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A Note on the Diffusion of Directed Polymers in a Random Environment

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Abstract. A simple martingale argument is presented which proves that directed polymers in random environments satisfy a central limit theorem for $d \geqq 3$ and if the disorder is small enough. This simplifies and extends an approach by J. Imbrie and T. Spencer.

1. Introduction

In a recent paper, Imbrie and Spencer [1] considered the following model of a random walk in a random environment. Let $\xi(t)$, $t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be an ordinary symmetric random walk on $\mathbf{Z}^d$ starting in 0 and let $h(t,x)$, $t \in \mathbb{N}$, $x \in \mathbf{Z}^d$, be i.i.d. random variables which are $+$ or -1 with probability 1/2 and also independent of ξ. We denote by $\langle\ \rangle$ the expectation with respect to ξ and by $E(\,.\,)$ the expectation with respect to the h-variables. Let $0 < \varepsilon < 1$ be fixed and for $T \in \mathbb{N}$,

$$\kappa(T) = \prod_{j=1}^{T} (1 + \varepsilon h(j, \xi(j))) \ .$$

Imbrie and Spencer proved the following result by a rather elaborate expansion technique:

Theorem 1. *If $\varepsilon > 0$ is small enough and $d \geqq 3$, then*

$$\lim_{T \to \infty} \langle |\xi(T)|^2 \kappa(T) \rangle / T \langle \kappa(T) \rangle = 1 \quad \textit{almost surely}$$

(here $|\ |$ is the Euclidean norm).

We give here a very simple proof based on martingale limit theorems. The result in [1] is somewhat stronger and includes also a convergence rate. Such rates can also be obtained by the method presented here. An inspection of the proof reveals that the convergence rate is $O(T^{-\delta})$ almost surely for $\delta < (d-2)/4$. Theorem 1 is a special case of a more general result which includes the central limit theorem which seems to be new. Let $\xi_1(T), \ldots, \xi_d(T)$ be the components of the random walk.

Theorem 2. *If* $\varepsilon>0$ *is small enough and* $d\geqq 3$, *then for all* $n_1,\ldots,n_d\in\mathbb{N}_0$,

$$\lim_{T\to\infty}\left\langle\prod_{j=1}^{d}\left(\frac{\xi_j(T)}{\sqrt{T}}\right)^{n_j}\kappa(T)\right\rangle\Big/\langle\kappa(T)\rangle=\prod_{j=1}^{d}\gamma(n_j)d^{-n_j/2}\quad \textit{almost surely}\ ,$$

where $\gamma(n)=0$ *if* n *is odd*, $\gamma(0)=1$, *and* $\gamma(2k)=1\cdot 3\cdot\ldots\cdot(2k-1)$.

This implies a central limit theorem. For a given realisation of the h variables, we define the probability measure μ_h^T on $\mathbb{R}^d$ by

$$\mu_h^T(A)=\langle 1_A(\xi(T)/\sqrt{T})\kappa(T)\rangle/\langle\kappa(T)\rangle\ .$$

Theorem 2 implies the

Corollary. *For almost all* h, μ_h^T *converges to the centered normal law with covariance matrix* $1/d$ *times the identity matrix.*

2. Proof

Let $\mathbf{F}_t$ be the σ-field generated by the variables $h(s,x)$, $s\leqq t$, $x\in\mathbf{Z}^d$.

Lemma 1. $\langle\kappa(t)\rangle$ *is a nonnegative* $(\mathbf{F}_t)$-*martingale satisfying* $E(\langle\kappa(t)\rangle)=1$.

