

## A NOTE ON TASKINEN'S COUNTEREXAMPLES ON THE PROBLEM OF TOPOLOGIES OF GROTHENDIECK

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By the work of Taskinen (see [4, 5]), we know that there is a Fréchet space  $E$  such that  $L_b(E, l_2)$  is not a  $(DF)$ -space. Moreover there is a Fréchet–Montel space  $F$  such that  $F'_b \hat{\otimes}_\varepsilon F'_b$  is not  $(DF)$ . In this second example, the duality theorem of Buchwalter (cf. [2, §45.3]) can be applied to obtain that  $F'_b \hat{\otimes}_\varepsilon F'_b \cong (F \hat{\otimes}_\pi F)'_{co}$  and hence  $F'_b \hat{\otimes}_\varepsilon F'_b$  is a  $(gDF)$ -space (cf. [1, Ch. 12 or 3, Ch. 8]). The  $(gDF)$ -spaces were introduced by several authors to extend the  $(DF)$ -spaces of Grothendieck and to provide an adequate frame to consider strict topologies.

It seemed to be open whether, for Fréchet spaces  $E$  and  $F$ , the spaces  $L_b(E, F'_b)$  and  $E'_b \hat{\otimes}_\varepsilon F'_b$  were  $(gDF)$ . In this short note we observe that the constructions of Taskinen in [4] can be adapted to give a negative answer to these two questions.

Our notations are standard and can be seen in [1, 2, 3].

Let  $X$  be a Fréchet space with a basis of absolutely convex 0-neighbourhoods  $(U_n)$  with  $2U_{n+1} \subseteq U_n$ . We denote by  $B(X)$  the set of all bounded subsets of  $X$ . Let  $A$  be a saturated subset of  $B(X)$  whose union covers  $X$ . We denote by  $\tau_A$  the topology on  $X'$  of uniform convergence on the elements of  $A$ . Using the characterizations of  $(gDF)$ -spaces and a standard argument by polarization, one gets

**Lemma 1.** *Let  $X$  be a Fréchet space. Then  $(X', \tau_A)$  is  $(gDF)$  if and only if for every sequence of absolutely convex subsets  $C_n \in A$ ,  $n \in \mathbb{N}$ , there is  $C \in A$  such that  $\cup(C_n \cap U_n; n \in \mathbb{N}) \subseteq C$ .*

Let  $E$  and  $F$  be Fréchet spaces with decreasing sequences of 0-neighbourhoods  $(W_n)$  and  $(V_n)$  such that  $(n^{-1}W_n)$  and  $(n^{-1}V_n)$  are basis of 0-neighbourhoods in  $E$  and  $F$  respectively. Then  $L_b(E, F'_b)$  is topologically isomorphic to the dual  $(E \hat{\otimes}_\pi F)'$  endowed with the topology of uniform convergence on the sets  $\overline{\Gamma(A \otimes B)}$ ,  $A \in B(E)$ ,  $B \in B(F)$ . Then we have:

**Corollary 2.**  *$L_b(E, F'_b)$  is  $(gDF)$  if and only if for every  $(B_n) \subseteq B(E)$ ,  $(C_n) \subseteq B(F)$ , there are  $A \in B(E)$ ,  $B \in B(F)$  such that  $\cup(\overline{\Gamma(B_n \otimes C_n)} \cap \overline{\Gamma(2^{-n}W_n \otimes V_n)}; n \in \mathbb{N}) \subseteq \overline{\Gamma(A \otimes B)}$ , all the closures taken in  $E \hat{\otimes}_\pi F$ .*

Observe that by symmetry the condition above is also equivalent to  $L_b(F, E'_b)$  being  $(gDF)$ .

We will use the notations of [4, §4] and we will take  $M_n = M'_n = l_2^n$ . Hence we can

suppose that the projection constants  $\rho(n)$  from  $G_n$  onto  $M_n$  satisfy

$$\sqrt{\frac{2n}{\pi}} < \rho(n) < 2\sqrt{n} \quad (\text{see [4, p. 22]}).$$

**Example 3.** *There is a Fréchet space  $E$  such that  $L_b(E, l_2)$  is not (gDF).*

This will be a consequence of Corollary 2, once we prove that for  $F=l_2$ ,  $V$  the unit ball of  $F$ , and  $E$  the Fréchet space of Taskinen [4, §4.4], we have that the set

$$\cup(\overline{\Gamma(2^{-n}W_n \otimes V)} \cap \overline{\Gamma(W_{n+1} \otimes 2^{4n}V)}): n \in \mathbb{N}$$

is not contained in  $\overline{\Gamma(A \otimes B)}$  for all  $A \in B(E)$ ,  $B \in B(F)$ , the closures taken in  $E \hat{\otimes}_\pi F$  (observe that  $W_{n+1} \in B(E)$ ).

