A note on the effect of surface contamination in water wave damping

By JOSÉ A. NICOLÁS AND JOSÉ M. VEGA
ETSI Aeronáuticos, Universidad Politécnica de Madrid, Plaza Cardenal Cisneros, 3, 28040 Madrid, Spain

Asymptotic formulas are derived for the effect of contamination on surface wave damping in a brimful circular cylinder; viscosity is assumed to be small and contamination is modelled through Marangoni elasticity with insoluble surfactant. It is seen that an appropriately chosen finite Marangoni elasticity provides an explanation for a significant amount of the unexplained additional damping rate in a well-known experiment by Henderson & Miles (1994); discrepancies are within 15%, significantly lower than those encountered by Henderson & Miles (1994) under the assumption of inextensible film.

1. Introduction

The precise theoretical prediction (and explanation) of water wave damping has been a long standing open problem (Henderson & Miles 1994, hereinafter referred to as HM94). Its solution is essential for various purposes, including the appropriate modelling of weakly-nonlinear water wave dynamics. Three sources of damping are readily identified:

(*a*) *Viscous dissipation in both the oscillatory boundary layers and the bulk*. The latter is second order as viscosity goes to zero and was systematically ignored in the literature. But for the usual small-but-fixed values of viscosity this effect must not be ignored if the contact line is pinned, as recently shown by Martel, Nicolás & Vega (1998, hereinafter referred to as MNV), who obtained results in good agreement with the experiments in HM94; see also Howell *et al.* (2000) for further comparisons.

(*b*) *Contact line dynamics*, which is a not-well-understood effect and is modelled by phenomenological formulas (see, e.g., HM94) and is avoided by pinning the contact line.

(*c*) *Surface contamination*, again a not-well-understood effect that is most likely to be present in water unless much care is taken in the experimental setup. It is modelled by phenomenological formulas (Dorrestein 1951; Levich 1962; Miles 1967) that also apply to thin films of highly viscous Newtonian fluids on the surface (Jenkins & Dysthe 1997).

For pinned contact line, HM94 assumed that the free surface is inextensible and obtained results whose error ranged in the interval 20–80%, depending on the mode. Following MNV, Miles & Henderson (1998) included the effect of viscous dissipation in the bulk to obtain slightly better, but still not-good-enough results. The error was larger for lower modes, and this point is essential to anticipate what must be done to obtain better results, namely to take a finite Marangoni elasticity. This is known to be a good candidate for additional damping, as first shown by Dorrestein

(1951), but it was disregarded in HM94 after a too-simple asymptotic estimate of the role of finite elasticity for high-order modes, which led them to expect that it would increase with the wavenumber. But the opposite is true if the wavelength is large compared to the capillary length (which was the case in the experiment), as seen by a careful look at well-known formulas for free contact lines (Miles 1967, 1991), which should also give the qualitative behaviour for the fixed-contact-line case. Similar asymptotically correct formulas for fixed contact lines are not available and will be derived below. These could be useful for safe comparison with experiments and thus help to elucidate the role of contamination (and, more generally, of surfactants) on surface wave damping.

For simplicity we consider a circular cylinder of radius R and depth d, but the results below are straightforwardly extended to other geometries. As in MNV, we use R and the gravitational time $(R/g)^{1/2}$ for non-dimensionalization, linearize around the quiescent state and make the usual mode decomposition, $(\boldsymbol{u}, w, p, f) = (\boldsymbol{u}, w, p, f)\exp(\Omega t)$, where $\boldsymbol{u}$ and w are the horizontal and vertical velocity components, p is the pressure and f is the free-surface deflection, to obtain

$$\nabla \cdot \boldsymbol{u} + w_z = 0, \tag{1.1}$$

$$\Omega \boldsymbol{u} = -\nabla p + C(\Delta \boldsymbol{u} + \boldsymbol{u}_{zz}), \qquad \Omega w = -p_z + C(\Delta w + w_{zz}), \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{0}, \qquad w = 0 \quad \text{at } z = -\Lambda \quad \text{and} \quad \text{at } r = 1, \tag{1.3}$$

$$w = \Omega f, \qquad p - f + B^{-1}\Delta f = 2Cw_z \quad \text{at } z = 0, \tag{1.4}$$

