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A NOTE ON THE COMBINATORIAL PRINCIPLES $\diamond(E)$

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ABSTRACT. Shelah has proved that $\diamond$ does not imply that $\diamond(E)$ holds for every stationary set $E \subseteq \omega_1$. We prove that, in the other direction, whenever $\diamond(E)$ holds there are disjoint stationary sets $F, G \subseteq E$ such that both $\diamond(F)$ and $\diamond(G)$ hold.

1. Introduction. Recall that if $E \subseteq \omega_1$, $\diamond(E)$ asserts the existence of a sequence $\langle S_\alpha | \alpha \in E \rangle$ such that $S_\alpha \subseteq \alpha$ and, whenever $X \subseteq \omega_1$, then the set $\{\alpha \in E | X \cap \alpha = S_\alpha\}$ is stationary. $\diamond$ is the principle $\diamond(\omega_1)$. For background information we refer the reader to our paper [**1**].

It was open for several years whether $\diamond$ implies that $\diamond(E)$ holds for any stationary set $E \subseteq \omega_1$. The main reason why it was thought by some that this was the case was that the proof of $\diamond$ from $V = L$ is almost identical to the proof of each instance of $\diamond(E)$ from $V = L$. However, it was finally proved by Shelah in [**3**] that it is possible for there to be disjoint stationary sets E and F such that $\diamond(E)$ holds (whence $\diamond$ holds, of course) and $\diamond(F)$ fails. Shelah's proof uses a new forcing technique. We were subsequently able to find a proof using the well-known technique of iterated Souslin forcing. Our proof appears in [**2**].

Now, in both the Shelah proof and our proof mentioned above, one fixes a pair E, F of disjoint stationary sets in advance and then force to obtain $\diamond(E)$ and $\neg\diamond(F)$ in a boolean extension, keeping E and F stationary. Hence the two proofs do not tell us whether $\diamond$ is strictly weaker than all *nontrivial* instances of $\diamond(E)$. Nontrivial? Well, it is clear that if $E \subseteq \omega_1$ contains a closed and unbounded set, then $\diamond(E)$ and $\diamond$ are equivalent. But what if $E \subseteq \omega_1$ is both stationary and co-stationary? This is what we mean by the nontrivial case. In this paper we show that $\diamond$ does in fact imply many "nontrivial" instances of $\diamond(E)$.

2. The result. Our proof depends upon the following result, which has been known to us for many years.

Let $\mathcal{I}$ denote the set of all subsets, E, of ω_1 for which $\diamond(E)$ fails.

2.1 LEMMA. *$\mathcal{I}$ is a countably complete ideal on ω_1.*

PROOF. Clearly, if $E \in \mathcal{I}$ and $F \subseteq E$, then $F \in \mathcal{I}$. We show that if

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$$E = \bigcup_{n=0}^{\infty} E_n,$$

where $E_n \in \mathcal{I}, n = 0, 1, 2, \ldots,$ then $E \in \mathcal{I}$.

Let $\langle S_\alpha | \alpha \in E \rangle$ be such that $S_\alpha \subseteq \alpha$. We show that $\langle S_\alpha | \alpha \in E \rangle$ cannot be a $\Diamond(E)$ sequence, which proves the lemma, of course.

Fix some bijection

$$j: \omega_1 \times \omega \leftrightarrow \omega_1,$$

such that whenever $\alpha \in \omega_1$ is a limit ordinal, then

$$j \upharpoonright (\alpha \times \omega): \alpha \times \omega \leftrightarrow \alpha.$$

For $\alpha \in E, n \in \omega$, set

$$S_\alpha^n = \{\xi \in \alpha | j(\xi, n) \in S_\alpha\}.$$

Since $E_n \in \mathcal{I}$, $\langle S_\alpha^n | \alpha \in E_n \rangle$ is not a $\Diamond(E_n)$-sequence, so we can find a set $X_n \subseteq \omega_1$ and a closed unbounded set $C_n \subseteq \omega_1$ such that

(i) $\alpha \in C_n \Rightarrow \lim(\alpha)$;

(ii) $\alpha \in C_n \cap E_n \to X_n \cap \alpha \neq S_\alpha^n$.

Let $C = \bigcap_{n=0}^{\infty} C_n$. Then C is closed and unbounded in ω_1 and:

(iii) $\alpha \in C \to \lim(\alpha)$;

(iv) $\alpha \in C \cap E_n \to X_n \cap \alpha \neq S_\alpha^n$.

Define $X \subseteq \omega_1$ by $X = \{j(\xi, n) | \xi \in X_n\}$. We complete the proof by showing that

$$\alpha \in C \cap E \to X \cap \alpha \neq S_\alpha.$$

Let $\alpha \in C \cap E$. Pick n so that $\alpha \in E_n$. Suppose that $X \cap \alpha = S_\alpha$. Then, since $\lim(\alpha)$,

$$\begin{aligned} S_\alpha^n &= \{\xi \in \alpha | j(\xi, n) \in S_\alpha\} = \{\xi \in \alpha | j(\xi, n) \in X \cap \alpha\} \\ &= \{\xi \in \omega_1 | j(\xi, n) \in X\} \cap \alpha \\ &= X_n \cap \alpha, \quad \text{contrary to (iv).} \quad \square \end{aligned}$$

REMARK. Shelah has observed that $\mathcal{I}$ is in fact a normal ideal.

2.2 THEOREM. *Assume $\Diamond(E)$. Then there are disjoint stationary sets $F, G \subseteq E$ such that $\Diamond(F)$ and $\Diamond(G)$ both hold.*

PROOF. Let

$$\mathcal{I}_E = \{E \cap F | F \in \mathcal{I}\}.$$

By 2.1, $\mathcal{I}_E$ is a countably complete ideal on E. Since $\Diamond(E)$ holds, $\mathcal{I}_E$ is clearly nonprincipal. Since ω_1 is not a measurable cardinal (i.e. since we know that ω_1 cannot carry a nonprincipal, countably complete *prime* ideal) there must be a set $F \subseteq E$ such that $F, E - F \notin \mathcal{I}_E$. Thus $\Diamond(F)$ and $\Diamond(E - F)$ hold. $\square$

We finish with two remarks. Firstly, since no countably complete ideal on ω_1 can be $\aleph_1$-saturated, the above proof shows that, in fact, $\Diamond$ implies the existence of a family $E_\alpha, \alpha < \omega_1$, of disjoint stationary sets such that $\Diamond(E_\alpha)$

holds for each α. Secondly, we may replace ω_1 in Theorem 2.2 by any uncountable regular cardinal κ. The proof is the same except when κ is a measurable cardinal. In this case we use the fact that no κ-complete prime ideal on κ can be second-order definable.

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