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A note on superoscillations associated with Bessel beams

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Abstract
Waves involving Bessel functions can oscillate faster than their band-limited Fourier transforms suggest, with the superoscillations being fastest near phase singularities. Different waves representing a 'flyby' close to a phase singularity are analysed. These can superoscillate similarly, despite being differently normalized, or not normalizable at all.

Keywords: vortices, weak values, singularities

1. Introduction

The Bessel wave

$$\psi(\boldsymbol{r};l) = J_l(r)\exp(\mathrm{i}l\phi) \qquad (\boldsymbol{r} = \{x,y\} = r\{\cos\phi,\sin\phi\}), \tag{1}$$

with an optical vortex (phase singularity) of strength l at the origin $r = 0$, can represent an exact solution of the free-space Helmholtz equation in the $\boldsymbol{r}$ plane, with wavenumber $k = 1$, that is wavelength $\lambda = 2\pi$ (or equivalently, a general wavenumber $k = 2\pi/(\text{wavelength }\lambda)$ with distances measured in units of $\lambda/2\pi$). Alternatively, it can represent a plane section of a Bessel beam propagating in the z direction in three dimensions, with wavenumber k_0 and additional phase factor $\exp(\mathrm{i}z\sqrt{k_0^2/k^2-1})$. For these Bessel waves, the vortex strength l also represents the angular momentum, though for general waves the two concepts are unrelated [1].

As I discussed briefly elsewhere [2], the fact that in the vicinity of the vortex ψ oscillates arbitrarily fast, and therefore faster than the wavelength $\lambda = 2\pi$, means that this wave is an example of a superoscillatory function, that is, a band-limited function varying on scales smaller than its largest Fourier component (here $k = 1$): it is 'faster than Fourier' [3–8]. And the fact that J_l vanishes at the origin as r^l illustrates the general phenomenon that functions are exponentially weak (here as a function of l) where they superoscillate.

My aim here is to explore and illustrate this connection in a little more detail. In a sense this study of superoscillation near individual vortices is complementary to statistical analyses of random waves [9, 10], which showed that superoscillation (in a sense to be described in section 2) is unexpectedly common: in the plane, one-third of the area is superoscillatory.

2. Superoscillations in the plane

A characterization of the superoscillatory behaviour of ψ is the local wavevector, that is, the local phase gradient, equal to the quantum weak value [11, 12] of the momentum operator $\hat{\boldsymbol{k}} \equiv -\mathrm{i}\nabla$ (see section 2.1 of [7]). For (1) this is particularly simple—just the azimuthal (vortex) flow

$$\boldsymbol{k}_\mathrm{w}(\boldsymbol{r}) = \nabla\arg\psi = \frac{\mathrm{Re}\,\psi^*\hat{\boldsymbol{k}}\psi}{|\psi|^2} = \frac{1}{|\psi|^2}\langle\psi|\frac{1}{2}(\hat{\boldsymbol{k}}\delta(\hat{\boldsymbol{r}}-\boldsymbol{r}) + \delta(\hat{\boldsymbol{r}}-\boldsymbol{r})\hat{\boldsymbol{k}})|\psi\rangle = \frac{l}{r}\boldsymbol{e}_\phi. \tag{2}$$

Superoscillations correspond to $|\boldsymbol{k}_\mathrm{w}| > 1$, that is $r < l$. This radius corresponds to the crossover of the real function $J_l(r)$ from exponentially increasing to oscillatory. For $r > l$ the wave is not superoscillatory, and the oscillations of $J_l(r)$ are slower than 1, tending asymptotically, that is for $r \gg l$, to being proportional to $\cos(r - l\pi/2 - \pi/4)$ [13], that is, varying on the scale of the wavelength.