$$C^{1/2}(\boldsymbol{u}_z + \nabla w) = (\gamma/\Omega)\nabla(\nabla \cdot \boldsymbol{u}) \quad \text{at } z = 0, \tag{1.5}$$

$$f = 0 \quad \text{at } r = 1, \qquad \int_0^{2\pi}\int_0^1 f(r,\theta) r \, \mathrm{d}r \, \mathrm{d}\theta = 0, \tag{1.6}$$

where ∇, $\nabla\cdot$ and Δ are the *horizontal* gradient, divergence and Laplacian operators, and we use a polar coordinate system in the horizontal plane, with associated unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$. The problem depends on the slenderness $\Lambda = d/R$, the gravitational Reynolds number $C^{-1} = (gR^3)^{1/2}/\nu$, the Bond number $B = \rho g R^2/\sigma$ and the non-dimensional Marangoni elasticity of the free surface γ, defined as

$$\gamma = \Gamma_0(\mathrm{d}\sigma/\mathrm{d}\Gamma_0)C^{1/2}/(\mu\sqrt{gR}), \tag{1.7}$$

where $\Gamma_0(\mathrm{d}\sigma/\mathrm{d}\Gamma_0)$ is a dimensional measure of Marangoni elasticity in terms of the surface tension σ (denoted in Henderson 1998 as π) and the surfactant concentration Γ_0. For convenience we scale γ with $C^{1/2}$ and consider the limit

$$C \to 0, \qquad B^{-1} = O(1), \tag{1.8}$$

which is realistic for water (and many liquids) except for containers with millimetric depth.

The only requirement for the validity of the linear approximation above is that the steepness of the surface wave is small, as seen when using the strained coordinate $\eta = (z - F)\Lambda/(\Lambda + F)$, where F is the free-surface deflection. In particular, F can be large compared to the thickness of the free surface boundary layer (see below), even though the unperturbed free surface $z = 0$ is outside this boundary layer in the original variables.

2. Damping rate and frequency in a brimful circular cylinder

In the limit (1.8) the solutions of (1.1)–(1.6) exhibit oscillatory boundary layers near the solid walls and the free surface, whose thicknesses are of the order of $C^{1/2}$. The eigenvalue Ω and the associated eigenfunctions in the bulk (outside the boundary layers) are expanded as

$$\Omega = \Omega_0 + C^{1/2}\Omega_1 + C\Omega_2 \cdots, \tag{2.1}$$

$$(\boldsymbol{u}, w, p, f) = (\boldsymbol{u}_0, w_0, p_0, f_0) + C^{1/2}(\boldsymbol{u}_1, w_1, p_1, f_1) + \cdots. \tag{2.2}$$

The leading-order terms are given by the usual strictly inviscid problem, which is solvable in semi-analytical form (HM94; MNV). Marangoni elasticity produces a jump in the tangential velocity across the free-surface boundary layer (instead of the jump in tangential stress encountered for clean surfaces), where the solution is

$$\boldsymbol{u} = \boldsymbol{u}_0^0 + (\tilde{\boldsymbol{u}}_0^0 - \boldsymbol{u}_0^0)\exp(\Omega_0^{1/2}\eta) + O(C^{1/2}), \tag{2.3}$$

in terms of the stretched coordinate $\eta = z/C^{1/2}$. Here $\tilde{\boldsymbol{u}}_0^0$ is the horizontal velocity at the free surface $\eta = 0$ and $\boldsymbol{u}_0^0$ is the horizontal velocity of the outer inviscid flow at $z = 0$. To leading order, (1.5) and (2.3) yield

$$\gamma\nabla(\nabla\cdot\tilde{\boldsymbol{u}}_0^0) = \Omega_0^{3/2}(\tilde{\boldsymbol{u}}_0^0 - \boldsymbol{u}_0^0) \quad \text{if } 0 \leqslant r < 1, \qquad \tilde{\boldsymbol{u}}_0^0\cdot\boldsymbol{e}_r = 0 \quad \text{at} \quad r = 1. \tag{2.4}$$

