{\infty} \sum_{n=s_k - s_{k-1}}^{s_k} \left| \sum_{m=1}^{n-1} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p$$

$$\le \sum_{k=2}^{\infty} \sum_{n=s_k - s_{k-1}}^{s_k} \left(\sum_{m=1}^{\infty} n_{k-1}^{-2} h^{-\frac{1}{q}} |x_m y_{n-m}| \right)^p, \text{ where } h = \min\{m, n-m\}$$

$$\le \sum_{k=2}^{\infty} \sum_{n=s_k - s_{k-1}}^{s_k} n_{k-1}^{-2p} M^p < \infty.$$

Combining (1.4) and (1.5) we obtain that (1.3) is satisfied. QED

References

1. Embry, Mary. Strictly Cyclic Operator Algebras on a Banach Space, to appear in Pac. J. Math.

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