It is worth remarking that for the analogue of (1) representing a wave from a source, that is, the Hankel wave

$$\psi_\mathrm{H}(\boldsymbol{r};l) = H_l^{(1)}(r)\exp(\mathrm{i}l\phi), \tag{3}$$

the local wavevector contains a radial component:

$$\boldsymbol{k}_\mathrm{w}(\boldsymbol{r}) = \frac{l}{r}\boldsymbol{e}_\phi + \mathrm{Im}\,\partial_r\log H_l^{(1)}(r)\boldsymbol{e}_r. \tag{4}$$

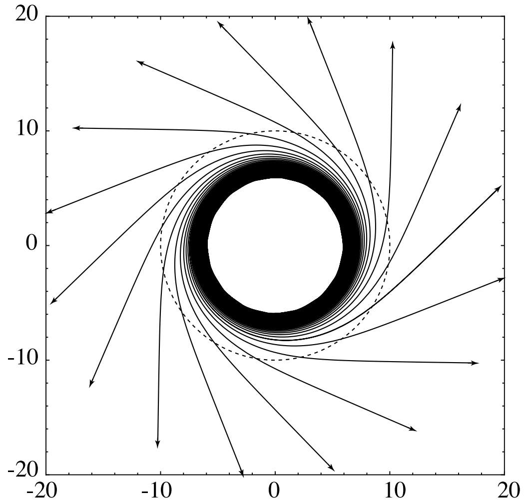

Figure 1. Lines of the wavevector (4) for Hankel wave (3) with $l = 10$ and, showing tight spiralling inside the circle $r = l$ (dashed), radiating outwards for $r > r_c$ with the lines asymptotically tangent to the circle $r = l$.

This represents slow spiralling out from the origin (figure 1), near which the azimuthal component dominates, the distance between successive windings being $O(r^3)$ [14]. There is a rapid but smooth transition near the circle $r = l$, from slow to fast outward spiralling, asymptotically (i.e. for $r \gg l$) tangential to the circle l, with the radial component dominating. (For a sink, the Bessel function is $H_l^{(2)}(r)$ and the spiralling is inwards.)

Returning to the wave (1), this has no zeros in the superoscillatory region $r < l$ except the strength l vortex at the origin. For $r > l$ there are the non-generic circular nodal lines at the zeros of J_l. However, ψ can be made generic by the simple perturbation

$$\psi(\boldsymbol{r}; l, \varepsilon) = J_l(r)\exp(\mathrm{i}l\phi) + \varepsilon J_0(kr). \tag{5}$$

This splits the zero at $r = 0$ into a ring of l strength 1 vortices, which for small ε are located near

$$r = 2(\varepsilon l!)^{1/l}, \qquad \phi = \frac{(2n-1)\pi}{l} \qquad (1 \le n \le l). \tag{6}$$

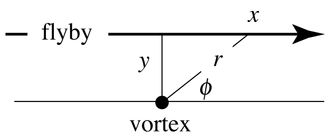

Figure 3. Geometry and coordinates for one-dimensional flyby.

The superoscillatory phase behaviour of (1) is thus converted into sub-wavelength intensity variations; figure 2 illustrates how rich a structure there can be within one square wavelength.

3. Superoscillations in a one-dimensional flyby

In standard weak measurement theory [11, 12, 15], superoscillations occur as functions of a single variable, and several recipes are known [3, 16]. The Bessel wave (1) can provide another example, by regarding it as a function of x for fixed y, corresponding to flyby of the vortex (figure 3):

$$\psi(x; y, l) = \exp\left(\mathrm{i}\, l \arctan\frac{y}{x}\right) J_l\left(\sqrt{x^2+y^2}\right). \tag{7}$$

The weak momentum (local wavenumber) is

$$k(x) = \partial_x \arg\psi = \partial_x l \arctan\frac{y}{x} = -\frac{ly}{x^2+y^2}. \tag{8}$$

Which is superoscillatory where $|k(x)| > 1$, that is

$$|x| < \sqrt{y(l-y)}. \tag{9}$$

This flyby interval lies within the circle with radius l. Of course we must choose $y < l$.

The strongest superoscillations are near $x = 0$, where $k(0) = l/y$. These oscillations are faster than Fourier by the factor l/y. Figure 4 illustrates the crossover between the superoscillatory behaviour for small x and the 'normal' oscillations in the region $\sqrt{x^2+y^2} > l$ where the Bessel function oscillates.