Since (see MNV)

$$\boldsymbol{u}_0^0 = -\sum_{n=0}^{\infty}\frac{a_n[\lambda_n J_m'(\lambda_n r)\boldsymbol{e}_r + mr^{-1}J_m(\lambda_n r)\boldsymbol{e}_\theta]\exp(\mathrm{i}m\theta)}{\Omega_0 J_m(\lambda_n)}, \tag{2.5}$$

where m is the azimuthal wavenumber of the eigenmode considered, $\lambda_0 = 0$, $\lambda_1, \ldots,$ are the roots of $J_m'(\lambda_n) = 0$ and a_n is as defined in MNV (in (A15*a*)–(A15*b*)), the solution of (2.4) is readily calculated as

$$\tilde{\boldsymbol{u}}_0^0 = -\Omega_0^{1/2}\sum_{n=0}^{\infty}\frac{a_n[\lambda_n J_m'(\lambda_n r)\boldsymbol{e}_r + mr^{-1}J_m(\lambda_n r)\boldsymbol{e}_\theta]\exp(\mathrm{i}m\theta)}{(\Omega_0^{3/2} + \gamma\lambda_n^2)J_m(\lambda_n)}. \tag{2.6}$$

Note that $\tilde{\boldsymbol{u}}_0^0$ and $\boldsymbol{u}_0^0$ are not proportional to each other because $\nabla(\nabla\cdot\boldsymbol{u}_0^0)$ is not proportional to $\boldsymbol{u}_0^0$. Then the simpler boundary condition suggested in HM94 (p. 286, (1.2)) does not apply for pinned ends and must be taken as a phenomenological one; but the computations for inextensible film in HM94 and Miles & Henderson (1998) do not rely on this boundary condition, and coincide with the one derived below in the limit $\gamma \to \infty$.

In order to calculate Ω_1 and Ω_2 we use the following *solvability condition*, which is the natural extension of that introduced in MNV:

$$(\Omega - \Omega_0)I_1 = -I_2 - I_3 - I_4, \tag{2.7}$$

where I_1, I_2 and I_3 are as defined and calculated in MNV and

$$I_4 = -C^{1/2}\Omega_0^{1/2}\int_0^{2\pi}\int_0^1 \bar{\boldsymbol{u}}_0^0\cdot(\tilde{\boldsymbol{u}}_0^0 - \boldsymbol{u}_0^0)r\,\mathrm{d}r\,\mathrm{d}\theta + O(C) \tag{2.8}$$

accounts for the effect of contamination. Note that the solid walls and the free surface are inside the boundary layers, and thus the velocity components u and v must be taken from the solution in the boundary layers; then $|I_3|$ and $|I_4|$ are seen to be of order $C^{1/2}$, while $|I_2| \sim C$ and $|I_1| \sim 1$. When (2.1) is substituted into (2.7) we obtain

$C^{1/2}\Omega_1 + C\Omega_2 + \cdots = -(I_2 + I_3 + I_4)/I_1$. At order $C^{1/2}$ this expression yields

$$\Omega_1 = \Omega_1^{\text{wall}} + \Omega_1^{\text{cont}}, \tag{2.9}$$

where Ω_1^{wall} comes from I_3 and exactly coincides with the $O(C^{1/2})$ term in MNV and $\Omega_1^{\text{cont}} = -I_4/I_1$; or invoking the expression for I_1 in MNV and using (2.5), (2.6) and (2.8),

$$\Omega_1^{\text{cont}} = -\left[\sum_{n=0}^{\infty}(\lambda_n^2 - m^2)\lambda_n^{-1}a_n^2 \tanh(\lambda_n \Lambda)\right]^{-1} \sum_{n=0}^{\infty} \frac{\gamma \Omega_0^{1/2}\lambda_n^2(\lambda_n^2 - m^2)a_n^2}{2(\Omega_0^{3/2} + \gamma\lambda_n^2)}. \tag{2.10}$$

The real part of $-\Omega_1^{\text{wall}}$ comes from viscous dissipation in the Stokes boundary layers attached to the walls, and the real part of $-\Omega_1^{\text{cont}}$ comes from viscous dissipation enhanced by contamination in both the boundary layer attached to the free surface and the free surface itself. Also, as $\gamma \to \infty$, $\tilde{\boldsymbol{u}}_0^0 \to \boldsymbol{0}$ (see (2.6)), and (2.10) provides the same result as that in HM94 for inextensible film.

