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A NOTE ON STRONGLY $*$-CLEAN RINGS

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Abstract. A $*$-ring R is called (strongly) $*$-clean if every element of R is the sum of a projection and a unit (which commute with each other). In this note, some properties of $*$-clean rings are considered. In particular, a new class of $*$-clean rings which called strongly π-$*$-regular are introduced. It is shown that R is strongly π-$*$-regular if and only if R is π-regular and every idempotent of R is a projection if and only if $R/J(R)$ is strongly regular with $J(R)$ nil, and every idempotent of $R/J(R)$ is lifted to a central projection of R. In addition, the stable range conditions of $*$-clean rings are discussed, and equivalent conditions among $*$-rings related to $*$-cleanness are obtained.

1. Introduction

Rings in which every element is the product of a unit and an idempotent are said to be *unit regular*. Recall that an element of a ring R is *clean* if it is the sum of an idempotent and a unit, and R is *clean* if every element of R is clean (see [12]). Clean rings were introduced by Nicholson in relation to exchange rings and have been extensively studied since then. Recently, Wang et al. [16] showed that unit regular rings have idempotent stable range one (i.e., whenever $aR + bR = R$ with $a, b \in R$, there exists $e^2 = e \in R$ such that $a + be \in U(R)$, written $\mathrm{isr}(R) = 1$ for short), and rings with $\mathrm{isr}(R) = 1$ are clean. In 1999, Nicholson [13] called an element of a ring R *strongly clean* if it is the sum of a unit and an idempotent that commute with each other, and R is *strongly clean* if each of its elements is strongly clean. Clearly, a strongly clean ring is clean, and the converse holds for an abelian ring (that is, all idempotents in the ring are central). Local rings and strongly π-regular rings are well-known examples of strongly clean rings.

A ring R is a *$*$-ring* (or *ring with involution*) if there exists an operation $* : R \to R$ such that for all $x,\ y \in R$

$$(x + y)^* = x^* + y^*,\quad (xy)^* = y^*x^*,\quad \text{and } (x^*)^* = x.$$

An element p of a $*$-ring is a *projection* if $p^2 = p = p^*$. Obviously, 0 and 1 are

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projections of any $*$-ring. A $*$-ring R is *$*$-regular* [2] if for every x in R there exists a projection p such that $xR = pR$. Following Vaš [15], an element of a $*$-ring R is (*strongly*) *$*$-clean* if it can be expressed as the sum of a unit and a projection (that commute), and R is (*strongly*) *$*$-clean* if all of its elements are (strongly) $*$-clean. Clearly, $*$-clean rings are clean and strongly $*$-clean rings are strongly clean. It was shown in [7, 11] that there exists a clean $*$-ring but not $*$-clean, and unit regular $*$-regular rings (which called *$*$-unit regular* rings in [7]) need not be strongly $*$-clean, which answered two questions raised by Vaš in [15].

In this note, we continue the study of (strongly) $*$-clean rings. In Section 2, several basic properties of (strongly) $*$-clean rings are investigated. Motivated by the close relationship between strong π-regularity and strong cleanness, we introduce the concept of strongly π-$*$-regular rings in Section 3. The structure of strongly π-$*$-regular rings is considered and some properties of extensions are discussed. As we know, it is still an open question that whether a strongly clean ring has idempotent stable range one, or even has stable range one (see [13]). In Section 4, we extend $\mathrm{isr}(R) = 1$ to the $*$-version. We call a $*$-ring R have *projection stable range one* (written $\mathrm{psr}(R) = 1$) if, for any $a, b \in R$, $aR + bR = R$ implies that $a + bp$ is a unit of R for some projection $p \in R$. It is shown that if R is strongly $*$-clean, then $\mathrm{psr}(R) = 1$, and if $\mathrm{psr}(R) = 1$, then R is $*$-clean. Furthermore, several equivalent conditions among (strongly) clean rings, (strongly) $*$-clean rings and $*$-rings with projection (idempotent) stable range one are obtained.

Throughout this paper, rings are associative with unity. Let R be a ring. The set of all idempotents, all nilpotents and all units of R are denoted by $Id(R)$, R^{nil} and $U(R)$, respectively. For $a \in R$, the commutant of a is denoted by $\mathrm{comm}(a) = \{x \in R : ax = xa\}$. We write $M_n(R)$ for the ring of all $n \times n$ matrices over R whose identity element we write as I_n. Let $\mathbb{Z}_n$ be the ring of integers modulo n. For a $*$-ring R, the symbol $P(R)$ stands for the set of all projections of R.

2. $*$-clean rings

In this section, some basic properties of $*$-clean rings are discussed, and several examples related to $*$-cleanness are given.

Example 2.1. (1) Units, elements in $J(R)$ and nilpotents of a $*$-ring R are $*$-clean.

(2) Idempotents of a $*$-regular rings are $*$-clean.

