A NOTE ON THE ASYMPTOTIC DISTRIBUTIONS OF UNIT ROOT TESTS IN THE ADDITIVE OUTLIER MODEL WITH BREAKS*

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Resumo

Este artigo discute testes para uma raíz unitária permitindo a possibilidade de uma quebra única no intercepto e/ou na inclinação da função de tendência do modelo de outlier aditivo discutido em Perron (1989). Detectamos e corrigimos um erro na função de distribuição assintótica do teste proposto neste caso. A modificação feita nos permite construir uma estatística com a mesma distribuição assintótica da encontrada em Perron (1989). Discutimos, também, a propriedade de aproximações assintóticas e várias extensões onde o ponto de quebra é desconhecido.

Abstract

This note discusses tests for a unit root allowing the possibility of a one-time change in the intercept and/or the slope of the trend function in the additive outlier model considered in Perron (1989). We discuss and correct an error in the stated asymptotic distributions of the tests in this case. We propose a simple modification of the procedure which yields statistics having the same asymptotic distributions as stated in Perron (1989). We also discuss the adequacy of the asymptotic approximations and various extensions to the case where the break-point is unknown with corresponding asymptotic critical values.

Tests for a unit root allowing for the possible presence of a one-time change in the intercept and/or slope of a series were proposed by Perron (1989) (henceforth referred to as P89). He considered two general classes of models: a) the *additive outlier model*, appropriate

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when the change is sudden, and b) the *innovational outlier model*, appropriate when the change is gradual. In this note, we discuss an error in the treatment of the asymptotic distributions of the tests associated with the *additive outlier model.* We point out that the asymptotic distributions of these statistics are different than those stated in P89 and also that they depend on the correlation structure of the data when a change in intercept is involved, even if the appropriate order of the autoregression is used. Fortunately, in these cases, a simple modification is available which yields statistics having the same asymptotic distributions (invariant to nuisance parameters) as stated in P89. This transformation is discussed, as well as the asymptotic approximation that is related. We also discuss extensions to the case where the breakpoint is unknown and present corresponding asymptotic critical values.

The present note contains an extended discussion and proofs of assertions stated in Perron (1992). It covers cases dealing with trending data, where as Perron and Vogelsang (1992) cover similar corrections and extensions to Perron (1990) for the case of non-trending data. The case of trending data offers some interesting contrasts, especially when a change in slope is involved with both segments of the trend joined at the time of change. Here, the two-step method suggested in Perron (1989) is still valid. The asymptotic distributions are, however, different from those stated earlier for a known breakpoint and also different from the limiting distribution stated in Zivot and Andrews (1992) for an unknown breakpoint. We provide tabulated critical values, in this and other settings, that should be useful for applications.

1. The models and the statistics.

The *additive outlier models* allow for a sudden change in the intercept and/or slope of a series $\{y_t\}_1^T$, say, at time $T_b (1 < T_b < T)$. Model A (the *crash model*) specifies a change in the intercept, model B a change in the slope (restricting the segments to be joined) and Model C allows for both a change in intercept and slope. The models are specified, respectively, as (for $t = 1, ..., T$):

$$y_t = \mu + \beta t + \gamma DU_t + Z_t, \tag{1.A}$$
$$y_t = \mu + \beta t + \theta DT_t^* + Z_t, \tag{1.B}$$
$$y_t = \mu + \beta t + \gamma DU_t + \theta DT_t + Z_t, \tag{1.C}$$

where $DU_t = 1$, $DT_t^* = t - T_b$ and $DT_t = t$ if $t > T_b$ and $DU_t = DT_t^* = DT_t = 0$ otherwise. The noise component Z_t is assumed to be a finite order ARMA(p,q) process of the form $A(L)Z_t = B(L)v_t(Z_0 = 0)$ with $v_t \sim i.i.d.(0, \sigma_v^2)$ with finite fourth moment. It is assumed that all the roots of $B(z) = 0$ are strictly outside the unit circle and that the polynomial $A(z) = 0$ has at most one root on the unit circle with all others strictly outside. Denote by α the sum of the autoregressive coefficients, $1 - A(1)$, and write $A(L) = (1 - \alpha L)A^*(L)$. Under the null hypothesis $\alpha = 1$, and under the alternative hypothesis $\alpha < 1$. Using this notation, we can write $Z_t = \alpha Z_{t-1} + e_t$ where $e_t = A^*(L)^{-1}B(L)v_t$.

Let $\{\tilde{y}_t^i; i = A, B, C\}$ be the residuals from a regression of y_t on $\{1, t, DU_t\}(i = A)$, $\{1, t, DT_t^*\}(i = B)$, $\{1, t, DU_t, DT_t\}(i = C)$. The tests based on the *additive outlier models* from P89 are the normalized bias $T(\tilde{\alpha}^i - 1)$ and the t-statistic for testing the null hypothesis $\alpha = 1$, denoted by $t_{\tilde{\alpha}^i}$, in the following second step regression:

$$\tilde{y}_t^i = \tilde{\alpha}^i \tilde{y}_{t-1}^i + \tilde{u}_t^i, \qquad (i = A, B, C) \quad t = 2, ..., T. \tag{2}$$

2. The limiting distributions.

Let $w_i(r)$ be the projection residual of a Wiener process $w(r)$ on the subspace generated by the functions $\{1, r, du(r)\}(i = A)$, $\{1, r, dr^*(r)\}(i = B)$ and $\{1, r, du(r), dr(r)\}(i = C)$ where $du(r) = 1$, $dr^*(r) = r - \lambda$, $dr(r) = r$ if $r > \lambda$ and $du(r) = dr^*(r) = dr(r) = 0$ otherwise. Here, $\lambda = [T_b/T]$ is the ratio of pre-break sample size to total sample size. Denoting by "$\Longrightarrow$" weak convergence in distribution, the limiting distributions of $T(\tilde{\alpha}^i - 1)$ and $t_{\tilde{\alpha}^i}$ are, instead of those stated in Theorem 2 of P89 (with $g_i(i = A, B, C), \psi_1, \psi_4, \psi_5, D_4$ and D_{12} as defined in that Theorem, see also Perron (1992) where g_B and g_C are correctly stated to be defined by $g_B = \lambda^3/3$ and $g_C = (1-\lambda)^3/12$):

$$T(\tilde{\alpha}^i - 1) \Longrightarrow [\int_0^1 w_i(r)dw(r) + \delta + L_i M_i][\int_0^1 w_i(r)^2 dr]^{-1}, \tag{3}$$