As in MNV, the $O(C)$ correction in (2.1) can be written as

$$\Omega_2 = \Omega_2^{\text{bulk}} + \Omega_2^{\text{wall}} + \Omega_2^{\text{cont}}, \tag{2.11}$$

where $-\Omega_2^{\text{bulk}}$ and $-\Omega_2^{\text{wall}}$ are real, and come from viscous dissipation in the bulk and a correction to viscous dissipation in the wall boundary layers respectively; they exactly coincide with their counterparts calculated in MNV. Ω_2^{cont} comes from the $O(C)$ correction provided by I_4 and a part of I_2, and could be calculated by a (somewhat tedious) procedure like that followed in MNV to calculate Ω_2^{wall}. Here we only point out that $|\Omega_1^{\text{wall}}|$ and $-\Omega_2^{\text{wall}}$ are typically (i.e. except for small Λ) small compared to $-\Omega_2^{\text{bulk}}$; as explained in MNV, this is so because the boundary layers attached to the solid walls are fairly weak, namely the jump in tangential velocity across these layers is small (as compared to the velocity in the bulk). Thus, even though it is asymptotically inconsistent, if we ignore Ω_2^{wall} but retain Ω_2^{bulk} we obtain numerically good approximations (see MNV; Miles & Henderson 1998). But there is not a similar argument to retain Ω_2^{bulk} and neglect Ω_2^{cont} (which is also asymptotically inconsistent) because the boundary layer near the interface is no longer weak.

Here we have completely neglected the effects of surfactant solubility and surface viscosity, which are of independent interest and are accounted for by replacing the boundary condition (1.5) by

$$C^{1/2}(\boldsymbol{u}_z + \nabla w) = [\delta_1 + \gamma/(\Omega + \gamma_1\Omega^{1/2})]\nabla(\nabla \cdot \boldsymbol{u}) + \delta_2 \Delta \boldsymbol{u} \quad \text{at} \quad z = 0. \tag{2.12}$$

γ_1, δ_1 and δ_2 account for surfactant solubility and dilatational and shear surface viscosities respectively. The effects of γ_1 and δ_1 are readily taken into account by just replacing γ/Ω_0 by $\gamma^* = \delta_1 + \gamma/(\Omega_0 + \gamma_1\Omega_0^{1/2})$ in (2.4), (2.6) and (2.10). The effect of shear viscosity instead requires replacing the boundary condition in (2.4) by the no-slip condition $\tilde{\boldsymbol{u}}_0^0 = \boldsymbol{0}$ (and, of course, adding $\delta_2 \Delta \boldsymbol{u}$ to the left-hand side of the equation). The solution of the resulting two-dimensional problem for general δ_2 is much more involved and is omited here. But if $\delta_2 \ll 1$ this problem exhibits a Stokes-like boundary layer in the free surface near the contact line (where the azimuthal surface velocity component slows down to zero) that provides the leading-order effect of shear surface viscosity. The resulting expression for Ω_1^{cont} is obtained by just replacing in (2.10) the second factor on the right-hand side by

$$\sum_{n=0}^{\infty} \frac{\gamma^* \lambda_n^2(\lambda_n^2 - m^2)a_n^2}{2(\Omega_0^{1/2} + \gamma^*\lambda_n^2)} + \sqrt{\frac{\delta_2}{\delta^*}} \sum_{n=0}^{\infty} \frac{m^2\Omega_0^{1/2}a_n}{\Omega_0^{1/2} + \gamma^*\lambda_n^2} \sum_{n=0}^{\infty} \bar{a}_n + O\left(\frac{\delta_2}{|\delta^*|}\right), \tag{2.13}$$