Proof. (1) It is obvious.

(2) Let R be $*$-regular and $e \in Id(R)$. Then there exists a projection p such that $(1-e)R = pR$. So we have $1 - e = p(1 - e)$ and $p = (1 - e)p$, and hence $ep = 0$. Note that $(e-p)(e-p) = e-ep-pe+p = e+p(1-e) = e+(1-e) = 1$. So $e - p \in U(R)$, and $e = p + (e - p)$ is $*$-clean in R. □

By Example 2.1, every local ring with involution $*$ is $*$-clean. In [15], Vaš asked whether there is an example of a $*$-ring that is clean but not $*$-clean. It was answered affirmatively in [7] and [11]. In fact, one can construct some counterexamples based on the following.

Example 2.2. Let R be a boolean $*$-ring. Then R is $*$-clean if and only if $* = 1_R$ is the identity map of R. In particular, $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with $(a, b)^* = (b, a)$ is clean but not $*$-clean.

Proof. Note that every boolean ring is clean. Suppose that R is $*$-clean. Given any $a \in R$. Then $-a = p + u = p + 1 = p - 1$ for some $p \in P(R)$. So we have $a = 1 - p \in P(R)$. Thus, $a^* = a$, which implies $* = 1_R$. Conversely, if $* = 1_R$, then every idempotent of R is a projection. Thus, R is $*$-clean. $\square$

Lemma 2.3. *Let R be a $*$-ring. If $2 \in U(R)$, then for any $u^2 = 1$, $u^* = u \in R$ if and only if every idempotent of R is a projection.*

Proof. ($\Rightarrow$) Let $e \in Id(R)$. Then $(1 - 2e)^2 = 1$. So we have $2e = 2e^*$, and thus $2(e - e^*) = 0$. Since $2 \in U(R)$, $e = e^*$. As desired.

($\Leftarrow$) Given $u \in R$ with $u^2 = 1$. Then $\frac{u+1}{2} \in Id(R)$ since $(\frac{u+1}{2})^2 = \frac{u^2+2u+1}{4} = \frac{u+1}{2}$. Since every idempotent of R is a projection, it follows from $(\frac{u+1}{2})^* = \frac{u+1}{2}$ that $u^* = u$. $\square$

The $*$-ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in Example 2.2 reveals that "$2 \in U(R)$" in Lemma 2.3 cannot be removed.

Corollary 2.4. *Let R be a $*$-ring with $2 \in U(R)$. The following are equivalent*:

(1) *R is clean and every unit of R is self-adjoint (i.e., $u^* = u$ for every $u \in U(R)$).*
(2) *R is $*$-clean and $* = 1_R$.*

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2). Let $a \in R$. Then $a = e + u$ for some $e \in Id(R)$ and $u \in U(R)$. Note that $(1 - 2e)^2 = 1$. By Lemma 2.3, $e^* = e$. Thus $a \in R$ is $*$-clean and $a^* = a$, and so $* = 1_R$. $\square$

Recall that an element t of a $*$-ring R is *self-adjoint square root of* 1 if $t^2 = 1$ and $t^* = t$.

Theorem 2.5. *Let R be a $*$-ring, the following are equivalent*:

(1) *R is $*$-clean and $2 \in U(R)$.*
(2) *Every element of R is a sum of a unit and a self-adjoint square root of 1.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R$. Then $\frac{1+a}{2} = p + u$ for some $p \in P(R)$ and $u \in U(R)$. It follows that $a = (2p-1)+2u$ where $(2p-1)^* = 2p-1$, $(2p-1)^2 = 1$ and $2u \in U(R)$.

(2) $\Rightarrow$ (1). We first show that $2 \in U(R)$. By hypothesis, $1 = x + v$ with $x^2 = 1$ and $v \in U(R)$. So we have $(1 - v)^2 = x^2 = 1$, which implies that

$v^2 = 2v$. Since v is a unit, $v = 2 \in U(R)$. Given any $a \in R$, then there exist $y, w \in R$ satisfying $2a - 1 = y + w$ with $y^* = y$, $y^2 = 1$ and $w \in U(R)$. Thus, $a = \frac{y+1}{2} + \frac{w}{2}$ is a $*$-clean expression since $(\frac{y+1}{2})^* = \frac{y+1}{2}, (\frac{y+1}{2})^2 = \frac{y+1}{2}$ and $\frac{w}{2} \in U(R)$. □

Camillo and Yu [5] showed that if R is a ring in which 2 is a unit, then R is clean if and only if every element of R is the sum of a unit and a square root of 1. Indeed, by the proof of Theorem 2.5, the condition $2 \in U(R)$ is also necessary.