$$t_{\tilde{\alpha}}i \Longrightarrow [\int_0^1 w_i(r)dw(r) + \delta + L_i M_i] [(\sigma_e^2/\sigma^2 + L_i^2)\int_0^1 w_i(r)^2 dr]^{-1/2}, \tag{4}$$

for $i = A, C$, where $L_A = D_4/[(1-\lambda)\lambda] + \psi_1/(2g_A)$, $M_A = w(\lambda) - \lambda^{-1}\int_0^\lambda w(r)dr - \lambda\psi_1/(2g_A)$, $L_C = D_4/[(1-\lambda)\lambda] + 6D_{12}/\lambda^2 - (1-\lambda)\psi_5/(2g_C)$, $M_C = w(\lambda) - \lambda^{-1}\int_0^\lambda w(r)dr - 6D_{12}/\lambda^2$. Also $\delta = (\sigma^2 - \sigma_e^2)/2\sigma^2$ where $\sigma^2 = \lim_{T\to\infty} T^{-1}E[S_T^2]$ with $S_T = \sum_{t=1}^T e_t$ and $\sigma_e^2 = \lim_{T\to\infty} T^{-1}\sum_{t=1}^T E(e_t^2)$. For Model B:

$$T(\tilde{\alpha}^B - 1) \Longrightarrow [\int_0^1 w_B(r)dw(r) + \delta + (\psi_4/g_B)\int_\lambda^1 w_B(r)dr] [\int_0^1 w_B(r)^2 dr]^{-1}, \tag{5}$$

$$t_{\tilde{\alpha}}B \Longrightarrow (\sigma/\sigma_e)[\int_0^1 w_B(r)dw(r) + \delta + (\psi_4/g_B)\int_\lambda^1 w_B(r)dr] [\int_0^1 w_B(r)^2 dr]^{-1/2}. \tag{6}$$

All proofs are in the Appendix. Consider first results pertaining to Models A and C. The limiting distributions in (3) and (4) are free of nuisance parameters when the errors e_t are uncorrelated (in which case $\sigma_e = \sigma$) and percentage points could be tabulated. There are two problems with this approach. One is that the Phillips-Perron (1988) nonparametric correction is different and more complex than that stated in Section 4.1 of P89. More importantly, it can be shown that the use of the augmented regression:

$$\tilde{y}_t^i = \hat{\alpha}^i \tilde{y}_{t-1}^i + \sum_{j=1}^k \hat{c}_j \Delta\tilde{y}_{t-j}^i + \hat{u}_t, \tag{7}$$

does not eliminate the dependency of the t-statistic on nuisance parameters even when Z_t is an autoregressive process of known order. There is, however, a simple way to modify the second step regression (2) to avoid these problems. Consider first the modification:

$$\tilde{y}_t^i = \hat{\omega}^i D(TB)_t + \hat{\alpha}^i \tilde{y}_{t-1}^i + \hat{u}_t^i. \qquad (i = A, C) \quad (t = 2, \dots, T) \qquad (8)$$

Let $t_{\hat{\alpha}^i}$ be the t-statistic for testing $\alpha^i = 1$ in (8). It is shown in the Appendix that:

$$T(\hat{\alpha}^i - 1) \Longrightarrow (\int_0^1 w_i(r)dw(r) + \delta)(\int_0^1 w_i(r)^2 dr)^{-1}, \quad (i = A, C) \qquad (9)$$

$$t_{\hat{\alpha}^i} \Longrightarrow (\sigma/\sigma_e)(\int_0^1 w_i(r)dw(r) + \delta)(\int_0^1 w_i(r)^2 dr)^{-1/2}. \ (i = A, C) \qquad (10)$$

Hence, if the one-time dummy $D(TB)_t$ is included in the second step regression, the asymptotic distributions of the statistics are the same as those stated in P89, and in particular, the same as those associated with the *innovational outlier model* (regressions (12) and (14) in P89), When the errors are martingales differences $\sigma = \sigma_e$ and hence $\delta = 0$. In that case the limiting distributions stated in (9) and (10) are invariant to nuisance parameters. Critical values are given in Table 4 (A, B) and 6 (A, B) of P89.

Consider now the case of Model B. Here things are different. First, the limiting distributions are different from the *innovational outlier* case only insofar as an extra term (independent of nuisance parameters) is present in the numerator. Hence, on the one hand, the application of the Phillips-Perron transformations (as discussed in section 4.1 of P89) is still valid provided the asymptotic distribution in (8) and (9) with $\sigma_e = \sigma$ are used. Also, contrary to the case with Models A and C, the limiting distribution of the t-statistic, obtained using the augumented regression (7), is given by (6) with $\sigma_e = \sigma$ when the noise component is an autoregression.

Hence, the critical values of the limiting distributions (5) and (6) with $\sigma_e = \sigma$ can be used for inference purposes. These are different from those stated in P89 and are tabulated in Perron (1992). Table 1 reproduces the asymptotic distribution of $t_{\hat{a}}B$ and reports finite sample critical values to assess the adequacy of the asymptotic approximation (the data-generating process used being a random walk with N(0, 1) innovations and initial condition set at zero; 10,000 replications are used). The approximation is seen to be adequate for common sample sizes. Comparing the results with those in Tables 5.A and 5.B of P89, it is seen that the differences in the asymptotic distributions are mainly in the right tail, the left tail being very similar. Furthermore, the corrected asymptotic distribution is, unlike the other cases, clearly asymmetric around $\lambda = 0.5$ (again, especially given the behavior of the distribution in the right tail).

3. Extensions to more general error processes.

Consider first the case of model B. Applying the augumented regression (7) when the DGP is an AR(p) with a unit root leads to a t-statistic with an asymptotic distribution equivalent to that stated in (6) with $\sigma_e = \sigma$ (details of the proof are available on request). Hence, the limiting distribution is different from that tabulated in P89 but is otherwise free of nuisance parameters, and the appropriate critical values are those in Table 1 of this note. Similarly the corresponding Phillips-Perron $Z(t_{\tilde{\alpha}})$ and $Z(\tilde{\alpha})$ statistics are still valid if the asymptotic critical values in Tables 1 and 2 of this note are used.

