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A NOTE ON THE CONTINUITY OF THE INVERSE

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In his article [2] Wallace mentions the following problem: let X be an algebraic group with a locally compact Hausdorff topology such that the map of $X \times X$ into X which takes (x, y) into xy for all $x, y \in X$ is continuous. Then is X a topological group? The purpose of this note is to answer this question in the affirmative.

Lemma 1 is an immediate consequence of the continuity of multiplication, and the proof of Lemma 2 appears in [1]. The proofs of these lemmas will therefore be omitted.

LEMMA 1. *Let F be a filter on X such that $F \to x$ and $F^{-1} \to y$. Then $y \equiv x^{-1}$.*

LEMMA 2. *Let A be a compact subset of X. Then A^{-1} is closed.*

LEMMA 3. *Let E be a countable subset of X, and let x be a limit point of E. Then x^{-1} is a limit point of E^{-1}.*

PROOF. There is an ultra filter base $\mathfrak{U}$ on E such that $\mathfrak{U} \to x$. By Lemma 1 it is sufficient to show that there is $y \in X$ such that $\mathfrak{U}^{-1} \to y$. To this end it will be shown that there is a compact set C and a set $U \in \mathfrak{U}$ such that $U^{-1} \subset C$.

Let $B = E \cup \{x\}$ and $D = \bigcup_{n=-\infty}^{\infty} B^n$. Then D is a countable subgroup of X. Furthermore, if $A = \overline{D}$, then the continuity of multiplication implies that $A^2 \subset A$.

Now let V be a compact neighborhood of the identity. Then $\overline{D} = A$ implies that $A \subset DV^{-1}$. Thus $A = \bigcup [dV^{-1} \cap A \mid d \in D] = \bigcup [d(V^{-1} \cap A) \mid d \in D]$ since D is a group and $A^2 \subset A$. But $d(V^{-1} \cap A)$

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is closed for every $d \in D$ by Lemma 2. Moreover, A is a closed subset of a locally compact space and hence locally compact. This implies that the interior relative to A of one of the sets $d(V^{-1} \cap A)$ is not null. Hence there is an open set N of X and an element d of D such that $\emptyset \neq N \cap A \subset d(V^{-1} \cap A)$. Since $\bar{D} = A$, there exists $c \in D \cap N$. Thus $xc^{-1}(N \cap A) = xc^{-1}N \cap A$ is a neighborhood of x relative to A. Since $\mathfrak{U} \to x$, and $\mathfrak{U}$ is an ultra filter base on A, there exists $U \in \mathfrak{U}$ such that $U \subset xc^{-1}(N \cap A) \subset xc^{-1}dV^{-1}$. This implies that $U^{-1} \subset Vd^{-1}cx^{-1}$ which is compact. The proof is completed.

LEMMA 4. *Let A be a compact subset of X. Then A^{-1} is compact.*

PROOF. By Lemma 2 A^{-1} is closed. The proof will be completed by showing that A^{-1} can be covered by a finite number of translates of an arbitrary compact neighborhood, V, of the identity.

Assume this claim false. Then there is a sequence $\{x_n^{-1}\}$ contained in A^{-1} such that $x_n^{-1} \notin \bigcup [x_i^{-1}V \mid i = 1, \cdots, n-1]$. Set $E_n = [x_k \mid k \geqq n]$. By the compactness of A, there exists $x \in \bigcap [\bar{E}_n \mid n = 1 \cdots]$. Let U be a neighborhood of the identity such that $U^2 \subset V$. Since $x \in \bar{E}_1$, there is $x_m \in Ux$, whence $x^{-1} \in x_m^{-1}U$. Moreover $x \in \bar{E}_{m+1}$ implies by Lemma 3 that $x^{-1} \in E_{m+1}^{-1}$. Thus there is $n > m$ such that $x_n^{-1} \in x^{-1}U^2 \subset x_m^{-1}U^2 \subset x_m^{-1}V$, which contradicts the choice of x_n^{-1}.

THEOREM. *Let X be an algebraic group with a locally compact Hausdorff topology such that multiplication is continuous. Then X is a topological group.*

PROOF. Let U be an open neighborhood of the identity e. Let $\mathfrak{C}$ be the collection of compact neighborhoods of e. Then it must be shown that there exists $V \in \mathfrak{C}$ such that $V^{-1} \subset U$. Suppose this is not the case, i.e. $V^{-1} \cap U' \neq \emptyset$ for all $V \in \mathfrak{C}$. By Lemma 4 the family $(V^{-1} \cap U' / V \in \mathfrak{C})$ consists of compact sets. Since this family also has the finite intersection property, $\bigcap [V^{-1} \cap U' / V \in \mathfrak{C}] \neq \emptyset$. But $e = \bigcap [V^{-1} / V \in \mathfrak{C}] \supset \bigcap [V^{-1} \cap U' / V \in \mathfrak{C}]$ implies that $e = \bigcap [V^{-1} \cap \mathfrak{U}' / V \in \mathfrak{C}]$. This means in particular that $e \in U'$, whichis a contradiction. The proof is completed.

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