Proposition 2.6. *The following are equivalent for a $*$-ring R :*

(1) *R is $*$-clean and $0, 1$ are the only projections.*
(2) *R is clean ring and $0, 1$ are the only idempotents.*
(3) *R is a local ring.*

Proof. (2) ⇒ (3) follows from [14, Lemma 14] and (3) ⇒ (1) follows by Example 2.1.

(1) ⇒ (2). It suffices to show that the only idempotents in R are 0 and 1. For $e^2 = e \in R$, the hypothesis implies that $e = p + u$ where $p \in P(R) = \{0, 1\}$ and $u \in U(R)$. If $p = 0$, then $e = u$ is a unit, so $e = 1$. If $p = 1$, then $1 - e = -u \in U(R)$, and hence $e = 0$. As required. □

Let I be an ideal of a $*$-ring R. We call I is *$*$-invariant* if $I^* \subseteq I$. In this case, the involution $*$ of R can be extended to the factor ring R/I, which is still denoted by $*$.

Lemma 2.7. *Let R be $*$-clean. If I is a $*$-invariant ideal of R, then R/I is $*$-clean. In particular, $R/J(R)$ is $*$-clean.*

Proof. Since the homomorphism image of a projection (resp., unit) is also a projection (resp., unit), the result follows.

Next we only need to prove that $J(R)$ is $*$-invariant. For any $a^* \in (J(R))^*$, we show that $a^* \in J(R)$. Note that $a \in J(R)$. Take any $x \in R$. Then $1 - x^*a \in U(R)$. Thus $1 - a^*x = (1 - x^*a)^*$ is a unit of R, as desired. □

Let R be a $*$-ring. Then $*$ induces an involution of the power series ring $R[[x]]$, denoted by $*$, where $(\sum_{i=0}^{\infty} a_i x^i)^* = \sum_{i=0}^{\infty} a_i^* x^i$.

Proposition 2.8. *Let R be a $*$-ring. Then $R[[x]]$ is $*$-clean if and only if R is $*$-clean.*

Proof. Suppose that $R[[x]]$ is $*$-clean. Note that $R \cong R[[x]]/(x)$ and (x) is a $*$-invariant ideal of $R[[x]]$. By Lemma 2.7, R is $*$-clean. Conversely, assume that R is $*$-clean. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$. Write $a_0 = p + u$ with $p \in P(R)$ and $u \in U(R)$. Then $f(x) = p + (u + \sum_{i=1}^{\infty} a_i x^i)$, where $p \in P(R) \subseteq P(R[[x]])$ and $u + \sum_{i=1}^{\infty} a_i x^i \in U(R[[x]])$. Hence $f(x)$ is $*$-clean in $R[[x]]$. □

According to [14, Proposition 13], the polynomial ring $R[x]$ is never clean. Hence, $R[x]$ is not $*$-clean for any involution $*$.

3. Strongly π-$*$-regular rings

Strong π-regularity is closely related to strong cleanness. In this section, we introduce the notion of strongly π-$*$-regular rings which can be viewed as $*$-versions of strongly π-regular rings. The structure and properties of strongly π-$*$-regular rings are given.

Lemma 3.1 ([11, Lemma 2.1]). *Let R be a $*$-ring. If every idempotent of R is a projection, then R is abelian.*

Due to [7], an element a of a $*$-ring R is *strongly $*$-regular* if $a = pu = up$ with $p \in P(R)$ and $u \in U(R)$; R is *strongly $*$-regular* if each of its elements is strongly $*$-regular. By [7, Proposition 2.8], any strongly $*$-regular element is strongly $*$-clean.

Theorem 3.2. *Let R be a $*$-ring. Then the following are equivalent for $a \in R$:*

(1) *There exist $e \in P(R)$, $u \in U(R)$ and an integer $m \geq 1$ such that $a^m = eu$ and a, e, u commute with each other.*
(2) *There exist $f \in P(R)$, $v \in U(R)$ such that $a = f + v$, $fv = vf$ and $af \in R^{\text{nil}}$.*
(3) *There exists $p \in P(R)$ such that $p \in \text{comm}(a)$, $ap \in U(pRp)$ and $a(1-p) \in R^{\text{nil}}$.*
(4) *There exists $b \in \text{comm}(a)$ such that $(ab)^* = ab$, $b = bab$ and $a - a^2b \in R^{\text{nil}}$.*

Proof. (1) $\Rightarrow$ (2). Write $f = 1 - e$. Clearly, $f \in P(R)$ and $a^m - f \in U(R)$ with the inverse $u^{-1}e - f$. From $af = fa$, we have $a - f := v$ is a unit of R (since $(a-f)(a^{m-1} + a^{m-2}f + \cdots + af + f) = a^m - f \in U(R)$) and $fv = vf$. It is clear that $(af)^m = a^m f = 0$.

(2) $\Rightarrow$ (3). Set $p = 1 - f$. Then $p \in P(R)$, $ap = pa = vp \in U(pRp)$ and $a(1-p) = af \in R^{\text{nil}}$.