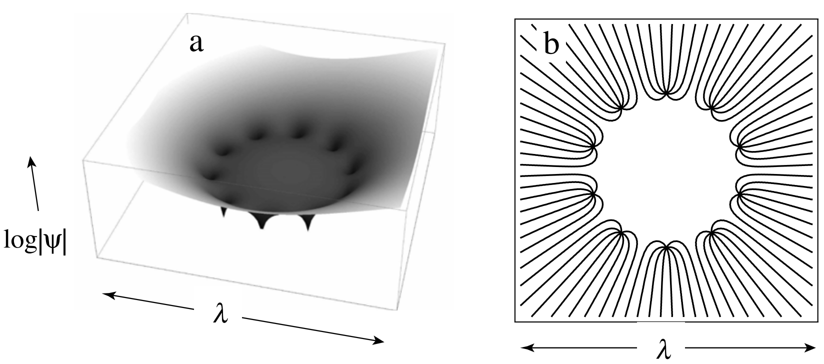

Figure 2. Perturbed wave (5) for $l = 10$ and $\varepsilon = 10^{-7}$, plotted for one square wavelength centred on $\boldsymbol{r} = 0$ for (a) $\log|\psi(\boldsymbol{r}; l, \varepsilon)|$ and (b) wavefronts $\arg\psi(\boldsymbol{r}; l, \varepsilon)$ at intervals of $\pi/4$. The radius in (6) is $r = 1.807 = 0.288\lambda$.

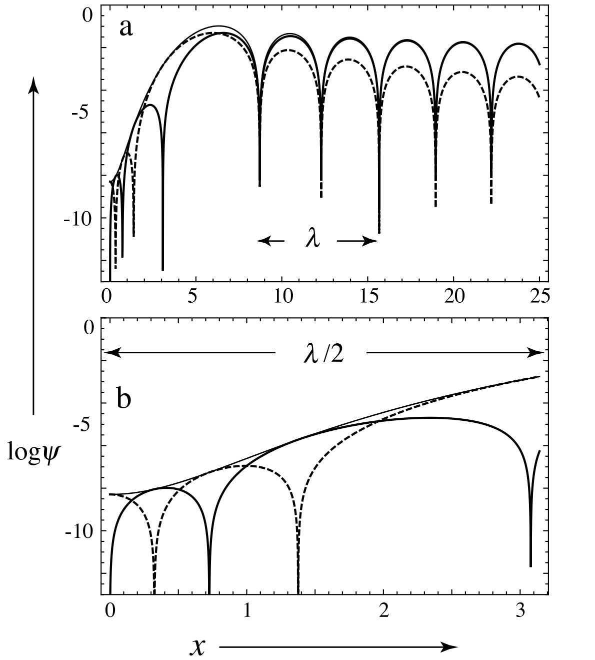

Figure 4. Flyby function (7) for $l = 5$, $y = 1$, so the superoscillations are faster than Fourier by a factor 5. Thin curves: $\log|\psi|$; thick curves: $\log|\mathrm{Re}\,\psi|$; dashed curves: $\log|\mathrm{Im}\,\psi|$.
(a) Shows the crossover between the superoscillations for small x and the Bessel oscillations, with wavelength $\lambda = 2\pi$, expected on the basis of the Fourier content of ψ. (b) Magnifies the superoscillatory region.

In the superoscillatory range, ψ as given by (7) has the asymptotic behaviour

$$\psi(x; y, l) \approx A(y, l) \exp\left(-\mathrm{i}\frac{lx}{y}\right) \times \exp\left(\frac{lx^2}{2y^2} + \mathrm{i}\frac{x^3}{3y^3}\right) \qquad (x < y), \tag{10}$$

where

$$A(l, y) = \mathrm{i}^l J_l(y) \approx \frac{1}{\sqrt{2\pi l}}\left(\frac{\mathrm{i}ey}{2l}\right)^l \qquad (l \gg 1,\ y \ll l). \tag{11}$$

This example exhibits the phenomenon, familiar from other superoscillatory functions [3, 16], of the function increasing antigaussianly away from the superoscillatory region, with the superoscillations gradually slowing; see figure 5.