Consider now the transformations to (8), for Models A and C, necessary for the limiting distributions of the tests to be invariant to nuisance parameters when Z_t is an ARMA(p, q). First, the extensions of the Phillips and Perron (1988) statistics, discussed in Section 4.1 of P89, remain valid provided $\tilde{\alpha}^i$ and $t_{\tilde{\alpha}}i$ in equations (6) and (7) of P89 are replaced by $\hat{a}^i$ and $t_{\hat{\alpha}}i$ from (8) above (similarly $\tilde{\sigma}^2$ and $\tilde{\sigma}_e^2$ need to be replaced by estimators based on the residuals from that regression). The asymptotic critical values are still those in Table 4.A-B and 6.A-B of P89. Secondly, the Said-Dickey (1984) extension, discussed in Section 4.1 of P89, remains valid for Models A and C provided the augmented regression (7) is replaced by:

$$\tilde{y}_t^i = \sum_{j=0}^{k} \hat{\omega}_j D(TB)_{t-j} + \hat{\alpha}^i \tilde{y}_{t-1}^i + \sum_{j=1}^{k} \hat{c}_j \Delta \tilde{y}_{t-j}^i + \hat{u}_t, \; (i = A, C) \quad (11)$$

The introduction of the dummies $\{D(TB)_{t-j}\}_{j=0}^{k}$ makes the asymptotic distribution of the t-statistic for $\alpha = 1$ in (11), $t_{\hat{\alpha}}i$, be that stated in (10) with $\sigma = \sigma_e$. The introduction of these dummies is sufficient to correct the problems discussed above for the *additive outlier model* and the critical values in Tables 4.B and 6.B of P89 are appropriate.

4. Extensions to unknown break points.

Several recent studies have considered extensions of the tests proposed in P89 to the case where the break point is unknown (see Banerjee, Lumsdaine and Stock (1992), Zivot and Andrews (1992) and Perron (1991)). Among the tests considered in these papers are minimal t-statistics obtained over all possible break points, *i.e.* $t_{\hat{\alpha}}i(\inf) \equiv \inf_{\lambda \epsilon \Lambda} t_{\hat{\alpha}}i(\lambda)$ where $t_{\hat{\alpha}}i(\lambda)$ is the t-statistic for testing $\alpha = 1$ in model i with a break point fixed at $[\lambda T]$ and Λ is a closed subset of the interval $(0, 1)$. Using results in Zivot and Andrews (1992), the limiting distribution of the minimal t-statistic from Model B (regression (7)) is:

$$t_{\hat{\alpha}}B(\inf) \Longrightarrow \inf_{\lambda \epsilon \Lambda}\{[\int_0^1 w_B(r)dw(r) + (\psi_4/g_B)\int_\lambda^1 w_B(r)dr] \\ [\int_0^1 w_B(r)^2 dr]^{-1/2}\}. \quad (12)$$

Tabulated critical values of the asymptotic distribution in (12) are presented in Table 3. These are obtained using simulation methods with a grid of 1,000 values for λ and 50,000 replications. The critical values of the corresponding asymptotic distributions of the t-statistics associated with Models A and C are presented in Zivot and Andrews (1992). Note that Table 3 of Zivot and Andrews (1992) does not provide the asymptotic distribution of $t_{\hat{\alpha}}B(\inf)$ as obtained from regression (7). They use a one step procedure which does not permit the change in slope to be present under the null hypothesis.

For completeness Table 4 gives the asymptotic critical values of the distribution of $\inf_{\lambda \epsilon \Lambda}[\int_0^1 w_B(r)dw(r) + (\psi_4/g_B)\int_\lambda^1 w_B(r)dr]$ $[\int_0^1 w_B(r)^2 dr]^{-1}$, for Model B. These critical values can be used when considering the minimal value (over all break points) of the Phillips-Perron $Z(\tilde{\alpha})$ statistic in the *additive outlier model* assuming the break point to be unknown. Also presented in Table 4 are the critical values of the distributions of $\inf_{\lambda\epsilon\Lambda}[\int_0^1 w_i(r)dw(r)\ (\int_0^1 w_i(r)^2 dr)^{-1}]$ for $i = A$, and C. These can be used with the minimal values of the $Z(\tilde{\alpha})$ statistics in Models A and C (provided equation (8) is used). These are referred to as the limiting distribution of $\inf_{\lambda\epsilon\Lambda} T(\hat{\alpha}^i(\lambda) - 1)$ since they correspond to the asymptotic distribution of the minimal normalized bias over all possible break points when the errors e_t are uncorrelated. In the case of Models A and C, $\hat{\alpha}^i$ is constructed from the regression (8), and for Model B it is constructed from regression (2).

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Mathematical Appendix

We first note that the exact distributions of the statistics of interest are invariant to the parameters μ, β, λ, and θ in (1.A)–(1.C). Therefore, without loss of generality, we derive the following asymptotic results under the simplified data-generating process, where e_t, a finite order ARMA(p, q) process, is as defined in the text:

$$y_t = y_{t-1} + e_t. \tag{A.1}$$

PROOF OF (3)–(4), MODEL A: Let $\tilde{y}_t^A$ be the residuals from a projection of y_t on $\{1, t, DU_t\}$ $(t = 1, ..., T)$. Straightforward algebra yields:

$$\begin{aligned} \tilde{y}_t^A &= y_t - \overline{Y}^a - (t - A_1)c, \quad t \leq T_b, \\ \tilde{y}_t^A &= y_t - \overline{Y}^b - (t - A_2)c, \quad t > T_b, \end{aligned} \tag{A.2}$$

where $\overline{Y}^a = T_b^{-1} \sum_{t=1}^{T_b} y_t$, $\overline{Y}^b = (T - T_b)^{-1} \sum_{t=T_b+1}^{T} y_t$, $A_1 = T_b^{-1} \sum_{t=1}^{T_b} t$, $A_2 = (T - T_b)^{-1} \sum_{t=T_b+1}^{T} t$, and $c = \{\sum_{t=1}^{T} t y_t - T_b \overline{Y}^a A_1 - (T - T_b)\overline{Y}^b A_2\}\{\sum_{t=1}^{T} t^2 - T_b A_1^2 - (T - T_b)A_2^2\}^{-1}$. We note the following convergence results that are easily obtained using results in P89: $T^{-1/2}\overline{Y}^a \Longrightarrow (\sigma/\lambda)\int_0^\lambda w(r)dr$, $T^{-1/2}\overline{Y}^b \Longrightarrow (\sigma/(1-\lambda))\int_\lambda^1 w(r)dr$, $T^{-1}A_1 \Longrightarrow \lambda/2$, $T^{-1}A_2 \Longrightarrow (1+\lambda)/2$, and $T^{1/2}c \Longrightarrow \sigma\psi_1/g_A$, where g_A and ψ_1 are as defined in P89 (Theorem 2). Using (A.1) and (A.2):

$$\tilde{y}_t^A - \tilde{y}_{t-1}^A = e_t - c, \quad (t \neq T_b + 1),$$

$$\tilde{y}_t^A - \tilde{y}_{t-1}^A = e_t - c + \overline{Y}^a - \overline{Y}^b + (A_2 - A_1)c, \quad (t = T_b + 1). \tag{A.3}$$