(3) $\Rightarrow$ (4). By (3), $aw = wa = p$ for some $w \in U(pRp)$. So we obtain $[a-(1-p)][w-(1-p)] = 1 - a(1-p) \in U(R)$ since $a(1-p)$ is nilpotent, which implies that $a - (1-p) \in U(R)$. Let $b = [a - (1-p)]^{-1}p$. Then $b \in \text{comm}(a)$, $bp = b$ and $ab = [a-(1-p)]b = p \in P(R)$. Thus $(ab)^* = ab$, $b = bp = bab$ and $a - a^2b = a(1-ab) = a(1-p) \in R^{\text{nil}}$.

(4) $\Rightarrow$ (1). Let $e = ab$. Then $(ab)^* = ab$ implies $e^* = e$, and $bab = b$ yields $e^2 = e$. So $e \in P(R)$. As $a - a^2b \in R^{\text{nil}}$, $a^m = a^m e$ for some integer $m \geq 1$. Take $u = a^m + (1-e)$ and $u' = b^m e + (1-e)$. Then $uu' = u'u = 1$. Hence, $u \in U(R)$ and $a^m = a^m e = ue$ with a, e, u commuting with each other. □

Recall that an element a of a ring R is *strongly π-regular* if $a^n \in a^{n+1}R \cap Ra^{n+1}$ for some $n \geq 1$ (equivalently, $a^n = eu$ with $e \in Id(R)$, $u \in U(R)$ and a, e, u all commute [13]); R is *strongly π-regular* if every element of R is strongly π-regular. Based on the above, we introduce the following concept.

Definition 3.3. Let R be a $*$-ring. An element $a \in R$ is called strongly π-$*$-regular if it satisfies the conditions in Theorem 3.2; R is called strongly π-$*$-regular if every element of R is strongly π-$*$-regular.

Corollary 3.4. *Any strongly $*$-regular element is strongly π-$*$-regular, and any strongly π-$*$-regular element is strongly $*$-clean.*

Example 3.5. (1) Let $R = \mathbb{Z}_4$ and $* = 1_R$. Then R is strongly π-$*$-regular. However, $2 \in R$ is not strongly $*$-regular.

(2) Let R be a local domain with involution $*$ and $J(R) \neq 0$. Note that $P(R) = Id(R) = \{0, 1\}$. So R is strongly $*$-clean by Proposition 2.6, but any power of a nonzero element in $J(R)$ can not expressed as the product of a projection and a unit.

Recall that a ring R is *π-regular* if for any $a \in R$, there exist $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$. Strongly π-regular rings and regular rings are π-regular (see [13]). A ring R is *directly finite* if $ab = 1$ implies $ba = 1$ for all $a, b \in R$. Abelian rings are directly finite.

Theorem 3.6. *The following are equivalent for a $*$-ring R :*

(1) *R is strongly π-$*$-regular.*
(2) *R is π-regular and every idempotent of R is a projection.*
(3) *For any $a \in R$, there exist $n \geq 1$ and $p \in P(R)$ such that $a^n R = pR$, and R is abelian.*
(4) *For any $a \in R$, there exists $n \geq 1$ such that a^n is strongly $*$-regular.*
(5) *For any $a \in R$, there exist $p \in P(R)$ and $u \in U(R)$ such that $a = p + u$, $ap \in R^{\mathrm{nil}}$; and $v^{-1}qv$ is a projection for all $v \in U(R)$ and all $q \in P(R)$.*

Proof. (1) $\Rightarrow$ (2). Note that every strongly π-$*$-regular ring is strongly π-regular and strongly $*$-clean. Thus R is a π-regular ring. By [11, Theorem 2.2], every idempotent of R is a projection.

(2) $\Rightarrow$ (3). For any $a \in R$, there exists $n \geq 1$ such that $a^n = a^n x a^n$ for some $x \in R$. Write $a^n x = p$. Then $p \in P(R)$ and $a^n = pa^n$. It is clear that $a^n R = pR$. In view of Lemma 3.1, R is abelian.

(3) $\Rightarrow$ (4). Let $e \in Id(R)$. Then $eR = pR$ for some $p \in P(R)$. Since R is abelian, we have $e = pe = ep = p$. Thus, every idempotent of R is a projection. Given $a \in R$, there exist $n \geq 1$ and $q \in P(R)$ such that $a^n R = qR$. So one gets $a^n = qa^n$ and $q = a^n x$ for some $x \in R$, which implies $a^n = a^n x a^n$. Next we show that $a^n - (1-q)$ is invertible. Note that $[a^n - (1-q)][xq - (1-q)] = 1$. Then $a^n - (1-q) := u \in U(R)$ since R is directly finite. Multiplying the equation $a^n - (1-q) = u$ by p yields $a^n = a^n q = uq = qu$, which implies that a^n is strongly $*$-regular.