4. Normalizing the flyby

The flyby function (7) is band-limited, because, from a standard integral representation of the Bessel function [13],

$$\psi(x; y, l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\ \exp\{\mathrm{i}(x\sin\theta + y\cos\theta - l\theta)\} = \frac{1}{2\pi}\int_{-1}^{1} \mathrm{d}q\, a(q)\exp(\mathrm{i}qx) \tag{12}$$

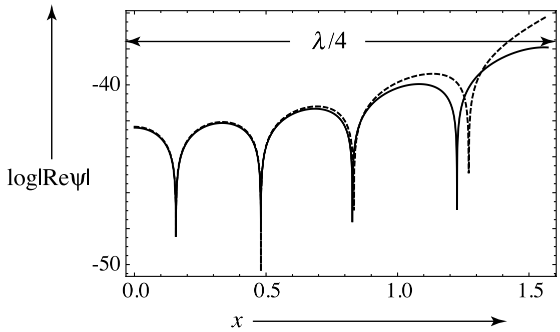

Figure 5. Superoscillatory region of flyby function, for $l = 20$, $y = 2$, so the superoscillations are faster than Fourier by a factor 10. Full curve, $\log|\mathrm{Re}\,\psi|$ for exact function (7); dashed curve: approximation (10).

where

$$a(q) = \frac{\exp(-\mathrm{i}l\sin^{-1}q)}{\sqrt{1-q^2}}\left[\exp\left(\mathrm{i}y\sqrt{1-q^2}\right) + (-1)^l\exp\left(-\mathrm{i}y\sqrt{1-q^2}\right)\right]. \tag{13}$$

But the flyby function is not square-integrable. One way to see this is from the large-argument asymptotics of the Bessel function [13], giving

$$|\psi(|x|; y, l)|^2 \approx \frac{2}{\pi|x|}\cos^2\left(x - \frac{1}{2}l\pi - \frac{1}{4}\pi\right) \qquad (|x| \gg l). \tag{14}$$

This implies that the normalization integral diverges logarithmically. The same conclusion follows from the integral of $|a(q)|^2$, which diverges at $q = \pm 1$ because of the denominator $\sqrt{1-q^2}$ in (13).

However, it is easy to create flyby functions from (7) that are square-integrable (this is desirable if the superoscillatory function is to be a model for a quantum wavefunction). One way is simply to take the imaginary part, because, from the decay of the phase factor in (7),

$$(\mathrm{Im}\,\psi(|x|; y, l))^2 \approx \frac{2l^2y^2}{\pi|x^3|}\cos^2\left(x - \frac{1}{2}l\pi - \frac{1}{4}\pi\right) \qquad (k|x| \gg l), \tag{15}$$

for which the normalization integral converges. In the Fourier representation, the counterpart of (12) is

$$\mathrm{Im}\,\psi(x; y, l) = \frac{1}{2\pi}\int_{-1}^{1}\mathrm{d}q\, a_{\mathrm{Im}}(q)\exp(\mathrm{i}qx), \tag{16}$$

where

$$a_{\mathrm{Im}}(q) = \frac{1}{2\mathrm{i}}(a(q) - a^*(-q)) = \frac{\sin(y\sqrt{1-q^2})}{\sqrt{1-q^2}}[\exp\{-\mathrm{i}l\sin^{-1}q\} - (-1)^l\exp\{\mathrm{i}l\sin^{-1}q\}], \tag{17}$$

cancelling the singularity at $q = \pm 1$. The faster decay of Im ψ is evident in figure 4(a).

An alternative approach, suggested by Professor Aharonov [17] is to take the derivative with respect to y:

$$\begin{aligned}\partial_y \psi(x; y, l) &= \exp\left(\mathrm{i}l \arctan\frac{y}{x}\right)\\ &\times\left(\frac{\mathrm{i}lx}{x^2+y^2}J_l\left(\sqrt{x^2+y^2}\right)+\frac{y}{\sqrt{x^2+y^2}}J_l'\left(\sqrt{x^2+y^2}\right)\right)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\,\cos\theta\,\exp\{\mathrm{i}(x\sin\theta + y\cos\theta - l\theta)\}\\ &= \frac{1}{2\pi}\int_{-1}^{1}\mathrm{d}q\sqrt{1-q^2}a(q)\exp(\mathrm{i}qx),\end{aligned} \tag{18}$$

from which it is clear that the derivative has killed the divergence, making the function normalizable. In fact, from Bessel asymptotics,

$$|\partial_y \psi(x; y, l)|^2 \sim 1/|x|^3 \qquad \text{as } |x| \to \infty, \tag{19}$$

so the convergence of the normalization integral for $\partial_y \psi$ is the same as that for Im ψ.