	$\sigma = 66\,(72.4)\,\mathrm{dyn\,cm^{-1}}$ $\gamma = 0.8$ This paper			Experiment		$\sigma = 66\,\mathrm{dyn\,cm^{-1}}$ $\gamma = \infty$ Approximation by H & M		
(m, q)	f	Δ	Δ_E/Δ	f_E	Δ_E	f	Δ	Δ_E/Δ
(1, 0)	4.61 (4.66)	5.02 (5.01)	1.15 (1.15)	4.63	5.8	4.64	3.19	1.8
(2, 0)	6.20 (6.31)	6.70 (6.73)	1.15 (1.15)	6.19	7.7	6.29	4.70	1.6
(0, 1)	6.69 (6.80)	6.22 (6.31)	1.16 (1.14)	6.68	7.2	6.70	4.45	1.6
(3, 0)	7.59 (7.76)	7.96 (8.01)	1.02 (1.01)	7.62	8.1	7.73	6.06	1.3
(1, 1)	8.32 (8.51)	7.76 (7.82)	1.15 (1.14)	8.37	8.9	8.49	6.12	1.4
(4, 0)	8.93 (9.18)	9.21 (9.28)	1.02 (1.01)	8.96	9.4	9.13	7.34	1.3

TABLE 1. Comparison with Henderson & Miles' (1994) experiment and theoretical prediction; f is the dimensional frequency (in c.p.s.) and Δ is the non-dimensional (see (3.1)) damping ratio.

where γ^* is as defined above and $\delta^* = \Omega_0^{1/2} + m^2\delta_1 + m^2\gamma^*$. This approximation is of interest because surface dilatational viscosity may be several orders of magnitude larger than surface shear viscosity (Lopez & Hirsa 1998).

3. Comparison with the experiment by Henderson & Miles

In order to compare with HM94, we must consider the dimensional frequency f and non-dimensional damping rate Δ:

$$f = (g/R)^{1/2}[\mathrm{Im}\,(\Omega)/2\pi], \quad \Delta = 2\left[gR/(\pi\nu f_0)\right]^{1/2}(-\mathrm{Re}\,\Omega), \tag{3.1}$$

where $f_0 = (g/R)^{1/2}[\mathrm{Im}\,(\Omega)/2\pi]$ is the dimensional inviscid frequency. We must estimate the Marangoni elasticity parameter γ and the surface tension σ; we assume that the density and kinematic viscosity are not affected by contamination. The effect of σ is only felt through the Bond number B, and since B^{-1} is small, the results below are fairly insensitive to σ in the reasonable range $\sigma = 60$–$72.4\,\mathrm{dyn\,cm^{-1}}$. Then we select $\gamma = 0.8$ as the value that gives a better agreement for damping rates and $\sigma = 62\,\mathrm{dyn\,cm^{-1}}$ as the value of surface tension that gives a better agreement for frequencies. With those values of γ and σ and the known values of the remaining dimensional parameters ($\rho = 1\,\mathrm{g\,cm^{-3}}$, $\nu = 0.01\,\mathrm{cm^2\,s^{-1}}$, $R = 2.766\,\mathrm{cm}$, $d = 3.80\,\mathrm{cm}$), the non-dimensional parameters Λ, B and C are 1.372, 120.93 and 6.94×10^{-5} respectively, and our estimates for f and Δ are as given in table 1. In order to illustrate that the results are fairly independent of σ we also give (between parentheses) the frequencies and damping rates obtained for $\sigma = 72.4\,\mathrm{dyn\,cm^{-1}}$; the insensitiveness to σ gives us little confidence in our guess for surface tension, which should be estimated by other means. As we anticipated in § 1, the comparison in table 1 shows that our estimate of the damping rates (and thus the approximation with a finite Marangoni elasticity) is reasonably good and significantly improves the approximation by HM94, who considered an inextensible film. Also note that our estimate on the frequencies is quite good. The slightly better results in Henderson & Miles (1998, table 2), included viscous dissipation in the bulk, and should be compared with a similar approximation including both finite Marangoni elasticity and viscous dissipation in the bulk; but for the reasons explained at the end of § 3 (and remarked below, in (*a*)) we are not considering such an approximation.

Some remarks are now in order:

(*a*) The main source of damping in our theoretical results in table 1 comes from the term Ω_1^{cont}, accounting for contamination. Thus, when trying to proceed with higher-order, $O(C)$-terms, there is no reason to retain Ω_2^{bulk} (which was calculated in MNV) but neglect Ω_2^{cont} (which is not calculated here).

(*b*) The value of γ in table 1 was in fact selected as that giving the maximum value of $|\Omega_1^{\mathrm{cont}}|$ for the first ($(m, q) = (1, 0)$) mode, which was the one exhibiting the maximum discrepancy in the HM94 calculations. This maximum exceeds the value of $|\Omega_1^{\mathrm{cont}}|$ for inextensible film by a factor of 1.89 (instead of the factor 2 encountered for free contact lines, Miles 1967) and, when using our formulas at the end of § 2, is seen to decrease if either surfactant solubility or (dilatational or shear) surface viscosity is present; thus these three effects would worsen the fit if taken into account.