Let $\tilde{Y}^{A\prime} = (\tilde{y}_2^A, ..., \tilde{y}_T^A)$, $\tilde{Y}_{-1} = (\tilde{y}_1^A, ..., \tilde{y}_{T-1}^A)$, $E' = (e_2, ..., e_T)$, $i' = (1, 1, ..., 1)$, and $D(TB)$ a $(T-1)$ by 1 vector with 1 if $t = T_b + 1$ and 0 elsewhere. We can write (A.3) as:

$$\tilde{Y}^A - \tilde{Y}_{-1}^A = E - ic + D(TB)(\overline{Y}^a - \overline{Y}^b + (A_2 - A_1)c). \tag{A.4}$$

Using this notation, we have

$$T(\tilde{\alpha}^A - 1) = T^{-1}\tilde{Y}_{-1}^{A\prime}(\tilde{Y}^A - \tilde{Y}_{-1}^A)/T^{-2}\tilde{Y}_{-1}^{A\prime}\tilde{Y}_{-1}^{A\prime}\tilde{Y}_{-1}^A.$$

Consider first the numerator of $T(\tilde{\alpha}^A - 1)$:

$$\begin{aligned} T^{-1}\tilde{Y}_{-1}^{A\prime}(\tilde{Y}^A - \tilde{Y}_{-1}^A) &= T^{-1}\tilde{Y}_{-1}^{A\prime}(E - ic + D(TB)(\overline{Y}^a - \overline{Y}^b \\ &\quad + (A_2 - A_1)c) \\ &= T^{-1}\tilde{Y}_{-1}^{A\prime}E + T^{-1/2}\tilde{y}_{T_b}^A(T^{-1/2}\overline{Y}^a - T^{-1/2}\overline{Y}^b \\ &\quad + T^{-1}(A_2 - A_1)T^{1/2}c) + o_p(1). \end{aligned} \tag{A.5}$$

Using results in P89 (see also Zivot and Andrews (1992)), $T{-}1\tilde{Y}_{-1}^{A\prime}E \Longrightarrow \sigma^2 H_A/g_A \equiv \sigma^2[\int_0^1 w_A(r)dw(r) + \delta]$. Consider now the second term in (A.5). Using (A.2), we have:

$$\begin{aligned} T^{-1/2}\tilde{y}_{T_b}^A &= T^{-1/2}y_{T_b} - T^{-1/2}\overline{Y}^a - T^{-1}(T_b - A_1)T^{1/2}c \\ &\Longrightarrow \sigma[w(\lambda) - \lambda^{-1}\int_0^\lambda w(r)dr - \lambda\psi_1/(2g_A)] \equiv \sigma M_A; \end{aligned}$$

$$\begin{aligned} &T^{-1/2}\overline{Y}^a - T^{-1/2}\overline{Y}^b + T^{-1}(A_2 - A_1)T^{1/2}c \\ &\quad \Longrightarrow \sigma[\lambda^{-1}\int_0^\lambda w(r)dr - (1-\lambda)^{-1}\int_\lambda^1 w(r)dr + \psi_1/(2g_A)] \\ &\quad = \sigma[D_4/[\lambda(1-\lambda)] + \psi_1/(2g_A)] \equiv \sigma L_A, \end{aligned}$$

where D_4 is as defined in P89 (Theorem 2). Hence,

$$T^{-1}\tilde{Y}_{-1}^{A\prime}(\tilde{Y}^A - \tilde{Y}_{-1}^A) \Longrightarrow \sigma^2[H_A/g_A + L_A M_A]$$
$$\equiv \sigma^2[\int_0^1 w_A(r)dw(r) + \delta + L_A M_A]. \qquad \text{(A.6)}$$

Consider now the denominator of (A.4). As in P89, we have:

$$T^{-2}\tilde{Y}_{-1}^{A\prime}\tilde{Y}_{-1}^A \Longrightarrow \sigma^2 K_A/g_A \equiv \sigma^2 \int_0^1 w_A(r)^2 dr. \qquad \text{(A.7)}$$

This proves (3), with $i = A$, using (A.4), (A.6) and (A.7). To prove (4) we only need to further derive the limit of $\tilde{s}_A^2 = T^{-1}\sum_{t=2}^T \tilde{u}_t^2$ with $\tilde{u}_t$ the estimated residuals from (2).

$$\begin{aligned}
\tilde{s}_A^2 &= T^{-1}\sum_{t=2}^T(\tilde{y}_t^A - \tilde{\alpha}^A\tilde{y}_{t-1}^A)^2 \\
&= T^{-1}\sum_{t=2}^T(\tilde{y}_t^A - \tilde{y}_{t-1}^A)^2 - T^{-1}2T(\tilde{\alpha}^A - 1)T^{-1}\sum_{t=2}^T \tilde{y}_{t-1}^A(\tilde{y}_t^A - \tilde{y}_{t-1}^A) \\
&\quad + T^{-1}T^2(\tilde{\alpha}^A - 1)^2T^{-2}\sum_{t=2}^T(\tilde{y}_{t-1}^A)^2 \\
&= T^{-1}\sum_{t=2}^T(\tilde{y}_t^A - \tilde{y}_{t-1}^A)^2 + o_p(1) \\
&= T^{-1}\sum_{t=2}^T(e_t - c)^2 + T^{-1}[\overline{Y}^a - \overline{Y}^b + (A_2 - A_1)c]^2 + o_p(1) \\
&\Longrightarrow \sigma_e^2 + \sigma^2 L_A^2.
\end{aligned} \qquad \text{(A.8)}$$

The proof of (4), with $i = A$, follows using (A.6) through (A.8), notbing that $t_{\tilde{\alpha}}A = T^{-1}\tilde{Y}_{-1}^{A\prime}(\tilde{Y}^A - \tilde{Y}_{-1}^A)/[\tilde{s}_A^2 T^{-2}\tilde{Y}_{-1}^{A\prime}\tilde{Y}_{-1}^A]^{1/2}$.