Proof. $E(\langle\kappa(t)\rangle)=1$ is obvious and

$$\begin{aligned}E(\langle\kappa(t)\rangle|\mathbf{F}_{t-1})&=(2d)^{-t}\sum_{\substack{\omega:0\to\\|\omega|=t}}E\left(\prod_{j=1}^{t}(1+\varepsilon h(j,\omega(j)))|\mathbf{F}_{t-1}\right)\\&=(2d)^{-t+1}\sum_{\substack{\omega:0\to\\|\omega|=t-1}}\prod_{j=1}^{t-1}(1+\varepsilon h(j,\omega(j)))=\langle\kappa(t-1)\rangle\ .\end{aligned}$$

The summation is over nearest neighbor paths, $\omega=(\omega(0),\omega(1),\ldots,\omega(s))$. $\omega:0\to$ stands for $\omega(0)=0$, and $|\omega|$ is the length s.

Lemma 2. $\langle\kappa(t)\rangle$ *converges* a.s. *to a random variable* ζ *satisfying*

$$E(\zeta)=1\quad\textit{and}\quad P(\zeta=0)=0\ .$$

Proof. $\langle\kappa(t)\rangle$ converges a.s. by the martingale limit theorem (see e.g. [2, Theorem II-2–9]), say to ζ.

We consider two independent copies of the random walk $\xi^{(1)}$, $\xi^{(2)}$ with corresponding quantities

$$\kappa^{(i)}(t)=\prod_{j=1}^{t}(1+\varepsilon h(j,\xi^{(i)}(j)))\ .$$

The h variables remain independent of $\xi^{(1)}$ and $\xi^{(2)}$. Then

$$\begin{aligned}E(\langle\kappa(t)\rangle^2)&=E(\langle\kappa^{(1)}(t)\rangle\langle\kappa^{(2)}(t)\rangle)=E(\langle\kappa^{(1)}(t)\kappa^{(2)}(t)\rangle)\\&=\left\langle E\left(\prod_{j=1}^{t}(1+\varepsilon h(j,\xi^{(1)}(j)))(1+\varepsilon h(j,\xi^{(2)}(j)))\right)\right\rangle\\&=\langle(1+\varepsilon^2)^{n_t(\xi^{(1)},\xi^{(2)})}\rangle\ ,\end{aligned}$$

where

$$n_t(\xi^{(1)}, \xi^{(2)}) = \sum_{s=1}^{t} 1_{\xi^{(1)}(s)=\xi^{(2)}(s)} \leqq n_\infty(\xi^{(1)}, \xi^{(2)}) \ .$$

The law of n_∞ is the same as the number of visits of a single random walk to 0 (of course, not a single nearest neighbor random walk but nevertheless, one with symmetric jump distribution). A random walk in dimension $d \geqq 3$ has after every visit to 0 a positive probability of never returning to 0. Therefore, n_∞ has an exponential moment. So it follows that for small enough $\varepsilon > 0$,

$$\sup_t E(\langle \kappa(t) \rangle^2) < \infty \ .$$

We can conclude that $\langle \kappa(t) \rangle$ converges to ζ in L_2 and L_1 (see [2, Proposition IV-2–7]). Therefore, $E(\zeta) = 1$ and from this, we see that $P(\zeta = 0) \neq 1$. It is easy to see that the event $\{\zeta = 0\}$ belongs to the tail field

$$\bigcap_t \sigma(h(s, x) : s \geqq t, x \in \mathbf{Z}^d)$$

(although ζ is certainly not tail measurable!). To see this, we write for $T > t$,

$$\langle \kappa(T) \rangle = (2d)^{-t} \sum_x \sum_{\substack{\omega: 0 \to x \\ |\omega| = t}} \prod_{s=1}^{t} (1 + \varepsilon h(s, \omega(s)))(2d)^{-T+t}$$

$$\cdot \sum_{\substack{\omega: x \to \\ |\omega| = T-t}} \prod_{s=1}^{T-t} (1 + \varepsilon h(t+s, \omega(s))) \ ,$$

where the sum over x extends to those reachable from 0 in t steps. This converges to 0 for $T \to \infty$ if and only if the second part converges to 0 for any x reachable from 0 in t steps. Therefore, $\{\zeta = 0\}$ is a tail event and by Kolmogoroffs $0-1$-law and from $P(\zeta = 0) \neq 1$ it follows that $P(\zeta = 0) = 0$, proving the lemma.

