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A note on supplementary difference sets

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Abstract

Let S_1, S_2,⋯, S_n be subsets of G, a finite abelian group of order v, containing k_1, k_2,...,kn elements respectively. Write T_i for the totality of all differences between elements of S_i (with repetitions), and T for the totality of elements of all the T_i. We will denote this by T= T_1 & T_2 & ... & T_n. If T contains each non-zero element of G a fixed number of times, lambda say, then the sets S_1, S_2, ..., S_n will be called n-{v; k_1, k_2, ..., k_n ; lambda} supplementary difference sets.

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A Note on Supplementary Difference Sets

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Let $S_1, S_2, \ldots, S_n$ be subsets of G, a finite abelian group of order v, containing $k_1, k_2, \ldots, k_n$ elements respectively. Write T_i for the totality of all differences between elements of S_i (with repetitions), and T for the totality of elements of all the T_i. We will denote this by $T = T_1$ & T_2 &... & T_n. If T contains each non-zero element of G a fixed number of times, λ say, then the sets $S_1, S_2, \ldots, S_n$ will be called $n-\{v; k_1, k_2, \ldots, k_n; \lambda\}$ *supplementary difference sets.*

If $k_1 = k_2 = \cdots = k_n = k$ we will write $n-\{v; k; \lambda\}$ to denote the supplementary difference sets. If $k_1 = k_2 = \cdots = k_i$, $k_{i+1} = k_{i+2} = \cdots = k_{i+j}, \ldots, k_l = \cdots = k_n$ then sometimes we write $n-\{v; i{:}k_1, j{:}k_{i+1}, \ldots; \lambda\}$. It can be easily seen by counting the differences that the parameters of $n-\{v; k_1, k_2, \ldots, k_n; \lambda\}$ supplementary difference sets satisfy

$$\lambda(v-1) = \sum_{j=1}^{n} k_j(k_j - 1).$$

We use braces, { }, to denote sets and square brackets, [ ], to denote collections where repetitions may remain.

We now let $v = 4r(2\lambda+1)+1 = p^\gamma$, where p is a prime and further let

$$H_i = \{x^{4rj+i} : 0 \leqslant j \leqslant 2\lambda\}, \qquad i = 0, 1, \ldots, 4r-1$$

with x a primitive element of $GF(v)$. Write

$$L = H_{2i_1} \cup H_{2i_2} \cup \cdots \cup H_{2i_m}$$

for some m, $0 < m < 2r$, where the i_j are distinct integers. Now we consider the differences between elements of H_{2i}, that is, the collection

$$\begin{aligned} &[x^{4rj+2i} - x^{4rl+2i} : j \neq l, 0 \leqslant j, l \leqslant 2\lambda] \qquad (1)\\ &= \{x^{4rj+2i} : 0 \leqslant j \leqslant 2\lambda\} \text{ times } [1 - x^{4r(l-j)} : l \neq j, 0 \leqslant l \leqslant 2\lambda]\\ &= H_{2i} \text{ times } [1 - x^{4r(l-j)} : l \neq j, 0 \leqslant l \leqslant 2\lambda] \end{aligned}$$

and, since any element of a group multiplied onto a coset gives a coset, this expression must represent cosets with certain multiplicities, say b_k, write

$$= \mathop{\&}_{k=0}^{4r-1} b_k H_k, \qquad (2)$$

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where, since H_{2i} has $2\lambda+1$ elements, the number of elements in (1) is $2\lambda(2\lambda+1)$ and the number of elements in (2) is $\sum_{k=0}^{4r-1} b_k(2\lambda+1)$. So

$$\sum_{k=0}^{4r-1} b_k = 2\lambda .$$

Now $2\lambda+1$ is odd, so $-1 \in H_{2r}$. Then if $x^a - x^b$ appears in (1) so does $x^b - x^a$. Thus whenever an element y occurs so does $-y$ and $y \in H_c \Rightarrow -y \in H_{c+2r}$. Thus $b_k = b_{k+2r}$.

The differences between elements of H_{2i} and H_{2k} are given by the collection

$$[x^{4rj+2i} - x^{4rl+2k} : 0 \leqslant j, l \leqslant 2\lambda] \tag{3}$$
$$= \{x^{4rj+2i} : 0 \leqslant j \leqslant 2\lambda\} \text{ times } [1 - x^{4r(l-j)+2(k-i)} : 0 \leqslant l \leqslant 2\lambda]$$
$$= H_{2i} \text{ times } [1 - x^{4r(l-j)+2(k-i)} : 0 \leqslant l \leqslant 2\lambda]$$
$$= \mathop{\&}_{n=0}^{4r-1} c_n H_n \tag{4}$$

where c_n give the multiplicities. By the same reasoning as before,

$$\sum_{n=0}^{4r-1} c_n = 2\lambda + 1 .$$

Now consider the differences from L, that is

$$[\text{differences from } H_{2i_j} : j = 1, 2, \ldots, m] \tag{5}$$
$$\& [\text{differences from } H_{2i_j} - H_{2i_k} : i_j \neq i_k, 0 \leqslant i_j, i_k \leqslant m] \tag{6}$$
$$= \mathop{\&}_{k=0}^{4r-1} a_k H_k \qquad \text{using (2) and (4)}. \tag{7}$$

Counting elements we see (5) and (6) have $m(2\lambda+1)\ (m(2\lambda+1)-1)$ and (7) has $(2\lambda+1)\sum_{k=0}^{4r-1} a_k$ elements. Hence

$$\sum_{k=0}^{4r-1} a_k = m\big(m(2\lambda + 1) - 1\big).$$

Finally, we note that in (6) if $H_a - H_b$ occurs so does $H_b - H_a$ so if y occurs so does $-y$ and as before we see that

$$a_k = a_{k+2r} . \tag{8}$$

Write

$$\left.\begin{aligned} w &= \sum_{k=0}^{r-1} a_{2k} - \sum_{k=0}^{r-1} a_{2k+1} \\ z &= (w, w+m), \quad s = |w+m|/z, \quad t = |w|/z . \end{aligned}\right\} \tag{9}$$

We now show, using L to construct sets of size $m(2\lambda+1)$ and $m(2\lambda+1)+1$, how to find some supplementary difference sets.