(4) $\Rightarrow$ (5). For $e \in Id(R)$, $e = qv = vq$ for some $q \in P(R)$ and $v \in U(R)$ by the assumption. Then $e = qv = e^2 = qv^2$, and so we obtain $q = qv = e$, which implies that every idempotent of R is a projection. Clearly, $v^{-1}qv$ is a projection for all $v \in U(R)$ and all $q \in P(R)$. Given $a \in R$ as in (4), $a^n =$

$(1-p)w = w(1-p)$ for some $p \in P(R)$ and $w \in U(R)$. Note that R is abelian. So we have $a^n p = (ap)^n = 0$ and $(a-p)[a^{n-1}w^{-1}(1-p) - \sum_{i=0}^{n} a^i p] = 1$, and hence $a - p \in U(R)$ as R is directly finite.

(5) $\Rightarrow$ (1). By (5), every element of R is $*$-clean. In view of [11, Theorem 2.2], R is abelian. Thus R is strongly π-$*$-regular by Theorem 3.2(2). □

Corollary 3.7. *Let R be a $*$-ring. The following are equivalent*:

(1) *R is strongly π-$*$-regular.*
(2) *$R/J(R)$ is strongly π-$*$-regular with $J(R)$ nil, every projection of R is central and every projection of $R/J(R)$ is lifted to a projection of R.*
(3) *$R/J(R)$ is strongly $*$-regular with $J(R)$ nil, and every idempotent of $R/J(R)$ is lifted to a central projection of R.*

Proof. Write $\overline{R} = R/J(R)$. By Lemma 2.7, $\overline{R}$ is a $*$-ring.

(1) $\Rightarrow$ (2). Clearly, $\overline{R}$ is strongly π-$*$-regular. As R is strongly π-regular, for any $a \in J(R)$, there exist $m \geq 1$, $e \in Id(R)$ and $u \in U(R)$ such that $e = a^m u \in J(R)$. So $a^m = eu^{-1} = 0$, which implies that $J(R)$ is nil. Note that R is strongly $*$-clean. So the rest follows from [11, Corollary 2.11].

(2) $\Rightarrow$ (3). By virtue of [11, Corollar 2.11], $\overline{R}$ is reduced (i.e., $\overline{R}^{\text{nil}} = 0$), and every idempotent of $\overline{R}$ is lifted to a central projection of R. So we only need to prove that $\overline{R}$ is strongly $*$-regular. Given any $x \in \overline{R}$. By Theorem 3.2, there exist $p \in P(\overline{R})$ and $v \in U(\overline{R})$ such that $a = p + v$, $vp = pv$ and $ap \in \overline{R}^{\text{nil}} = 0$. It follows that $a = a(1-p) = v(1-p) = (1-p)v$ is strongly $*$-regular in $\overline{R}$.

(3) $\Rightarrow$ (1). Since $\overline{R}$ is strongly regular, it is reduced clean. By [11, Corollary 2.11], every idempotent of R is a projection. Note that $J(R)$ is nil and $\overline{R}$ is π-regular. So R is π-regular by [1, Theorem 4]. In view of Theorem 3.6, R is strongly π-$*$-regular. □

Corollary 3.8. *Let R be a $*$-ring. Then R is strongly $*$-clean and π-regular if and only if R is strongly π-$*$-regular.*

Proof. If R is strongly $*$-clean and π-regular, by [11, Theorem 2.2], idempotents of R are projections. So R is strongly π-$*$-regular by Theorem 3.6. The other direction is clear. □

For a $*$-ring R, the matrix ring $M_n(R)$ has a natural involution inherited from R : if $A = (a_{ij}) \in M_n(R)$, A^* is the transpose of (a_{ij}^*) (i.e., $A^* = (a_{ij}^*)^T = (a_{ji}^*)$). Henceforth we consider $M_n(R)$ as a $*$-ring with respect to this natural involution.

Corollary 3.9. *Let R be a $*$-ring. Then $M_n(R)$ is not strongly π-$*$-regular for any $n \geq 2$.*

Let R be a $*$-ring and $S = pRp$ with $p \in P(R)$. Then the restriction of $*$ on S will be an involution of S, which is also denoted by $*$.

Corollary 3.10. *If R is strongly π-$*$-regular, then so is eRe for any $e \in Id(R)$.*

Proof. Let $S = eRe$ with $e \in Id(R)$. By hypothesis, e is a projection of R. So S is a $*$-ring. It is well known that S is strongly π-regular (see also [4, Lemma 39]). Clearly, every idempotent of S ($\subseteq R$) is a projection. So the result follows by Theorem 3.6. □

Let RG be the group ring of a group G over a ring R. According to [11, Lemma 2.12], the map $* :\ RG \to RG$ given by $(\sum_g a_g g)^* = \sum_g a_g^* g^{-1}$ is an involution of RG, and is denoted by $*$ again.

