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A NOTE ON STRONGLY E-REFLEXIVE INVERSE SEMIGROUPS

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ABSTRACT. In contrast to the semilattice of groups case, an inverse semigroup S which is the union of strongly E-reflexive inverse subsemigroups need not be strongly E-reflexive. If, however, the union is saturated with respect to the Green's relation $\mathfrak{D}$, and in particular if the union is a disjoint one, then S is indeed strongly E-reflexive. This is established by showing that $\mathfrak{D}$-saturated inverse subsemigroups have certain pleasant properties. Finally, in contrast to the E-unitary case, it is shown that the class of strongly E-reflexive inverse semigroups is not closed under free inverse products.

The reader is referred to **[1]**, **[2]** for the basic theory of inverse semigroups, including the theory of free inverse products. Recall from **[4]**, **[5]** that an inverse semigroup S is said to be *strongly E-reflexive* whenever S is a semilattice of E-unitary inverse semigroups, or alternatively, whenever there exists a semilattice of groups congruence η on S such that only idempotents are linked to idempotents under η. In **[4]**, **[5]** we studied this class of semigroups and showed that many of the properties of semilattices of groups and of E-unitary inverse semigroups generalise to this class, albeit sometimes in a weaker form. We continue this line of investigation here.

In what is by now a classic theorem, Clifford showed that an inverse semigroup which is a union of groups is a semilattice of groups. We ask to what extent this is true for strongly E-reflexive inverse semigroups. It is already known that a semilattice of strongly E-reflexive inverse semigroups is again strongly E-reflexive **[5]**. The following simple example shows that we cannot hope for a full generalisation of Clifford's theorem.

Consider the bisimple inverse ω-semigroup $S(G, \alpha)$, where the endomorphism α of the group G is not injective. As noted in **[4**, p. 341], $S(G, \alpha)$ is not strongly E-reflexive. However, using **[1**, Lemma 1.31], it is easily seen that $S(G, \alpha)$ is a union of its maximal subgroups and copies of the bicyclic semigroup, and these are all E-unitary.

The restriction we require will now be given, and the example just noted would seem to indicate that it is the weakest possible.

Let S be an inverse semigroup with semilattice of idempotents E. Let U be an inverse subsemigroup of S which is $\mathfrak{D}$*-saturated* in the sense that $x \mathfrak{D} y \in U$ implies $x \in U$, where $\mathfrak{D}$ denotes the usual Green's relation on S. The maximal group homomorphic image of U is denoted by $\bar{U}$ with $\bar{u}$ denoting the image of u ($u \in U$). Let $U' = \{x \in S | x \geqslant u \text{ for some } u \in U\}$; note that U' may equal S.

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The first result shows that U' has some pleasant properties.

PROPOSITION. (i) *U' is an inverse subsemigroup of S which contains U, and $xy \in U'$ implies $x \in U'$ and $y \in U'$.*

(ii) *The rule: $x\phi = \bar{u}$ if $x \geqslant u \in U$ and $x\phi = 0$ otherwise, gives a well-defined homomorphism ϕ: $S \to \bar{U}^0$ such that $\phi | U$ is the canonical homomorphism onto $\bar{U}$.*

PROOF. (i) $xy \geqslant u \in U \Rightarrow xx^{-1} \geqslant xyy^{-1}x^{-1} \geqslant uu^{-1} \Rightarrow x \geqslant uu^{-1}x \ \mathcal{R}\ u \Rightarrow x \in U'$, since U is $\mathcal{D}$-saturated and $\mathcal{R} \subseteq \mathcal{D}$.

Dually, $y \in U'$. The remainder of the result is easily proven.

(ii) Suppose $x \in U'$ with $x \geqslant u \in U$ and $x \geqslant v \in U$. Then $u = ex, v = fx$ where $e = uu^{-1} \in U \cap E, f = vv^{-1} \in U \cap E$. Hence $efu = efv$, and $ef \in E \cap U$, so that $\bar{u} = \bar{v}$. It is then almost immediate that ϕ is well-defined. The rest of the result involves a little routine calculation, using (i).