To prove this it is enough to see that for every  $t_m \geq 1$ ,  $m \in \mathbb{N}$ , the set

$$\cup(\Gamma(2^{-n}W_n \otimes V) \cap \Gamma(2^{4n}W_{n+1} \otimes V)): n \in \mathbb{N}$$

is not contained in the closure of  $\Gamma((\cap(t_m W_m: m \in \mathbb{N})) \otimes V)$  in  $E \otimes_\pi F$ . First observe that the key Lemma 4.3 of [4] yields for  $s(n) := \rho(n)/4$  that  $\Gamma(U \otimes V) \cap \Gamma(2^{5n}\tilde{U} \otimes V) \not\subseteq \Gamma((s(2^{5n})U \cap 4s\tilde{U}) \otimes V)$  for all  $n \in \mathbb{N}$  and  $s > 0$ . Then

$$\Gamma(2^{-n}U \otimes V) \cap \Gamma(2^{4n}\tilde{U} \otimes V) \not\subseteq \Gamma(((s(2^{5n})/2^n)U \cap 2^{-n+2}s\tilde{U}) \otimes V)$$

for all  $n \in \mathbb{N}$  and  $s > 0$ . Since  $\lim \rho(n)/4n^{1/5} = \infty$ , there is  $n_0 \in \mathbb{N}$  such that  $s(2^{5n_0})/2^{n_0} \geq 2t_1$ . Then  $\Gamma(2^{-n_0}U \otimes V) \cap \Gamma(2^{4n_0}\tilde{U} \otimes V)$  is not contained in  $2\Gamma((t_1 U \cap t_{n_0+1}\tilde{U}) \otimes V)$ . Since

$$(\Gamma((\cap(t_m W_m: m \in \mathbb{N})) \otimes V) + \Gamma(W_{n_0+1} \otimes V)) \cap (E_{n_0} \otimes F) \subseteq 2\Gamma((t_1 U \cap t_{n_0+1}\tilde{U}) \otimes V)$$

([4; proof of 4.5]),  $\Gamma(W_{n_0+1} \otimes V)$  is a 0-neighbourhood in  $E \otimes_\pi F$ , and

$$[\Gamma[2^{-n_0}W_{n_0} \otimes V) \cap \Gamma(2^{4n_0}W_{n_0+1} \otimes V)] \cap (E_{n_0} \otimes F) = \Gamma(2^{-n_0}U \otimes V) \cap \Gamma(2^{4n_0}\tilde{U} \otimes V)$$

the conclusion follows.

**Example 4.** *There is a Fréchet–Schwartz space  $F$  with the bounded approximation property and a Fréchet space  $E$  such that  $F'_b \hat{\otimes}_\varepsilon E'_b \cong L_b(F, E'_b)$  is not (gDF).*

$F$  is the Fréchet–Schwartz and  $E$  is the Fréchet space constructed as in [4, §4.7]. In this case the proof is a little more involved. Using now the second key Lemma 4.8 in [4] one can prove that

$$\cup(\Gamma(2^{-n}W_n \otimes V_n) \cap (\cap(\Gamma(2^{4n}W_p^n \otimes V_p): p \geq n+1))): n \in \mathbb{N}$$

is not contained in the closure of  $\Gamma(A \otimes B)$  in  $E \otimes_\pi F$  for  $A \in B(E)$  and  $B \in B(F)$ .

Now since  $W_p = \tilde{U}$  for  $p \geq n+1$ , we can apply the positive result [4, §3.1] for  $(H, \tilde{h})$

and  $F$  to obtain for each  $n \in \mathbb{N}$ ,  $C_n \in B(F)$  such that

$$\cap(\Gamma(2^{4n}W_p^n \otimes V_p): p \geq n+1) \subseteq \overline{\Gamma(W_{n+1}^n \otimes C_n)}$$

the closure taken in  $E \hat{\otimes}_\pi F$ . Consequently, for  $(W_{n+1}^n) \subseteq B(E)$ ,  $(C_n) \subseteq B(F)$  the set

$$\cup(\overline{(\Gamma(2^{-n}W_n \otimes V_n) \cap \Gamma(W_{n+1}^n \otimes C_n))}: n \in \mathbb{N})$$

is not contained in  $\overline{\Gamma(A \otimes B)}$ , the closure taken in  $E \hat{\otimes}_\pi F$ . The conclusion follows from Corollary 2. □

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