(*c*) When using (1.7) we readily obtain an estimate of the Marangoni elasticity coefficient, $\Gamma_0(\mathrm{d}\sigma/\mathrm{d}\Gamma_0) \simeq 50\,\mathrm{dyn\,cm^{-1}}$, which is somewat larger than that giving a large jump in damping rate in the experiment by Henderson (1998), which was (Henderson 1998, figure 3) 14, 21 and $7\,\mathrm{dyn\,cm^{-1}}$ for lecithin, oleyl alcohol and diolein respectively. According to the conclusions in Henderson (1998), a value of $50\,\mathrm{dyn\,cm^{-1}}$ would give a surface saturated with surfactant and thus an inextensible film in this experiment, while this value gives an extensible film in our case. We do not have an explanation for this discrepancy. In this connection, our formulas above could be used for comparison if this experiment were performed with a pinned contact line; in this case the effect of capillary hysteresis, added by Henderson (1998) to get a good fit, would be unnecessary.

(*d*) Since we are using a phenomenological formula to model contamination that is based on several ad hoc assumptions, we may wonder whether it makes any sense to look for quite precise results (beyond the precision in table 1). In any event, the results in table 1 show that Marangoni elasticity could be a significant factor in the large damping rates measured in HM94 and might provide the main source of damping in related contaminated water wave experiments, as already conjectured by Van Dorn (1966). If this were confirmed (by further comparison with experiments, which are not at present available) we would have safe ground on which to quantitatively model the weakly-nonlinear dynamics of these surface waves and of the associated streaming flow. The latter would be greatly affected by the free-surface boundary layer, whose structure depends dramatically on Marangoni elasticity.

Research partially supported by DGES and NASA, under Grants PB97-0556 and NAG3-2152. We are indebted to Professor John Miles for useful comments and suggestions on an earlier version of the note.

REFERENCES

DORRESTEIN, R. 1951 General linearized theory of the effect of surface films on water ripples. *Proc. K. Ned. Akad. Wet.* B **54**, 260.

HENDERSON, D. M. 1998 Effects of surfactants on Faraday-wave dynamics. *J. Fluid Mech.* **365**, 89–107.

HENDERSON, D. M. & MILES, J. W. 1994 Surface-wave damping in a circular cylinder with a fixed contact line. *J. Fluid Mech.* **275**, 285–299 (referred to herein as HM94).

HOWELL, D., HEATH, T., MCKENNA, C., HWANG, W. & SCHATZ, M. F. 2000 Measurements of surface-wave damping in a container. *Phys. Fluids* **12**, 322–326.

JENKINS, A. D. & DYSTHE, K. B. 1997 The effective film viscosity coefficients of a thin floating fluid layer. *J. Fluid Mech.* **344**, 335–337.

Levich, V. G. 1962 *Physicochemical Hydrodynamics*. Prentice-Hall.

Lopez, J. M. & Hirsa, A. 1998 Direct determination of the dependence of the surface shear and dilatational viscosities on the thermodynamic state of the interface: Theoretical foundations. *J. Colloid Interface Sci.* **206**, 231–239.

Martel, C., Nicolás, J. A. & Vega, J. M. 1998 Surface-wave damping in a brimful circular cylinder. *J. Fluid Mech.* **360**, 213–228. See also Corrigendum, *J. Fluid Mech.* **373**, 379 (referred to herein as MNV).

Miles, J. W. 1967 Surface-wave damping in closed basins. *Proc. R. Soc. Lond.* A. **297**, 459–475.

Miles, J. W. 1991 A note on surface films and surface waves. *Wave Motion* **13**, 303–306.

Miles, J. W. & Henderson, D. M. 1998 A note on interior vs. boundary-layer damping of surface waves in a circular cylinder. *J. Fluid Mech.* **364**, 319–323.

Van Dorn, W. G. 1966 Boundary dissipation of oscillatory waves. *J. Fluid Mech.* **24**, 769–779.