THEOREM 1. *Let* $v=4r(2\lambda+1)+1=p^{\gamma}$, *where* p *is a prime and* $r=2^{\delta}$. *Then* s *copies of each of*

$$L_j = x^{2j}L, \quad j = 0, 1, \dots, r-1,$$

and t *copies of each of*

$$K_j = 0 \cup x^{2j+i}L, \quad j = 0, 1, \dots, r-1,$$

where s, t *and* w *are given by* (9), $i=0$ *if* (w *is negative and* $m>-w$), $i=1$ *otherwise, are*

$$r(s+t) - \{4r(2\lambda+1)+1;\, rt\colon m(2\lambda+1)+1;\, rs\colon m(2\lambda+1);\, \varphi\tfrac{1}{4}[m^2(2\lambda+1)(t+s)+m(t-s)]\}$$

supplementary difference sets.

Proof. Since $2\lambda+1$ is always odd, $-1 \in H_{2r}$, we have from (8) $a_k = a_{k+2r}$. The totality of differences from

$$L_j = H_{2i_1+2j} \cup H_{2i_2+2j} \cup \cdots \cup H_{2i_m+2j}$$

is x^{2j} times the totality of differences from L_0 or

$$\mathop{\&}_{k=0}^{4r-1} a_{4r-2j+k} H_k = \mathop{\&}_{k=0}^{2r-1} a_{2r-2j+k}(H_k \,\&\, H_{k+2r}).$$

So by taking all the differences from L_j, $j=0, 1, \dots, r-1$ we have

$$\begin{aligned} X &= \mathop{\&}_{i=0}^{2r-1} \left\{ \left(\sum_{k=0}^{r-1} a_{2k} \right) H_{2i} \,\&\, \left(\sum_{k=0}^{r-1} a_{2k+1} \right) H_{2i+1} \right\} \\ &= \mathop{\&}_{i=0}^{2r-1} (\alpha H_{2i} \,\&\, \beta H_{2i+1}). \end{aligned}$$

The totality of differences, then, from the sets

$$K_j = 0 \cup H_{2i_1+2j+1} \cup H_{2i_2+2j+1} \cup \cdots \cup H_{2j_m+2j+1}, \quad j = 0, 1, \dots, r-1,$$

is

$$Z = \mathop{\&}_{i=0}^{2r-1} (\beta H_{2i} \,\&\, (\alpha+m) H_{2i+1}).$$

There are four cases to consider:

(i) $\alpha \geqslant \beta$ and $\beta \geqslant \alpha + m$, which is impossible;

(ii) $\alpha \leqslant \beta$ and $\beta \leqslant \alpha + m$. Here $w = \alpha - \beta$ is negative and $m > \beta - \alpha = -w$.

So, if instead of the sets K_j we use the totality of differences from the sets $0 \cup L_j$, then we have the differences

$$Y = \mathop{\&}_{i=0}^{2r-1} \left((\alpha + m)\, H_{2i} \;\&\; \beta H_{2i+1a}\right).$$

Now s times X plus t times Y (where s and t are defined in (9)) gives $(\beta m/z)\ G$;

(iii) $\alpha < \beta$ and $\beta \geqslant \alpha + m$; and

(iv) $\alpha > \beta$ and $\beta \leqslant \alpha + m$.

In these last two cases s times X and t times Z gives

$$((\beta^2 - \alpha^2 - \alpha m)/z)\, G \quad \text{and} \quad ((\alpha^2 + \alpha m - \beta^2)/z)\, G$$

respectively.

Then, noting that by summing the elements of X in two ways we find $\alpha + \beta = \frac{1}{2} m [m(2\lambda + 1) - 1]$, we have the result of the theorem.

EXAMPLE. With $v = 41$, $r = 2$, $\lambda = 2$, and $m = 3$, $w = 1$, $s = 2$, $t = 1$ we find $6 - \{41; 2{:}16, 4{:}15; 33\}$ supplementary difference sets.

In the theorem the initial set L has been left reasonable undecided but if we choose another initial set.

$$M_j = H_{2j_1 + 2j} \cup H_{2j_2 + 2j} \cup \cdots \cup H_{2j_m + 2j} \qquad j = 0, 1, \ldots, r - 1$$

where all the j_a are distinct, we may get a different set of supplementary difference sets.

For example: with $v = 41$, $r = 2$, $\lambda = 2$, with $m = 2$ and the initial set $H_0 \cup H_2$ we get $w = 1$, $s = 3$, $t = 1$ and hence $8 - \{41; 2{:}11, 6{:}10; 19\}$ supplementary difference sets, while with the initial set $H_0 \cup H_4$ we get $w = -3$, $s = 1$, $t = 3$ and hence $8 - \{41; 6{:}11, 2{:}10; 21\}$ supplementary difference sets.

Finally we note that *balanced incomplete block designs* may be obtained from supplementary difference sets with two k values by using the results of Jennifer Wallis [2].

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