There has been some discussion [5, 6] of how to characterize or optimize the degree of superoscillation in functions ψ such as those considered here, based on comparing the value of ψ where it superoscillates with its inevitably much larger values elsewhere. For periodic superoscillatory functions, it is natural to define the 'superoscillation yield' as the ratio of the integral of $|\psi|^2$ over the superoscillation region to its integral over the period [5]. And at first sight a similar definition would seem natural for ψ defined over the whole real line $-\infty < x < \infty$. But it is not natural, because the value of $\int_{-\infty}^{\infty}\mathrm{d}x\,|\psi(x)|^2$ is dominated by the decay as $|x| \to \infty$, which can be different for two functions with essentially the same superoscillatory behaviour rising to essentially the same largest values.

To illustrate this, we can compare superoscillatory functions scaled so that the superoscillations near $x = 0$ are of order unity, that is

$$\psi_{\mathrm{sc}}(x; y, l) = \exp\left(\mathrm{i}\,l\arctan\frac{y}{x}\right)\frac{J_l(\sqrt{x^2+y^2})}{J_l(y)}, \tag{20}$$

and

$$\begin{aligned}\partial_y\psi_{\mathrm{sc}}(x; y, l) &\equiv \exp\left(\mathrm{i}l\arctan\frac{y}{x}\right)\\ &\times\left(\frac{\frac{\mathrm{i}lx}{x^2+y^2}J_l(\sqrt{x^2+y^2})+\frac{y}{\sqrt{x^2+y^2}}J_l'(\sqrt{x^2+y^2})}{J_l'(y)}\right).\end{aligned} \tag{21}$$

Figure 6 shows $|\psi_{\mathrm{sc}}|$, which is not normalizable, Im ψ_{sc}, which is, and Re $\partial_y\psi_{\mathrm{sc}}$, which is also normalizable but has a very different normalization integral. All three functions superoscillate similarly and rise to largest similar values near $x = \sqrt{l^2 - y^2}$ where the Bessel functions change from oscillatory to exponential, so it seems that their eventual decay is irrelevant to their superoscillatory behaviour. For such functions, defined on the whole real line, it might be preferable to define the degree of superoscillation differently, in terms of the ratio of the largest values to the superoscillatory values.

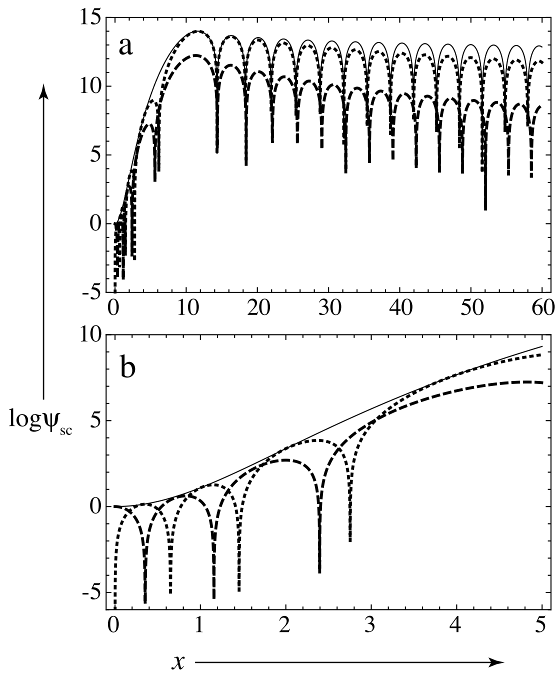

Figure 6. Scaled flyby functions (20) and (21) for $l = 10$, $y = 2$. Thin curve, $\log|\psi_{\mathrm{sc}}|$; dotted curve, $\log|\mathrm{Im}\,\psi_{\mathrm{sc}}|$; dashed curve, $\log|\mathrm{Re}\,\partial_y\psi_{\mathrm{sc}}|$. (a) Over a long range, showing the functions rising to large values and then decaying; (b) magnification in the superoscillatory region.

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