We create now a whole family of new martingales. If $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$, let

$$\varrho(\lambda) = \frac{1}{d} \sum_{j=1}^{d} \cosh(\lambda_j) \ .$$

It is well-known (and obvious) that

$$\exp\left(\sum_{j=1}^{d} \lambda_j \xi_j(t) - t \log \varrho(\lambda) \right)$$

is a martingale with respect to the filtration of the random walk (no h-variables are involved). This remains true when $\xi(t)$ is replaced by a more general d-dimensional random walk $\sum_{j=1}^{t} X(j)$, where $X(j)$ are i.i.d. with $\varrho(\lambda) = \langle \exp(\lambda \cdot X) \rangle < \infty$ for λ in a neighborhood of 0 in $\mathbb{R}^d$.

If $n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$, the polynomial $W_n(t, x)$ is defined by

$$\frac{\partial^{|n|}}{\partial \lambda_1^{n_1} \ldots \partial \lambda_d^{n_d}} \exp\left(\sum_{j=1}^{d} \lambda_j x_j - t \log \varrho(\lambda) \right) \Bigg|_{\lambda = 0} \ ,$$

where $|n|=n_1+n_2+\ldots+n_d$. We write

$$W_n(t,x)=\sum A_n(i_1,\ldots,i_d,j)x_1^{i_1}\ldots x_d^{i_d}t^j \ .$$

The coefficients A depend on the derivatives of $\log\varrho$ in 0.

Lemma 3. *For a general random walk with $\varrho(\lambda)<\infty$ for λ in a neighborhood of* 0 *and* $\langle X(j)\rangle=0$, *we have*
a) *if $i_1+\ldots+i_d+2j>|n|$, then $A_n(i_1,\ldots,i_d,j)=0$.*
b) *The coefficients with $i_1+\ldots+i_d+2j=|n|$ depend on the second derivatives of* $\log\varrho$ *at* 0.
c) *If $i_1+\ldots+i_d=|n|$, then $A_n(i_1,\ldots,i_d,0)=\delta_{i_1n_1}\delta_{i_2n_2}\ldots\delta_{i_dn_d}$.*

Proof. a) and c) are obvious and b) follows from the fact that $\partial\varrho/\partial\lambda_j$ at $\lambda=0$ equals 0.

$W_n(t,\xi(t))$ is a martingale for the filtration of the random walk, i.e.

$$\langle W_n(t,\xi(t))|\xi(s),\ s\leqq t-1\rangle=W_n(t-1,\xi(t-1)) \ .$$

Here $\langle\ |\xi(s),s\leqq t-1\rangle$ denotes conditional expectation given the path up to time $t-1$. Coming back to our special symmetric random walk, it follows that

$$Y_n(t)=\langle W_n(t,\xi(t))\kappa(t)\rangle$$

is a $(\mathbf{F}_t)$-martingale.

Lemma 4. *If $|n|\geqq 1$ then*

$$\lim_{t\to\infty} t^{-|n|/2}Y_n(t)=0 \quad \textit{almost surely} \ .$$

Proof. We show that the martingale

$$\sum_{s=1}^{t} s^{-|n|/2}(Y_n(s)-Y_n(s-1))$$

remains L_2-bounded. From this, it follows that it converges almost surely and from the Kronecker-lemma Lemma 4 follows:

$$\begin{aligned}E(\langle W(t,\xi)\kappa(t)-W(t-1,\xi)\kappa(t-1)\rangle^2)&=E(\langle W(t,\xi)\varepsilon\kappa(t-1)h(t,\xi(t))\rangle^2)\\&=E(\varepsilon^2\langle W(t,\xi^{(1)})\kappa^{(1)}(t-1)h(t,\xi^{(2)}(t))\,W(t,\xi^{(2)})\kappa^{(2)}(t-1)h(t,\xi^{(2)}(t))\rangle) \ ,\end{aligned}$$