PROOF OF (3)–(4) FOR MODEL C: Let $\tilde{y}_t^C$ be the residuals from a projection of y_t on $\{1, t, DU_t, DT_t\}$ $(t = 1, ..., T)$. Straightforward algebra yields: $\tilde{y}_t^C = y_t - \overline{Y}^a - (t - A_1)c_1$, if $t \leq T_b$, and $\tilde{y}_t^C = y_t - \overline{Y}^b - (t - A_2)(c_1 + c_2)$, if $t > T_b$, where $[c_1, c_2]' = (Z'Z)^{-1}Z'Y^*$ with:

$$Z = \begin{bmatrix} 1 - A_1 & 0 \\ \vdots & \vdots \\ T_b - A_1 & 0 \\ T_b + 1 - A_2 & T_b + 1 - A_2 \\ \vdots & \vdots \\ T - A_2 & T - A_2 \end{bmatrix}, \quad \text{and} \quad Y^* = \begin{bmatrix} y_1 - \overline{Y}^a \\ \vdots \\ y_{T_b} - \overline{Y}^a \\ y_{T_b+1} - \overline{Y}^b \\ \vdots \\ y_T - \overline{Y}^b \end{bmatrix} \tag{A.10}$$

We can write:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = D^{-1} \begin{bmatrix} (B_1 - B_2)Z_2 \\ B_2 Z_1 - B_1 Z_2 \end{bmatrix}, \text{ with } Z'Z \equiv \begin{bmatrix} Z_1 & Z_2 \\ Z_2 & Z_2 \end{bmatrix}, \ Z'Y^* = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $D = Z_1 Z_2 - Z_2^2$; $Z_1 = \sum_{t=1}^T t^2 - \lambda T A_1^2 - (1 - \lambda)TA_2^2$, $Z_2 = \sum_{t=T_b+1}^T t^2 - (1 - \lambda)TA_2^2$, $B_1 = \sum_{t=1}^T ty_t - \lambda T\overline{Y}^a A_1 - (1 - \lambda)T\overline{Y}^b A_2$, $B_2 = \sum_{t=T_b+1}^T ty_t - (1 - \lambda)T\overline{Y}^b A_2$. Using results in P89, we have: $T^{-1}A_1 \to \lambda/2$, $T^{-1}A_2 \to (1 + \lambda)/2$, $T^{-5/2}B_1 \Longrightarrow \sigma[\int_0^1 rw(r)dr - (\lambda/2)\int_0^\lambda w(r)dr - (1/2)(1 + \lambda)\int_\lambda^1 w(r)dr] \equiv \sigma D_{11}$, $T^{-5/2}B_2 = \sigma[\int_\lambda^1 rw(r)dr - (1/2)(1 + \lambda)\int_\lambda^1 w(r)dr] \equiv \sigma[D_{11} - D_{12}] \equiv -\sigma\psi_5$, $T^{-3}Z_1 \to (1 - \lambda)^3/12 + \lambda^3/12$, $T^{-3}Z_2 \to (1 - \lambda)^3/12$. Hence

$$T^{-6}D \to \lambda^3(1-\lambda)^3/144,$$

$$\begin{aligned} T^{1/2}c_1 &\Longrightarrow [\lambda^3(1-\lambda)^3/144]^{-1}\sigma[D_{11}-(D_{11}-D_{12})][(1-\lambda)^3/12] \\ &= \sigma 12 D_{12}/\lambda^3, \end{aligned}$$

$$\begin{aligned} T^{1/2}c_2 &\Longrightarrow [\lambda^3(1-\lambda)^3/144]^{-1}\sigma[-\psi_5\{(1-\lambda)^3/12+\lambda^3/12\} \\ &\qquad - D_{11}(1-\lambda)^3/12] \\ &= [\lambda^3(1-\lambda)^3/144]^{-1}\sigma[(D_{11}-D_{12})\{(1-\lambda)^3/12 \\ &\qquad + \lambda^3/12\} - D_{11}(1-\lambda)^3/12] \\ &= [\lambda^3(1-\lambda)^3/144]^{-1}\sigma[(D_{11}-D_{12})\lambda^3/12 - D_{12}(1-\lambda)^3/12] \\ &= -\sigma 12 D_{12}/\lambda^3 - \sigma 12\psi_5/(1-\lambda)^3 \equiv -\sigma 12 D_{12}/\lambda^3 - \sigma\psi_5/g_C, \end{aligned}$$

and $T^{1/2}(c_1+c_2) \Longrightarrow -\sigma\psi_5/g_C$ where g_C, ψ_5 and D_{12} are defined in P89 (Theorem 2) (see also Perron (1992)). The first-differences are given by:

$$\begin{aligned} \tilde{y}_t^C - \tilde{y}_{t-1}^C &= e_t - c_1, \quad t \le T_b, \\ &= e_t + \overline{Y}^a - \overline{Y}^b - (T_b+1-A_2)(c_1+c_2) + (T_b - A_1)c_1, \\ &\qquad t = T_b + 1, \\ &= e_t - (c_1+c_2), \ t > T_b + 1. \end{aligned}$$

Using a similar vector notation as before with the addition of $DU' = (0,\dots,0,1,\dots,1)$: $\tilde{Y}^C - \tilde{Y}_{-1}^C = E - c_1 i - c_2 DU + qD(TB)$, where $q = \overline{Y}^a - \overline{Y}^b - (T_b - A_2)(c_1+c_2) + (T_b - A_1)c_1$. Consider first the numerator of the normalized least-squares estimator $T(\tilde{\alpha}^C - 1)$:

$$\begin{aligned} T^{-1}\tilde{Y}_{-1}^{C\prime}(\tilde{Y}^C - \tilde{Y}_{-1}^C) &= T^{-1}\tilde{Y}_{-1}^{C\prime}[E - c_1 i - c_2 DU + qD(TB)] \\ &= T^{-1}\tilde{Y}_{-1}^{C\prime}E + T^{-1/2}qT^{-1/2}\tilde{y}_{T_b}^C + o_p(1). \end{aligned}$$

Similar to model A, we have

$$T^{-1}\tilde{Y}_{-1}^{C\prime}E \Longrightarrow \sigma^2 H_C/g_C \equiv \sigma^2 \int_0^1 w_C(r)dr(r) + \delta.$$