Corollary 3.11. *Let R be a $*$-ring with artinian prime factors, $2 \in J(R)$ and G be a locally finite 2-group. Then R is strongly π-$*$-regular if and only if RG is strongly π-$*$-regular.*

Proof. Assume that R is strongly π-$*$-regular. Then $Id(R) = P(R)$. In particular, R is abelian. So idempotents of R coincide with idempotents in RG by [8, Lemma 11], and hence every idempotent of RG is a projection. Since R is a ring with artinian prime factors and G is a locally finite 2-group, RG is a strongly π-regular ring by [10, Theorem 3.3]. In view of Theorem 3.6, RG is strongly π-$*$-regular.

Conversely, R is strongly π-regular by [10, Proposition 3.4]. Note that $Id(R) \subseteq Id(RG)$ and all idempotents of RG are projections. By Theorem 3.6, R is strongly π-$*$-regular. □

Let $\mathbb{C}$ be the complex filed. It is well known that for any $n \geq 1$, the matrix ring $M_n(\mathbb{C})$ is strongly π-regular. However, $M_n(\mathbb{C})$ is not strongly π-$*$-regular whenever $n \geq 2$ by Corollary 3.9. So it is interesting to determine when a matrix of $M_n(\mathbb{C})$ is strongly π-$*$-regular. The set of all $n \times 1$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^n$.

Example 3.12. Let $S = M_n(\mathbb{C})$ with $*$ the transpose operation. Then A is strongly π-$*$-regular if and only if there exist $e_1, e_2, \ldots, e_n \in \mathbb{C}^n$ such that $e_i^* e_j = 0$ for $i = 1, \ldots, r$; $j = r+1, \ldots, n$, and $A = P\left(\begin{smallmatrix} C & 0 \\ 0 & N \end{smallmatrix}\right)P^{-1}$ with $P = (e_1, e_2, \ldots, e_n) \in U(S)$, $C \in U(M_r(\mathbb{C}))$ and $N \in [M_{n-r}(\mathbb{C})]^{\text{nil}}$. In particular, any real symmetric matrix is strongly π-$*$-regular.

Proof. Given $A \in S$. Assume that $rank(A) = r$. By the Jordan canonical decomposition, there exists $P = (e_1, e_2, \ldots, e_n) \in U(S)$ such that

$$A = P \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} P^{-1},$$

where $e_i \in \mathbb{C}^n$ for all i, $C \in U(M_r(\mathbb{C}))$ and $N \in [M_{n-r}(\mathbb{C})]^{\text{nil}}$. Write

$$B = P \begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

Then one easily gets that $BA = AB,\ B = BAB$ and $A - A^2B = P\left(\begin{smallmatrix} 0 & 0 \\ 0 & N \end{smallmatrix}\right)P^{-1}$ is nilpotent. Note that B satisfies the above conditions is unique (see [3]). In view of Theorem 3.2, A is strongly π-$*$-regular if and only if $(AB)^* = AB$. Notice that $AB = P\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)P^{-1}$ and

$$\begin{aligned} &(AB)^* = AB \\ \Leftrightarrow\ &(P^{-1})^*\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)P^* = P\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)P^{-1} \\ \Leftrightarrow\ &(P^*P)^{-1}\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)(P^*P) = \left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ \Leftrightarrow\ &\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)(P^*P) = (P^*P)\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ \Leftrightarrow\ &P^*P = \left(\begin{smallmatrix} V_1 & 0 \\ 0 & V_2 \end{smallmatrix}\right) \text{ with } V_1 \in U(M_r(\mathbb{C})) \text{ and } V_2 \in U(M_{n-r}(\mathbb{C})) \\ \Leftrightarrow\ &e_i^* e_j = 0 \text{ for all } i \in \{1, 2, \ldots, r\},\ j \in \{r+1, r+2, \ldots, n\}, \end{aligned}$$

where

$$\begin{aligned} V_1 &= (e_1^*, e_2^*, \ldots, e_r^*)^T (e_1, e_2, \ldots, e_r); \\ V_2 &= (e_{r+1}^*, e_{r+2}^*, \ldots, e_n^*)^T (e_{r+1}, e_{r+2}, \ldots, e_n). \end{aligned}$$

If $A \in S$ is a real symmetric matrix, then there exists an orthogonal matrix P (i.e., $P^{-1} = P^T = P^*$) such that $A = P\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)P^{-1}$. So the result follows. □

In view of [2, Proposition 3], the involution of a $*$-regular ring R is proper (i.e., $x^*x = 0$ implies that $x = 0$ for all $x \in R$).