where $\xi^{(i)}$, $\kappa^{(i)}$ are as in the proof of Lemma 2 and we drop the index n for convenience. The above expression equals

$$\begin{aligned}&\varepsilon^2\langle W(t,\xi^{(1)})\,W(t,\xi^{(2)})\,(1+\varepsilon^2)^{n_{t-1}(\xi^{(1)},\xi^{(2)})}1_{\xi^{(1)}(t)=\xi^{(2)}(t)}\rangle\\&\leqq\varepsilon^2\langle W(t,\xi)^8\rangle^{1/4}\langle(1+\varepsilon^2)^{8n_\infty(\xi^{(1)},\xi^{(2)})}\rangle^{1/8}P(\xi^{(1)}(t)=\xi^{(2)}(t))^{3/4} \ .\end{aligned}$$

$P(\xi^{(1)}(t)=\xi^{(2)}(t))^{3/4}$ is of order $(t^{-d/2})^{3/4}\leqq t^{-9/8}$ and

$$\langle(1+\varepsilon^2)^{8n_\infty}\rangle$$

is finite for small enough $\varepsilon>0$.

Therefore, in order to show that

$$\sup_t E\left(\sum_{s=1}^{t} s^{-|n|/2}(Y_n(s)-Y_n(s-1))\right)^2 = \sup_t \sum_{s=1}^{t} s^{-|n|} E((Y_n(s)-Y_n(s-1))^2) < \infty \ ,$$

it suffices to prove

$$\langle W(t,\xi)^8 \rangle = 0(t^{4|n|}) \ .$$

This is obvious from Lemma 3a).

Proof of Theorem 2

The theorem is a consequence of Lemma 2–4. By induction, it follows from Lemma 3a), 3c), and 4 that

$$\sup_t \left\langle \prod_{j=1}^{d} \left(\frac{\xi_j(t)}{\sqrt{t}}\right)^{n_j} \kappa(t) \right\rangle < \infty \quad \text{almost surely} \ . \tag{2.1}$$

We introduce the polynomial $U_n(t,x)$ by deleting from W_n all summands

$$A(i_1,\dots,i_d,j)x_1^{i_1}\dots x_d^{i_d}t^j$$

with $i_1+\dots+i_d+2j<|n|$. We conclude from (2.1) and Lemma 4 that for $|n|\geqq 1$,

$$\lim_{t\to\infty} t^{-|n|/2}\langle U_n(t,\xi(t))\kappa(t)\rangle = 0 \quad \text{almost surely} \ ,$$

i.e.

$$\lim_{t\to\infty} \left\langle \sum_{i_1,\dots,i_d} A_n\left(i_1,\dots,i_d, \frac{|n|-i_1-\dots-i_d}{2}\right)\left(\frac{\xi_1(t)}{\sqrt{t}}\right)^{i_1}\dots\left(\frac{\xi_d(t)}{\sqrt{t}}\right)^{i_d}\kappa(t)\right\rangle = 0$$
$$\text{almost surely} \ , \tag{2.2}$$

where the sum extends over those $i_1,\dots,i_d$ with $|n|-i_1-\dots-i_d \geqq 0$ and even. Using Lemma 2, the theorem follows by induction. This can be seen by looking at

$$0 = \frac{\partial^{|n|}}{\partial\lambda_1^{n_1}\dots\partial\lambda_d^{n_d}} \left\langle \exp\left(\sum_{j=1}^{d}\lambda_j X_j - \frac{1}{2d}\sum_{j=1}^{d}\lambda_j^2\right)\right\rangle \ ,$$

where $X_1,\dots,X_d$ are i.i.d. normally distributed random variables with mean 0 and variance $1/d$. Because of Lemma 3b) this gives

$$\left\langle \sum A_n\left(i_1,\dots,i_d,\frac{|n|-i_1-\dots-i_d}{2}\right) X_1^{i_1}\dots X_d^{i_d}\right\rangle = 0 \ .$$

Comparing this with (2.2), the theorem follows.

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