Also:

$$
\begin{aligned}
T^{-1/2}q &= T^{-1/2}\overline{Y}^a - T^{-1/2}\overline{Y}^b - T^{-1}(T_b - A_2)T^{1/2}(c_1 + c_2) \\
&\quad + T^{-1}(T_b - A_1)T^{1/2}c_1 \\
&\Longrightarrow \sigma\{\lambda^{-1}\int_0^{\lambda} w(r)dr - (1-\lambda)^{-1}\int_{\lambda}^{1} w(r)dr \\
&\quad - (1-\lambda)\psi_5/(2g_C) + 6D_{12}/\lambda^2\} \\
&= \sigma\{D_4/[\lambda(1-\lambda)] - (1-\lambda)\psi_5/(2g_C) + 6D_{12}/\lambda^2\} \equiv \sigma L_C; \\
T^{-1/2}\tilde{y}^C_{T_b} &= T^{-1/2}y_{T_b} - T^{-1/2}\overline{Y}^a - T^{-1}(T_b - A_1)T^{1/2}c_1 \\
&\Longrightarrow \sigma\{w(\lambda) - \lambda^{-1}\int_0^{\lambda} w(r)dr - 6D_{12}/\lambda^2\} \equiv \sigma M_C.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
T^{-1}\tilde{Y}^{C\prime}_{-1}(\tilde{Y}^C - \tilde{Y}^C_{-1}) &\Longrightarrow \sigma^2\{H_C/g_C + L_C M_C\} \\
&\equiv \sigma^2\{\int_0^1 w_C(r)dw(r) + \delta + L_C M_C\}.
\end{aligned} \tag{A.11}
$$

The proof of (3), with $i = C$, follows using (A.11) and $T^{-2}\tilde{Y}^{C\prime}_{-1}\tilde{Y}^C_{-1} \Longrightarrow \sigma^2 K_C/g_C \equiv \sigma^2\int_0^1 w_C(r)^2 dr$. Derivations analogous to those for Model A show that $\tilde{s}^2_C \Longrightarrow \sigma^2_e + \sigma^2 L^2_C$.

Proof of (5)–(6), Model B: Let $\tilde{y}^B_t$ be the residuals from a projection of y_t on $\{1, t, DT^*_t\}(t = 1, ..., T)$. Straightforward algebra yields:

$$
\begin{aligned}
\tilde{y}^B_t &= y_t - \overline{Y} - (t - \bar{t})c_3 + \bar{t}^* c_4, \quad t \le T_b, \\
\tilde{y}^B_t &= y_t - \overline{Y} - (t - \bar{t})c_3 - (t - T_b - \bar{t}^*)c_4, \quad t > T_b,
\end{aligned} \tag{A.12}
$$

where $\overline{Y} = T^{-1}\sum_{t=1}^T y_t$, $\bar{t} = T^{-1}\sum_{t=1}^T t$, $\bar{t}^* = T^{-1}\sum_{t=1}^{T-T_b} t$. Note that $T^{-1/2}\overline{Y} \Longrightarrow \sigma\int_0^1 w(r)dr$, $T^{-1}\bar{t} \Longrightarrow 1/2$, $T^{-1}\bar{t}^* \Longrightarrow (1-\lambda)^2/2$. The variables c_3 and c_4 are defined by $[c_3, c_4]' = (W'W)^{-1}W'(Y - \overline{Y})$

where

$$W = \begin{bmatrix} 1-\bar{t} & -\bar{t}^* \\ \cdot & \cdot \\ \vdots & -\bar{t}^* \\ \vdots & 1-\bar{t}^* \\ & \vdots \\ T-\bar{t} & T-T_b-\bar{t}^* \end{bmatrix}$$

We have $T^{1/2}c_3 \Longrightarrow -\sigma\psi_3/g_B$ and $T^{1/2}c_4 \Longrightarrow -\sigma\psi_4/g_B$ with g_B, ψ_3 and ψ_4 as defined in P89 (Theorem 2, see also Perron (1992)). The first-differences are given by:

$$\begin{aligned} \tilde{y}_t^B - \tilde{y}_{t-1}^B &= e_t - c_3, \quad t \leq T_b, \\ &= e_t - (c_3 + c_4), \quad t > T_b, \end{aligned} \tag{A.13}$$

or in vector notation: $\tilde{Y}^B - \tilde{Y}_{-1}^B = E - c_3 i - c_4 DU$. Consider the numerator of $T(\tilde{\alpha}^B - 1)$:

$$\begin{aligned} T^{-1}\tilde{Y}_{-1}^{B\prime}(\tilde{Y}^B - \tilde{Y}_{-1}^B) &= T^{-1}\tilde{Y}_{-1}^{B\prime}(E - c_3 i - c_4 DU) \\ &= T^{-1}\tilde{Y}_{-1}^{B\prime}E - T^{1/2}c_4 T^{-3/2}\tilde{Y}_{-1}^{B\prime}DU + o_p(1) \\ &\Longrightarrow \sigma^2\{H_B/g_B + (\psi_4/g_B)\int_\lambda^1 w_B(r)dr\} \\ &\equiv \sigma^2\{\int_0^1 w_B(r)dw(r) + \delta + (\psi_4/g_B)\int_\lambda^1 w_B(r)dr\}. \end{aligned}$$

This proves (5) nothing that

$$T^{-2}\tilde{Y}_{-1}^{B\prime}\tilde{Y}_{-1}^B \Longrightarrow \sigma^2 K_B/g_B \equiv \sigma^2\int_0^1 w_B(r)^2 dr.$$

To prove (6), we simply need to show that $\tilde{s}_B^2 \to \sigma_e^2$. Using (A.13):

$$\tilde{s}_B^2 = T^{-1}\sum_{t=2}^{T}(\tilde{y}_t^B - \tilde{\alpha}^B \tilde{y}_{t-1}^B)^2 = T^{-1}\sum_{t=2}^{T}(\tilde{y}_t^B - \tilde{y}_{t-1}^B - (\tilde{\alpha}^B - 1)\tilde{y}_{t-1}^B)^2$$
$$= T^{-1}\sum_{t=2}^{T}(\tilde{y}_t^B - \tilde{y}_{t-1}^B)^2 + o_p(1) = T^{-1}\sum_{t=2}^{T_b}(e_t - c_3)^2 + T^{-1}\sum_{t=T_b+1}^{T}(e_t - c_3 - c_4)^2$$
$$= T^{-1}\sum_{t=2}^{T} e_t^2 + o_p(1) \to \sigma_e^2, \text{ since } c_3 = O_p(T^{-1/2}) \text{ and } c_4 = O_p(T^{-1/2}).$$