Remark 3.13. If R is strongly π-$*$-regular, then for any $x \in R$, $x^*x = 0$ implies $x \in R^{\text{nil}}$. Indeed, by Theorem 3.2, there exist $p \in P(R)$ and $u \in U(R)$ such that $x^m = pu = up$ for some $m \geq 1$. Then $0 = (x^*)^m x^m = (x^m)^* x^m = u^* pu$, and thus $p = 0$, whence $x^m = 0$.

4. Stable range conditions

In [13], Nicholson asked whether every strongly clean ring has stable range one, and it is still open. Recall that a ring R is said to *have idempotent stable range one* (written isr(R)=1) provided that for any $a,\ b \in R$, $aR + bR = R$ implies that $a + be \in U(R)$ for some $e \in Id(R)$ (see [6, 16]). If e is an arbitrary element of R (not necessary an idempotent), then R is said to *have stable range one.* Clearly, if isr$(R) = 1$, then R is clean and has stable range one. We extend the notion of isr$(R) = 1$ to $*$-versions.

Definition 4.1. A $*$-ring R is said to have projection stable range one (written psr$(R) = 1$) if for any $a,\ b \in R$, $aR + bR = R$ implies there exists $p \in P(R)$ such that $a + bp \in U(R)$.

The following result is motivated by [6, Proposition 2].

Proposition 4.2. *Let R be a $*$-ring. The following are equivalent*:

(1) *psr$(R) = 1$.*
(2) *For any $a,\ b \in R$, $aR + bR = R$ implies there exists $p \in P(R)$ such that $a + bp$ is right invertible.*

(3) *For any* $a,\ b \in R$, $aR + bR = R$ *implies there exists* $p \in P(R)$ *such that* $a + bp$ *is left invertible.*

Proof. The proof is similar to that of [6, Proposition 2].

(1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Let $a,\ b \in R$ with $aR + bR = R$. Then there is a projection $p \in R$ such that $a + bp = u$ is right invertible. Assume that $uw = 1$ for some $w \in R$. Then $wR + (1 - wu)R = R$. So the hypothesis implies there exists $q \in P(R)$ such that $w + (1 - wu)q$ is right invertible. Note that $u[w + (1 - wu)q] = 1$. Thus $w + (1 - wu)q$ is also left invertible, and hence invertible. This implies that $u \in U(R)$.

(3) $\Rightarrow$ (1). Given any $a, b \in R$ with $aR + bR = R$. Then there exists $p \in P(R)$ such that $a + bp$ is left invertible. We may let $v \in R$ with $v(a + bp) = 1$. Then $vR + 0R = R$. By hypothesis, we can find a projection q such that $v + 0q = v$ is left invertible. So v is a unit, which implies that $a + bp \in U(R)$. Therefore, $\mathrm{psr}(R) = 1$. □

For a $*$-ring R, it is clear that if $\mathrm{psr}(R) = 1$, then $\mathrm{isr}(R) = 1$. However, there exists a $*$-ring with $\mathrm{isr}(R) = 1$ but not satisfies $\mathrm{psr}(R) = 1$.

Example 4.3. Define the involution of $\mathbb{Z}_2$ by $* : x \mapsto x$. Let $S = M_2(\mathbb{Z}_2)$. Then S is a $*$-ring. In view of [16, Corollary 3.4], $\mathrm{isr}(S) = 1$ since S is unit regular. Notice that $P(S) = \{O, I_2, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$, and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) S + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) S = S$. However, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) P$ is not invertible for any $P \in P(S)$. Hence, $\mathrm{psr}(S) \neq 1$.

From Example 4.3, one can also find that the projection stable range one property cannot be inherited to the matrix ring.

Proposition 4.4. *Let* R *be a* $*$*-ring. If* $\mathrm{psr}(R) = 1$*, then* R *is* $*$*-clean.*

Proof. For any $a \in R$, the equation $aR + (-1)R = R$ implies that $a + (-1)p = u \in U(R)$ for some $p \in P(R)$. So $a = p + u$, and hence R is $*$-clean. □

According to [15, Proposition 4], the ring in Example 4.3 is $*$-clean. So we conclude that the converse of Proposition 4.4 is not true.

Following Nicholson [12], a ring R is *exchange* if for every $a \in R$, there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$. Clean rings are exchange, the converse holds whenever the rings are abelian. A $*$-ring R is called $*$*-abelian* if every projection of R is central [15].

Theorem 4.5. *Let* R *be a* $*$*-ring. The following are equivalent*:

(1) $\mathrm{psr}(R) = 1$ *and* R *is* $*$*-abelian.*
(2) *For any* $a,\ b \in R$, $aR + bR = R$ *implies there exists a projection* $p \in \mathrm{comm}(a)$ *such that* $a + bp \in U(R)$.
(3) $\mathrm{isr}(R) = 1$ *and every idempotent of* R *is a projection.*
(4) R *is clean* (*or exchange*) *and every idempotent of* R *is a projection.*
(5) R *is* $*$*-clean and* $*$*-abelian.*
(6) R *is strongly* $*$*-clean.*

(7) *For every* $a \in R$, *there exists a projection* $p \in aR$ *such that* $1 - p \in (1 - a)R$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are clear; (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) follows from [11, Theorem 2.2].