REMARK. Taking S to be a semilattice with more than two elements, we see that U need not be an ideal of U' in Proposition 1.

The proposition enables us to prove our main result.

THEOREM. *Let S be a union of $\mathcal{D}$-saturated strongly E-reflexive inverse subsemigroups S_i, $i \in I$. Then S is strongly E-reflexive.*

PROOF. Each S_i is a semilattice Λ_i of E-unitary inverse semigroups T_i^λ, $\lambda \in \Lambda_i$. It is easily shown that each T_i^λ is $\mathcal{D}$-saturated in S. Hence we may suppose without loss of generality that each S_i is E-unitary. For each $i \in I$, let ϕ_i: $S \to \bar{S}_i^0$ be the homomorphism defined as in (ii) above, and let T be the direct product of the $\bar{S}_i^0$. Then the ϕ_i induce a homomorphism ϕ: $S \to T$ with $s\phi$ having ith component $s\phi_i$, $i \in I$. Now $S\phi$ is a semilattice of groups, since T is. Suppose that $x\phi = e\phi$ for some $e \in E$, where $x \in S_i$ say. Then $x\phi_i$ is the identity element of $\bar{S}_i$, and since S_i is E-unitary it follows that $x \in E$; whence the result.

COROLLARY. *Let S be a disjoint union of strongly E-reflexive inverse subsemigroups. Then S itself is strongly E-reflexive.*

PROOF. Clearly each of the inverse subsemigroups in question is $\mathcal{D}$-saturated in S.

REMARK. The elementary theory of inverse semigroups shows that an inverse semigroup S which is a union of groups is a disjoint union of its maximal subgroups H_e, $e \in E$, and that this is the $\mathcal{D}$-decomposition of S. Hence S is the union of the $\mathcal{D}$-saturated E-unitary inverse subsemigroups H_e. It is easy to show that the homomorphism ϕ in the proof of the theorem is injective in this case. Hence S is a subdirect product of the H_e with zero added possibly. From this one can deduce, again by elementary means, that S is a semilattice of groups with the multiplication defined by linking homomorphisms. Thus, modulo some elementary results, our theory restricts to Clifford's classic theorems.

Now let E be the semilattice $\{e, f, g\}$ where $e > g, f > g$, and e, f are incomparable. Let S be the semilattice of groups $G_e \cup G_f \cup G_g$ where G_e, G_g are trivial and G_f is the cyclic 2-group; let T be the semilattice of groups $H_e \cup H_f \cup H_g$

where H_g is the trivial group and H_e, H_f are copies of the cyclic 2-group (the multiplications being defined in the obvious way). Consider the word $w = eabc$ in the free inverse product P of S and T, where e is the identity element of G_e and $a[b, c]$ is the non-identity element of $H_e[G_f, H_f]$. If ψ is a semilattice of groups homomorphism on P, it is easily seen that $w\psi$ is an idempotent. On the other hand one can find a representation of S and T in $\mathscr{I}_5$, the symmetric inverse semigroup on five symbols, in which the image of w is not an idempotent. Hence w is not an idempotent, so that P is not strongly E-reflexive.

On the other hand McAlister [3] has shown that the free inverse product of two E-unitary inverse semigroups is again E-unitary.

References

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vols. 1, 2, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961 and 1967.

2. J. M. Howie, *An introduction to semigroup theory*, Academic Press, London, 1976.

3. D. B. McAlister, *Inverse semigroups generated by a pair of subgroups*, Proc. Roy. Soc. Edinburgh Sect. A. **77** (1977), 9–22.

4. L. O'Carroll, *Strongly E-reflexive inverse semigroups*, Proc. Edinburgh Math. Soc. **20** (1976–77), 339–354.

5. ______, *Strongly E-reflexive inverse semigroups*. II, Proc. Edinburgh Math. Soc. **21** (1978), 1–10.

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