Proof of (9)–(10): We prove the results for Model A only; the proof for Model C is entirely analogous and therefore omitted. Let $\breve{y}_{0,t}^A$ be the residuals from a regression of $\tilde{y}_t^A$ on $D(TB)_t (t = 2, ..., T)$ and let $\breve{y}_{1,t-1}^A$ be the residuals from a regression of $\tilde{y}_{t-1}^A$ on $D(TB)_t (t = 2, ..., T)$. We have, for $t = 2, ..., T$: $\breve{y}_{0,t}^A = \tilde{y}_t^A$ if $t \neq T_b + 1$ and 0 otherwise; $\breve{y}_{1,t-1}^A = \tilde{y}_{t-1}^A$ if $t \neq T_b + 1$ and 0 otherwise. Also, $\breve{y}_{0,t}^A - \breve{y}_{1,t-1}^A = e_t - c$ if $t \neq T_b + 1$ and 0 otherwise. The least-squares estimate and t-statistic from regression (8) are given by:

$$T(\hat{\alpha}^A - 1) = T^{-1}\sum_{t=2}^{T}\breve{y}_{1,t-1}^A(\breve{y}_{0,t}^A - \breve{y}_{1,t-1}^A)/T^{-2}\sum_{t=2}^{T}(\breve{y}_{1,t-1}^A)^2, \text{ and}$$
$$t_{\hat{\alpha}}A = T^{-1}\sum_{t=2}^{T}\breve{y}_{1,t-1}^A(\breve{y}_{0,t}^A - \breve{y}_{1,t-1}^A)/[\hat{s}_A^2 T^{-2}\sum_{t=2}^{T}(\breve{y}_{1,t-1}^A)^2]^{1/2},$$
$$\text{where } \hat{s}_A^2 = T^{-1}\sum_{t=2}^{T}\hat{u}_t^2$$

with $\hat{u}t$ the estimated residuals from (8). Consider first the numerator of $T(\hat{\alpha}^A - 1)$:

$$T^{-1}\sum_{t=2}^{T}\breve{y}^A_{1,t-1}(\breve{y}^A_{0,t} - \breve{y}^A_{1,t-1}) = T^{-1}\sum_{t=2}^{T_b}\tilde{y}^A_{t-1}(e_t - c)$$

$$+ T^{-1}\sum_{t=T_b+2}^{T}\tilde{y}^A_{t-1}(e_t - c)$$

$$= T^{-1}\sum_{t=2}^{T}\tilde{y}^A_{t-1}(e_t - c)$$

$$- T^{-1}(y_{T_b} - \overline{Y}^a - (T_b - A_1)c)(e_{T_b+1} - c)$$

$$= T^{-1}\sum_{t=2}^{T}\tilde{y}^A_{t-1}e_t + o_p(1), \text{ using } \sum_{t=1}^{T}\tilde{y}^A_t = 0,$$

$$\Longrightarrow \sigma^2 H_A/g_A \equiv \sigma^2\int_0^1 w_A(r)dr(r) + \delta. \qquad \text{(A.14)}$$

Similarly the limit of the denominator of $T(\hat{\alpha}^A - 1)$ is given by:

$$T^{-2}\sum_{t=2}^{T}(\breve{y}^A_{1,t-1})^2 = T^{-2}\sum_{t=2}^{T_b}(\tilde{y}^A_{t-1})^2 + T^{-2}\sum_{t=T_b+2}^{T}(\tilde{y}^A_{t-1})^2$$

$$= T^{-2}\sum_{t=2}^{T}(\tilde{y}^A_{t-1})^2 - T^{-2}(\tilde{y}^A_{T_b})^2$$

$$= T^{-2}\sum_{t=2}^{T}(\tilde{y}^A_{t-1})^2 + o_p(1)$$

$$\Longrightarrow \sigma^2 K_A/g_A \equiv \sigma^2\int_0^1 w_A(r)^2 dr. \qquad \text{(A.15)}$$

This proves (9) using (A.14) and (A.15). To prove (10), we show that

$\hat{s}^2_A \rightarrow \sigma^2_e$. We have:

$$\hat{s}^2_A = T^{-1}\sum_{t=2}^{T}(\breve{y}^A_{0,t} - \hat{\alpha}^A \breve{y}^A_{1,t-1})^2$$

$$= T^{-1}\sum_{t=2}^{T}(\breve{y}^A_{0,t} - \breve{y}^A_{1,t-1} - (\hat{\alpha}^A - 1)\breve{y}^A_{1,t-1})^2$$

$$= T^{-1}\sum_{t=2}^{T}(\breve{y}^A_{0,t} - \breve{y}^A_{1,t-1})^2$$

$$- T^{-1}2T(\hat{\alpha}^A - 1)T^{-1}\sum_{t=2}^{T}\breve{y}^A_{1,t-1}(\breve{y}^A_{0,t} - \breve{y}^A_{1,t-1})$$

$$+ T^{-1}T^2(\hat{\alpha}^A - 1)^2 T^{-2}\sum_{t=2}^{T}(\breve{y}^A_{1,t-1})^2$$

$$= T^{-1}\sum_{t=2}^{T}(\breve{y}^A_{0,t} - \breve{y}^A_{1,t-1})^2 + o_p(1), \text{ in view (A.14), (A.15) and (9),}$$

$$= T^{-1}\sum_{t=2}^{T_b}(e_t - c)^2 + T^{-1}\sum_{t=T_b+2}^{T}(e_t - c)^2 + o_p(1), \text{ using (A.13),}$$

$$= T^{-1}\sum_{t=2}^{T} e_t^2 + o_p(1), \text{ since } c = O_p(T^{-1/2}),$$

$$\rightarrow \sigma^2_e, \text{ as required.}$$

Table 1.

Percentage Points of the Distribution of $t_{\tilde{\alpha}}$; Model B.