(2) $\Rightarrow$ (3). We only need to show that all idempotents are projections. Let $e \in Id(R)$. Then $eR + (-1)R = R$. So there exists $p \in P(R)$ such that $ep = pe$ and $e - p \in U(R)$. Note that $(e-p)(1-e-p) = (1-e-p)(e-p) = 0$. Thus, $e = 1 - p \in P(R)$. Therefore, every idempotent of R is a projection.

(7) $\Rightarrow$ (1). Let $e \in Id(R)$. Then there exists a projection $p \in eR$ such that $1-p \in (1-e)R$. So we obtain $p = ep$ and $1-p = (1-e)(1-p)$. It follows that $e = p$, and thus $Id(R) = P(R)$. In view of Lemma 3.1, R is abelian. Note that R is exchange. Then by [6, Theorem 12], $\mathrm{isr}(R) = 1$, and hence $\mathrm{psr}(R) = 1$. $\square$

It is still unknown that whether strongly clean rings have stable range one ([13]). However, we have an affirmative answer of their $*$-versions.

Corollary 4.6. *If* R *is a strongly* $*$*-clean ring, then* $\mathrm{psr}(R) = 1$.

The following example will reveal that the converse of Corollary 4.6 does not hold.

Example 4.7. Let $S = M_2(\mathbb{Z}_3)$. The involution of S is defined by $A \to A^*$, where A^* is the transpose of $A \in S$. Then S is not strongly $*$-clean by [7, Theorem 2.3]. Since S is unit regular, $\mathrm{isr}(S) = 1$ by [16, Corollary 3.4]. In view of [9, Lemma 7], we have

$$Id(S) = \{O, I_2, \left(\begin{smallmatrix} x & y \\ z & 1-x \end{smallmatrix}\right) \text{ with } yz = x - x^2\},$$

and

$$P(S) = \{O, I_2, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)\}.$$

We next prove that $\mathrm{psr}(S) = 1$. Assume on the contrary. Then there exist $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $A' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ with $AS + A'S = S$ but $A + A'P$ is not a unit for any $P \in P(S)$. That is,

$$\det(A + A'P) = 0.$$

This implies the following system of equations:

$$ad - bc = 0 \quad \text{(i)}, \qquad ad' - b'c = 0 \quad \text{(ii)},$$
$$a'd - bc' = 0 \quad \text{(iii)}, \qquad a'd' - b'c' = 0 \quad \text{(iv)},$$
$$ac' - a'c = bd' - b'd \quad \text{(v)}.$$

On the other hand, as $\mathrm{isr}(S) = 1$, there exists $E \in Id(S) \setminus P(S)$ such that $A + A'E \in U(S)$. Then E must be of the form $\left(\begin{smallmatrix} x & y \\ z & 1-x \end{smallmatrix}\right)$ where $yz = x - x^2$. By Eqs. (i)-(iv), we obtain

$$\det(A + A'E) = (ac' - a'c)y - (bd' - b'd)z.$$

Next we show that $ac' - a'c = bd' - b'd = 0$.

Case 1. $c \neq 0$. Multiplying Eq. (v) by c and by substituting $b'c = ad'$, we have $(ac' - a'c)c = bd'c - b'dc = (bc - ad)d' = 0$ by Eq. (i). Thus, $ac' - a'c = bd' - b'd = 0$.

Case 2. $d \neq 0$. Multiplying Eq. (v) by d and by substituting $a'd = bc'$, we have $(bd' - b'd)d = ac'd - a'cd = (ad - bc)c' = 0$ by Eq. (i). So $ac' - a'c = bd' - b'd = 0$.

Case 3. $c = d = 0$. From Eqs. (ii) and (iii), we get $ad' = bc' = 0$. If $b \neq 0$, then $c' = 0$, it follows that $ac' - a'c = 0$. If $a \neq 0$, then $d' = 0$, and so $bd' - b'd = 0$. Thus $ac' - a'c = bd' - b'd = 0$.

Therefore, $\det(A + A'E) = (ac' - a'c)y - (bd' - b'd)z = 0$ for any case, which contradicts $A + A'E \in U(S)$. Hence, $\mathrm{psr}(R) = 1$.

By Theorem 4.5, we have the following result immediately.

Corollary 4.8. *Let R be a $*$-ring. If $Id(R) = P(R)$, then the following are equivalent*:

(1) *R is (strongly) clean.*
(2) *R is exchange.*
(3) *R is (strongly) $*$-clean.*
(4) $\mathrm{isr}(R) = 1$.
(5) $\mathrm{psr}(R) = 1$.

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