λ	T	1.0%	2.5%	5.0%	10.0%	90.0%	95.0%	97.5%	99.0%
0.1	50	−4.30	−3.90	−3.58	−3.23	−1.28	−1.01	−0.73	−0.43
	100	−4.15	−3.84	−3.58	−3.24	−1.30	−1.02	−0.76	−0.45
	200	−4.07	−3.77	−3.48	−3.21	−1.27	−0.98	−0.70	−0.37
	1000	−4.13	−3.81	−3.52	−3.22	−1.19	−0.86	−0.56	−0.24
	∞	−4.15	−3.81	−3.52	−3.23	−1.19	−0.85	−0.55	−0.22
0.2	50	−4.55	−4.12	−3.81	−3.47	−1.44	−1.16	−0.88	−0.61
	100	−4.43	−4.11	−3.78	−3.46	−1.43	−1.16	−0.90	−0.58
	200	−4.23	−3.97	−3.71	−3.40	−1.41	−1.13	−0.84	−0.49
	1000	−4.34	−4.00	−3.71	−3.41	−1.36	−1.02	−0.70	−0.35
	∞	−4.34	−4.01	−3.72	−3.41	−1.35	−1.01	−0.70	−0.33
0.3	50	−4.67	−4.25	−3.95	−3.62	−1.59	−1.32	−1.06	−0.77
	100	−4.54	−4.22	−3.92	−3.60	−1.60	−1.32	−1.07	−0.73
	200	−4.32	−4.06	−3.80	−3.52	−1.56	−1.28	−1.01	−0.65
	1000	−4.41	−4.09	−3.84	−3.54	−1.52	−1.19	−0.88	−0.54
	∞	−4.41	−4.14	−3.85	−3.54	−1.52	−1.19	−0.87	−0.53
0.4	50	−4.73	−4.34	−4.05	−3.71	−1.74	−1.47	−1.22	−0.94
	100	−4.57	−4.23	−3.97	−3.65	−1.74	−1.47	−1.21	−0.91
	200	−4.44	−4.12	−3.88	−3.57	−1.69	−1.42	−1.17	−0.86
	1000	−4.46	−4.13	−3.90	−3.61	−1.69	−1.35	−1.06	−0.68
	∞	−4.48	−4.15	−3.91	−3.61	−1.69	−1.35	−1.06	−0.68

Table 1.

Continuação.

λ	T	1.0%	2.5%	5.0%	10.0%	90.0%	95.0%	97.5%	99.0%
0.5	50	−4.77	−4.40	−4.09	−3.78	−1.85	−1.60	−1.38	−1.12
	100	−4.57	−4.27	−3.99	−3.69	−1.83	−1.58	−1.36	−1.10
	200	−4.47	−4.17	−3.91	−3.63	−1.80	−1.55	−1.32	−1.03
	1000	−4.48	−4.16	−3.92	−3.64	−1.79	−1.48	−1.20	−0.87
	∞	−4.49	−4.17	−3.93	−3.65	−1.80	−1.47	−1.21	−0.85
0.6	50	−4.76	−4.40	−4.10	−3.79	−1.90	−1.68	−1.47	−1.25
	100	−4.59	−4.31	−3.99	−3.69	−1.88	−1.67	−1.45	−1.23
	200	−4.45	−4.17	−3.92	−3.64	−1.85	−1.62	−1.40	−1.14
	1000	−4.48	−4.16	−3.93	−3.64	−1.85	−1.56	−1.30	−0.99
	∞	−4.50	−4.18	−3.94	−3.65	−1.85	−1.56	−1.29	−0.96
0.7	50	−4.78	−4.38	−4.08	−3.76	−1.89	−1.69	−1.52	−1.33
	100	−4.55	−4.24	−3.95	−3.65	−1.88	−1.67	−1.50	−1.31
	200	−4.43	−4.13	−3.89	−3.60	−1.83	−1.61	−1.43	−1.21
	1000	−4.45	−4.12	−3.90	−3.60	−1.83	−1.57	−1.35	−1.06
	∞	−4.49	−4.13	−3.89	−3.60	−1.84	−1.57	−1.35	−1.06
0.8	50	−4.71	−4.30	−3.99	−3.68	−1.84	−1.65	−1.50	−1.37
	100	−4.53	−4.17	−3.89	−3.57	−1.82	−1.62	−1.47	−1.30
	200	−4.42	−4.09	−3.82	−3.52	−1.79	−1.58	−1.42	−1.23
	1000	−4.39	−4.10	−3.84	−3.54	−1.76	−1.54	−1.34	−1.11
	∞	−4.41	−4.09	−3.83	−3.55	−1.76	−1.53	−1.33	−1.11
0.9	50	−4.55	−4.15	−3.84	−3.52	−1.80	−1.63	−1.50	−1.38
	100	−4.43	−4.08	−3.78	−3.47	−1.76	−1.58	−1.44	−1.30
	200	−4.28	−3.98	−3.70	−3.41	−1.72	−1.53	−1.40	−1.25
	1000	−4.28	−3.96	−3.72	−3.43	−1.66	−1.46	−1.29	−1.09
	∞	−4.29	−3.95	−3.72	−3.42	−1.67	−1.45	−1.26	−1.06

Table 2.
Percentage Points of the Asymptotic Distribution of $T(\tilde{\alpha}-1)$
Two-step procedure for Model B.

λ	1.0%	2.5%	5.0%	10.0%	90.0%	95.0%	97.5%	99.0%
0.1	−34.27	−29.24	−24.90	−21.00	−3.61	−2.40	−1.48	−0.59
0.2	−37.02	−31.57	−27.45	−23.09	−4.44	−3.13	−2.04	−0.92
0.3	−37.89	−33.10	−29.10	−24.55	−5.41	−3.92	−2.70	−1.55
0.4	−38.95	−33.48	−29.60	−25.33	−6.39	−4.69	−3.47	−2.09
0.5	−38.80	−33.52	−29.94	−25.57	−6.99	−5.34	−3.99	−2.70
0.6	−38.42	−33.84	−29.74	−25.57	−7.19	−5.52	−4.34	−3.01
0.7	−38.02	−33.10	−28.97	−24.69	−6.77	−5.16	−4.09	−3.12
0.8	−37.26	−32.05	−27.97	−23.79	−5.96	−4.68	−3.79	−2.87
0.9	−35.19	−30.27	−26.07	−21.94	−5.16	−4.10	−3.37	−2.57

Table 3.
Percentage Points of the Asymptotic Distribution of $\inf_{\lambda\epsilon\Lambda} t_{\tilde{\alpha}}(\lambda)$.
Two step procedure for Model B.

1.0%	2.5%	5.0%	10.0%	90.0%	95.0%	97.5%	99.0%
−4.91	−4.62	−4.36	−4.07	−2.32	−2.12	−1.96	−1.78

Table 4.
Percentage Points of the Asymptotic Distribution
of $\inf_{\lambda\epsilon\Lambda} T(\tilde{\alpha}^i(\lambda)-1)$.

	1.0%	2.5%	5.0%	10.0%	90.0%	95.0%	97.5%	99.0%
Model A	−47.84	−42.51	−38.16	−33.87	−13.10	−11.36	−9.94	−8.40
Model B	−46.54	−41.10	−36.51	−31.89	−10.18	−8.47	−7.12	−5.92
Model C	−56.23	−50.11	−45.21	−40.17	−16.48	−14.39	−12.69	−11.00